GENERALIZED FIBONACCI NUMBERS WITH FIVE PARAMETERS

by

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In this paper, we define five parameters generalization of Fibonacci numbers that generalizes Fibonacci, Pell, Modified Pell, Jacobsthal, Narayana, Padovan, k-Fibonacci, k-Pell, Modified k-Pell, k-Jacobsthal numbers and Fibonacci p-numbers, distance Fibonacci numbers, (2, k)-distance Fibonacci numbers, generalized (k, r)-Fibonacci numbers in the distance sense by extending the definition of a distance in the recurrence relation with two parameters and adding three parameters in the definition of this distance, simultaneously. Tiling and combinatorial interpretations of generalized Fibonacci numbers are presented, and explicit formulas that allow us to calculate the nth number are given. Also generating functions and some identities for these numbers are obtained.

Key words: *combinatorial identities, distance Fibonacci numbers, tilings, generalized Fibonacci numbers, sums formulas*

Introduction

The Fibonacci numbers, which have enticed the attention of various researchers in modern science for many years, are terms of well-known recurrence sequence defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with initial terms $F_0 = 0$, $F_1 = 1$. Many recurrence sequences have been presented in different ways as generalizations of the Fibonacci numbers [1]. One of the ways of generalization is to add an integer to the recurrence relation of the Fibonacci numbers. For instance, Pell numbers, $P_n = 2P_{n-1} + P_{n-2}$ with $P_0 = 0$, $P_1 = 1$ and Jacobsthal numbers, $J_n = J_{n-1} + 2J_{n-2}$ with $J_0 = 0$, $J_1 = 1$ are defined for $n \geq 2$ [2]. In the other way, an arbitrary parameter is considered as the distance between the terms added and the positions of the terms in the recurrence relation. For some distance, it has been studied Narayana numbers, $N_n = N_{n-1} + N_{n-3}$ with $N_0 = 0$, $N_1 = N_2 = 1$, and Padovan numbers, $P_n = P_{n-2} + P_{n-3}$ with $P_0 = P_1 = P_2 = 1$ for $n \geq 3$ by using recurrence relations of the Fibonacci type [3, 4].

In another way of generalizing the Fibonacci numbers, arbitrary parameters are added to the recurrence relation of the Fibonacci numbers. For the parameter k in the recurrence relation, Falcon and Plaza [5] determined k-Fibonacci sequences as $F_{k.n} = kF_{k.n-1} + F_{k.n-2}$ with initial terms $F_{k.0} = 0$, $F_{k.1} = 1$. Mikkawy and Sogabe [6] introduced a new family of the k-Fibonacci sequence. Tasyurdu *et al.* [7] also presented the period and many features of this new family. These studies have been a source of inspiration for many researchers and k-sequences of well-known number sequences are presented as various studies called types or families of generalized Fibonacci numbers. For more details on generalizations, see [8-15].

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In other generalizations of the Fibonacci numbers, one of terms of the recurrence relation is taken as the well-known Fibonacci number and the other term is taken in an arbitrary parameter distance. Its generalization known as Fibonacci p-numbers has led to generalizations of the Fibonacci numbers in the distance sense. The Fibonacci p-numbers are presented as $F_p(n) = F_p(n-1) + F_p(n-p-1)$ with initial terms $F_p(1) = F_p(2) = \ldots = F_p(p+1) = 1$ for $p \geq 0$, $n > p + 1$ [16], and the recurrence sequences, called generalized or distance Fibonacci numbers, are presented as follows:

– Generalized Fibonacci numbers [17]: $F(k, n) = F(k, n-1) + F(k, n-k)$ for $n \geq k + 1$ and $F(k, n) = n + 1$ for $0 \leq n \leq k$, the integer $k \geq 1$.
– Distance Fibonacci numbers [18]: $Fd(k, n) = Fd(k, n-k+1) + Fd(n-k)$ for $n \geq k$ and $Fd(k, n) = 1$ for $0 \leq n \leq k-1$, integers $k \geq 2$ and $n \geq 0$.
– (2, k)-distance Fibonacci numbers [19]: $F_2(k, n) = F_2(k, n-2) + F_2(k, n-k)$ for $n \geq k$ and $F_2(k, i) = 1$ for $i = 0,1, \ldots k-1$, integers $k \geq 1$ and $n \geq 0$.
– The Padovan p-numbers [20]: $Pap(n + p + 2) = Pap(n + p) + Pap(n)$ for $n \geq 1$ and $Pap(1) = Pap(2) = \ldots = Pap(p) = 0$, $Pap(p+1) = 1$, $Pap(p + 2) = 0$, integer $p = 2, 3, 4, \ldots$

Then, the following recurrence sequences are given by considering an arbitrary parameter in the recurrence relations of these distance Fibonacci numbers:

– Generalized (k, r)-Fibonacci numbers [21]: $F_{k,n}(r) = kF_{k,n-r}(r) + F_{k,n-2}(r)$ for $n \geq r$ and $F_{k,n}(r) = 1$, $n = 0, 1, 2, \ldots r-1$, except $F_{k,1}(1) = k$, integers $k \geq 1$, $n \geq 0$ and $r \geq 1$.
– The (k, p)-Fibonacci numbers [22]: $F_{k,p}(n) = pF_{k,p}(n-1) + (p-1)F_{k,p}(n-k+1) + F_{k,p}(n-k)$ for $n \geq k$ and $F_{k,p}(0) = 0$, $F_{k,p}(n) = p^{n-1}$ for $1 \leq n \leq k-1$, integers $k \geq 2$, $n \geq 0$ and a rational number $p \geq 1$.

More generalized versions of the Fibonacci numbers have been obtained by using different interpretations, such as the tiling and combinatorial interpretations, which are extensively used in researching generalized Fibonacci numbers and their properties. These numbers are interpreted as the number of distinct ways to tile a length n board, a $1\times n$ grid with cells labeled 1, 2, …, n, using squares and dominoes of various lengths. For instance, the nth Fibonacci number counts the number of distinct ways to tile a $1\times n$ board using 1×1 squares and 1×2 dominoes. The Fibonacci p-numbers count the number of distinct ways to tile a $1\times n$ board using various $1\times r$, r-ominoes from $r = 1$ up to $r = p + 1$. The combinatorial poofs are provided to derive well-known identities of the Fibonacci numbers via tiling using various schemes [23-25].

The aim of this study is to describe a new generalization of the Fibonacci numbers, including all the ways of generalization given one by one in the studies cited above and newly added, which extends the definition of the distance and number of parameters according to each term in recurrence relation. So, we define generalized Fibonacci numbers with five parameters. We obtain the general formulas, generating functions and identities for the generalized Fibonacci numbers. Also, tiling, combinatorial and set decomposition interpretations are presented, allowing us to derive these numbers and their properties.

Numbers $F^d_{s,t}(p,q;n)$

In this section, a new generalization with five parameters of the well-known Fibonacci, the Fibonacci type and the distance Fibonacci numbers, called generalized Fibonacci numbers, is presented such that the distance to the nth number is not only the parameter p with the previous term, but also the parameter q with the two previous term, and the added terms are the parameter d distance between themselves. Moreover, arbitrary parameters are added to the recurrence relation of these numbers and obtain their generating functions.

Definition 1: For integers *p, q, s, t* ≥ 1, *q* ≥ *p* and *d* = *q* – *p*, the *n*th number $F_{s,t}^{d}(p,q;n)$ is defined recursively by the recurrence relation

$$F_{s,t}^{d}(p,q;n)=s^{d}F_{s,t}^{d}(p,q;n-p)+tF_{s,t}^{d}(p,q;n-q), \quad n\geq q \tag{1}$$

with initial terms $F_{s,t}^{d}(p,q;n)=s^{d\lfloor n/p\rfloor}$ for $n=0,1,2,\ldots,q-1$. The sequences of the numbers $F_{s,t}^{d}(p,q;n)$ are denoted by $\{F_{s,t}^{d}(p,q;n)\}_{n\geq 0}$.

From the *Definition 1*, the special cases of the numbers $F_{s,t}^{d}(p,q;n)$ obtained according to the parameters *d, p, q, s, t* are given in tab 1.

Table 1. The special cases of the numbers $F_{s,t}^{d}(p,q;n)$

s	*t*	*p*	*q*	*d*	Symbol	*n*th $F_{s,t}^{d}(p,q;n)$ number
1	1	1	2	1	$F_{1,1}^{1}(1,2;n)=F_{n+1}$	F_n, *n*th Fibonacci number [1]
k	1	1	2	1	$F_{k,1}^{1}(1,2;n)=F_{k,n+1}$	$F_{k,n}$, *n*th *k*-Fibonacci number [5]
2	1	1	2	1	$F_{2,1}^{1}(1,2;n)=P_{n+1}$	P_n, *n*th Pell number [2]
2	*k*	1	2	1	$F_{2,k}^{1}(1,2;n)=P_{k,n+1}$	$P_{k,n}$, *n*th *k*-Pell number [13]
2	1	1	2	1	$F_{2,1}^{1}(1,2;n)=q_{n}$	q_n, *n*th modified Pell number [14]
2	*k*	1	2	1	$F_{2,k}^{1}(1,2;n)=q_{k,n}$	$q_{k,n}$, *n*th modified *k*-Pell number [14]
1	2	1	2	1	$F_{1,2}^{1}(1,2;n)=J_{n+1}$	J_n, *n*th Jacobsthal number [2]
k	2	1	2	1	$F_{k,2}^{1}(1,2;n)=J_{k,n+1}$	$J_{k,n}$, *n*th *k*-Jacobsthal number [15]
1	1	1	3	2	$F_{1,1}^{2}(1,3;n)=N_{n+1}$	N_n, *n*th Narayana number [3]
1	1	2	3	1	$F_{1,1}^{1}(2,3;n)=\mathcal{P}_{n}$	$\mathcal{P}_n$, *n*th Padovan number [4]
1	1	1	*p* + 1	*p*	$F_{1,1}^{p}(1,p+1;n)=F_{p}(n+1)$	$F_p(n)$, *n*th Fibonacci *p*-number [16]
1	1	*k* – 1	*k*	1	$F_{1,1}^{1}(k-1,k;n)=Fd(k,n)$	$Fd(k,n)$, *n*th distance Fibonacci number [18]
1	1	2	*k*	*k* – 2	$F_{1,1}^{k-2}(2,k;n)=F_{2}(k,n)$	$F_2(k,n)$, *n*th (2, *k*)-distance Fibonacci number [19]
1	*k*	2	*r*	*r* – 2	$F_{1,k}^{r-2}(2,r;n)=F_{k,n}(r)$	$F_{k,n}(r)$, *n*th generalized (*k, r*)-Fibonacci number [21]

Since all results given throughout the study are provided for all generalized Fibonacci numbers, the parameters *d, p, q, s, t* given in tab. 1 can be used in the theorem or corollary of any special cases of the generalized Fibonacci numbers.

Generating functions for $F_{s,t}^{d}(p,q;n)$

In this section, we introduce generating functions of the sequences $\{F_{s,t}^{d}(p,q;n)\}_{n\geq 0}$. Let $f_{s,t}^{d}(p,q;n)(x)$ be the generating functions of the sequences $\{F_{s,t}^{d}(p,q;n)\}_{n\geq 0}$. Then

$$f_{s,t}^{d}(p,q;n)(x)=\sum_{n=0}^{\infty}F_{s,t}^{d}(p,q;n)x^{n}.$$

Theorem 1: Let *d, p, q, s, t* ≥ 1, *q* > *p* and *d* = *q* – *p* be integers. Generating functions for the sequences $\{F_{s,t}^{d}(p,q;n)\}_{n\geq 0}$ are

$$f_{s,t}^{d}(p,q;n)(x)=\frac{1+s^{d\left\lfloor \frac{1}{p}\right\rfloor}x+\ldots+s^{d\left\lfloor \frac{q-1}{p}\right\rfloor}x^{q-1}-s^{d}\left(x^{p}+s^{d\left\lfloor \frac{1}{p}\right\rfloor}x^{p+1}+\ldots+s^{d\left\lfloor \frac{q-p-1}{p}\right\rfloor}x^{q-1}\right)}{1-s^{d}x^{p}-tx^{q}}\ .$$

Proof. If $f_{s,t}^{d}(p,q;n)(x)$ is the generating functions of the sequences $\{F_{s,t}^{d}(p,q;n)\}_{n\geq 0}$, using *Definition 1* we get

$$f_{s,t}^{d}(p,q;n)(x)-s^{d}x^{p}f_{s,t}^{d}(p,q;n)(x)-tx^{q}f_{s,t}^{d}(p,q;n)(x)=$$

$$=\sum_{n=0}^{\infty}F_{s,t}^{d}(p,q;n)x^{n}-s^{d}\sum_{n=0}^{\infty}F_{s,t}^{d}(p,q;n)x^{n+p}-t\sum_{n=0}^{\infty}F_{s,t}^{d}(p,q;n)x^{n+q}=$$

$$=\sum_{n=0}^{q-1}F_{s,t}^{d}(p,q;n)x^{n}+\sum_{n=q}^{\infty}F_{s,t}^{d}(p,q;n)x^{n}-s^{d}\sum_{n=0}^{q-p-1}F_{s,t}^{d}(p,q;n)x^{n+p}-$$

$$-s^{d}\sum_{n=q-p}^{\infty}F_{s,t}^{d}(p,q;n)x^{n+p}-t\sum_{n=0}^{\infty}F_{s,t}^{d}(p,q;n)x^{n+q}=$$

$$=\sum_{n=0}^{q-1}F_{s,t}^{d}(p,q;n)x^{n}-s^{d}\sum_{n=0}^{q-p-1}F_{s,t}^{d}(p,q;n)x^{n+p}+$$

$$+\left[\sum_{n=0}^{\infty}F_{s,t}^{d}(p,q;n+q)x^{n+q}-s^{d}\sum_{n=0}^{\infty}F_{s,t}^{d}(p,q;n+q-p)x^{n+q}-t\sum_{n=0}^{\infty}F_{s,t}^{d}(p,q;n)x^{n+q}\right]=$$

$$=1+s^{d\left\lfloor \frac{1}{p}\right\rfloor}x+\ldots+s^{d\left\lfloor \frac{q-1}{p}\right\rfloor}x^{q-1}-s^{d}\left(x^{p}+s^{d\left\lfloor \frac{1}{p}\right\rfloor}x^{p+1}+\ldots+s^{d\left\lfloor \frac{q-p-1}{p}\right\rfloor}x^{q-1}\right)$$

with $F_{s,t}^{d}(p,q;n)=s^{d\left\lfloor \frac{n}{p}\right\rfloor}$ for $n=0,1,2,\ldots,q-1$ and so

$$f_{s,t}^{d}(p,q;n)(x)=\frac{1+s^{d\left\lfloor \frac{1}{p}\right\rfloor}x+\ldots+s^{d\left\lfloor \frac{q-1}{p}\right\rfloor}x^{q-1}-s^{d}\left(x^{p}+s^{d\left\lfloor \frac{1}{p}\right\rfloor}x^{p+1}+\ldots+s^{d\left\lfloor \frac{q-p-1}{p}\right\rfloor}x^{q-1}\right)}{1-s^{d}x^{p}-tx^{q}}$$

which completes the proof.

Theorem 1 generalizes previously obtained generating functions for sequences of the Fibonacci numbers [8], the *k*-Fibonacci numbers [5], the Pell and the *k*-Pell numbers [13], the Modified Pell and the Modified *k*-Pell numbers [14], the Jacobsthal and the *k*-Jacobsthal numbers [15], the Narayana numbers [11], the Padovan number [26], the Fibonacci *p*-numbers [27], the generalized (*k*, *r*)-Fibonacci numbers [21]. In addition, the following corollary is proven for the Fibonacci numbers in the distance sense.

Corollary 1. Let $k > 2$ be integer. Then, we have

– Generating functions of the distance Fibonacci numbers where

$f_{1,1}^{1}(k-1,k;n)(x) = Fd(k,n)(x)$ are

$$Fd(k,n)(x) = \frac{\sum_{n=0}^{k-2} x^n}{1-x^{k-1}-x^k}$$

– Generating functions of the (2, k)-distance Fibonacci numbers where $f_{1,1}^{k-2}(2,k;n)(x) = F_2(k,n)(x)$ are

$$F_2(k,n)(x) = \frac{1+x}{1-x^2-x^k}\ .$$

Interpretation of the numbers $F_{s,t}^{d}(p,q;n)$

In this section, the numbers $F_{s,t}^{d}(p,q;n)$ are expressed with different interpretations such that tiling, combinatorial and set decomposition interpretations which allow to derive properties of them via tiling, combinatorial and special set decomposition proof. Also, using these interpretations, various general formulas for the nth number of sequences $\{F_{s,t}^{d}(p,q;n)\}_{n\geq 0}$ are obtained.

Set decomposition methods for the $F_{s,t}^{d}(p,q;n)$

We now represent the numbers $F_{s,t}^{d}(p,q;n)$ related to special decompositions of the set of n integers. Assume that $d, p, q, s, t \geq 1$ are integers and $X_n = \{1, 2, \ldots, n\}$ is the set of n integers. Let $\mathcal{M} = \{M_i : i \in I\}$ be the family of subsets of the set X_n such that subsets M_i contain consecutive integers and satisfy the following conditions

i. $|M_i| = p$ or $|M_i| = q$ for $i \in I$,
ii. If $|M_i| = p$ it may be one of s^d different colors,
iii. If $|M_i| = q$ it may be one of t different colors,
iv. $M_i \cap M_j = \emptyset$ for $i \neq j,\ i, j \in I$,
v. $n - p + 1 \leq |\bigcup_{i\in I} M_i| \leq n$,
vi. If $r \notin M_i$, then either $r = n$ or $r+1 \notin M_i$ for $i \in I$

where $q > p$ and $n \geq p$.

Each the family $\mathcal{M}$ is a color decomposition of the set of n integers related to s^d and t colors and is called as a (s, t)-decomposition of the set X_n. Let $\mathcal{M}^{p,q}(n)$ be the number of all (s, t)-decomposition of the set X_n. By the previous conditions, either $\{1,2,\ldots,p\} \in \mathcal{M}$ or $\{1,2,\ldots,q\} \in \mathcal{M}$. Let $\mathcal{M}^{p}(n)$ be the number of (s, t)-decompositions of the set X_n such that $\{1,2,\ldots,p\} \in \mathcal{M}$ and let $\mathcal{M}^{q}(n)$ be the number of (s, t)-decompositions of the set X_n such that $\{1,2,\ldots,q\} \in \mathcal{M}$. Since these two cases are mutually exclusive, by the addition principle

$$\mathcal{M}^{p,q}(n) = \mathcal{M}^{p}(n) + \mathcal{M}^{q}(n) \tag{2}$$

and we directly obtain the following theorem.

Theorem 2. Let $d, p, q, s, t \geq 1$ be integers. For $q > p$, $n \geq p$, $\mathcal{M}^{p,q}(n) = F_{s,t}^{d}(p,q;n)$.

P*roof.* By induction. Let $d, p, q, s, t \geq 1$, $q > p$ and $n \geq p$ be integers and $X_n = \{1, 2, \ldots, n\}$. If $n < q$, then $F_{s,t}^{d}(p,q;n) = s^{d\lfloor n/p \rfloor}$. For $p \leq n < q$, $\mathcal{M}^{p,q}(n) = F_{s,t}^{d}(p,q;n)$ since $|M_i| = p$ for $i \in I$ and it may be one of s^d different colors. Suppose that $n \geq q$ and $\mathcal{M}^{p,q}(n) = F_{s,t}^{d}(p,q;n)$ holds for n. We show that $\mathcal{M}^{p,q}(n+1) = F_{s,t}^{d}(p,q;n+1)$ is true for n + 1. Suppose $\{1,2,\ldots,p\} \in \mathcal{M}$ and add $\{1, 2, \ldots p\}$ to each of the families of subsets of the set $X_{n+1-p} = \{p + 1, p + 2, \ldots, n + 1\}$, so X_{n+1} has $s^d \mathcal{M}^{p,q}(n+1-p) = \mathcal{M}^{p}(n+1)$ with a choice of

s^d different colors. Suppose $\{1, 2, \ldots, q\} \in \mathcal{M}$ and add $\{1, 2, \ldots q\}$ to each of the families of subsets of the set $X_{n+1-q} = \{q + 1, q + 2, \ldots, n + 1\}$, so X_{n+1} has $t\mathcal{M}^{p,q}(n+1-q) = \mathcal{M}^q(n+1)$ with a choice of t different colors. On the other hand, using the induction's hypothesis we have $\mathcal{M}^{p,q}(n+1-p) = F^d_{s,t}(p,q;n+1-p)$ and $\mathcal{M}^{p,q}(n+1-q) = F^d_{s,t}(p,q;n+1-q)$. In addition, from eqs. (1) and (2), we obtain

$$\begin{aligned}\mathcal{M}^{p,q}(n+1) &= \mathcal{M}^p(n+1) + \mathcal{M}^q(n+1)\\ &= s^d\mathcal{M}^{p,q}(n+1-p) + t\mathcal{M}^{p,q}(n+1-q)\\ &= s^d F^d_{s,t}(p,q;n+1-p) + tF^d_{s,t}(p,q;n+1-q)\\ &= F^d_{s,t}(p,q;n+1)\end{aligned}$$

and the theorem is proved.

This decomposition interpretation in sets allows us to express the numbers $F^d_{s,t}(p,q;n)$ with the tiling interpretations. We now present the tiling approach to the nth term of sequences $\{F^d_{s,t}(p,q;n)\}_{n\geq 0}$.

Tiling methods for the $F^d_{s,t}(p,q;n)$

Let us consider a board of length n, called (1, n)-rectangle or n-rectangle, to represent the numbers $F^d_{s,t}(p,q;n)$. For integers $d, p, q, s, t \geq 1$, we use p-rectangles and q-rectangles to tile these n-rectangles where there are s^d different colors for the p-rectangles and t different colors for the q-rectangles. Suppose we begin from the leftmost when placing cells, in this case an r-rectangle remains at the end for $r \in \{0, 1, \ldots, p-1\}$. Also, it is obvious that the tiling either begins with a p-rectangle and the number of distinct ways to tile an n-rectangle is $s^d F^d_{s,t}(p,q;n-p)$ with a choice of s^d different colors or it begins with a q-rectangle and the number of distinct ways to tile an n-rectangle is $tF^d_{s,t}(p,q;n-q)$ with a choice of t different colors. Thus, $F^d_{s,t}(p,q;n) = s^d F^d_{s,t}(p,q;n-p) + tF^d_{s,t}(p,q;n-q)$ in *Definition 1* is obtained.

Theorem 3. Let $d, p, q, s, t \geq 1$ be integers. For $q > p$, $n \geq p$, $F^d_{s,t}(p,q;n)$ counts the number of distinct ways to tile an n-rectangle using colored p-rectangles and colored q-rectangles, where there are s^d different colors for the p-rectangles and t different colors for the q-rectangles.

Proof. By induction. Let $d, p, q, s, t \geq 1$ be integers. For $q > p$, $n \geq p$, indicate by $(p,q)_n$ the number of distinct ways to tile an n-rectangle using s^d different colored p-rectangles and t different colored q-rectangles. To complete the proof, we show that $(p,q)_n = F^d_{s,t}(p,q;n)$. If $p \leq n < q$, then all cells in tilings are p-rectangle and it is obtained in exactly $s^{d\lfloor n/p \rfloor}$ ways. So $(p,q)_n = s^{d\lfloor n/p \rfloor} = F^d_{p,q}(p,q;n)$ for $p \leq n < q$. Suppose that $n \geq q$ and $(p,q)_n = F^d_{s,t}(p,q;n)$ holds for n. We show that $(p,q)_{n+1} = F^d_{s,t}(p,q;n+1)$ is true for $n+1$. From now on, there are two options according to first cell. Let $(p)_{n+1}$ be the number of tilings first cell being the p-rectangles and let $(q)_{n+1}$ be the number of tilings first being the q-rectangles for the $(n + 1)$-rectangle. Since these two cases are mutually exclusive, by the addition principle $(p,q)_{n+1} = (p)_{n+1} + (q)_{n+1}$. If first cell is a p-rectangle or a q-rectangle, we can simply remove the first cell to get a tiling of lengths $n + 1 - p$ or $n + 1 - q$, and this is the same as the number of distinct ways to tile an $(n+1-p)$-rectangle or an $(n+1-q)$-rectangle according to this first cell being a p-rectangle or a q-rectangle, respectively. Then $s^d(p,q)_{n+1-p} = (p)_{n+1}$ or $t(p,q)_{n+1-q} = (q)_{n+1}$. On the other hand, using the induction's hypothesis we have

$(p,q)_{(n+1-p)} = F^d_{s,t}(p,q;n+1-p)$ and $(p,q)_{(n+1-q)} = F^d_{s,t}(p,q;n+1-q)$. In addition, from eq. (1), we obtain

$$\begin{aligned}(p,q)_{n+1} &= (p)_{n+1} + (q)_{n+1} \\ &= s^d (p,q)_{(n+1-p)} + t(p,q)_{(n+1-q)} \\ &= s^d F^d_{s,t}(p,q;n+1-p) + tF^d_{s,t}(p,q;n+1-q) \\ &= F^d_{s,t}(p,q;n+1)\end{aligned}$$

and the theorem is proved.

Example 1. There are 16 distinct ways to tile a 5-rectangle using colored 2-rectangles and colored 3-rectangles, where there are 2 different colors for the 2-rectangles and 3 different colors for the 3-rectangles where $d=1,\ p=2,\ q=3,\ s=2,\ t=3$, all tilings are provided in fig. 1. Note that the colorless 1-rectangle always remains at the end. So, we have $F^1_{2,3}(2,3;5)=16$.

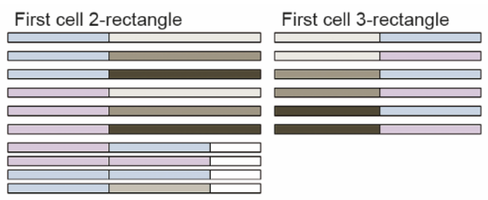

Figure 1. Tiling interpretations of $F^1_{2,3}(2,3;5)$ number

Combinatorial Methods for the $F^d_{s,t}(p,q;n)$

We now give explicit formulas that allow us to calculate the *n*th term of sequences $\{F^d_{p,q}(p,q;n)\}_{n\ge 0}$ using the combinatorial approach. For integers $d,p,q,s,t \ge 1$, $q>p$ and $n \ge p$, assume that $\mathcal{Y} = \{(m_1, m_2, \cdots, m_i, r) \mid i \in I \text{ and } 0 \le r \le p-1\}$ is the set of the ordered compositions of an *n*-rectangle and satisfies the following conditions

i. $m_i \in \{p,q\}$ for $i \in I$
ii. $n-p+1 \le \Sigma_{i\in I} m_i \le n$
iii. $r \in \{0,1,\ldots,p-1\}$ is always the last component of ordered composition

where there are s^d different colors for the components *p* and *t* different colors for the components *q*. Each element of the set $\mathcal{Y}$ is a composition of an *n*-rectangle and $(m_1, m_2, \cdots, m_i)$ is called as a *m*-order, except *r*. Let $\mathcal{Y}_{p,q}(n)$ be the number of all *m*-orders for the *n*-rectangle. By the previous conditions, either $m_1 = p$ or $m_1 = q$. Let $\mathcal{Y}_p(n)$ be the number of all *m*-orders such that $m_1 = p$ and let $\mathcal{Y}_q(n)$ be the number of all *m*-orders such that $m_1 = q$. Since these two cases are mutually exclusive, by the addition principle

$$\mathcal{Y}_{p,q}(n) = \mathcal{Y}_p(n) + \mathcal{Y}_q(n)\,. \tag{3}$$

If first component is *p* or *q*, we can simply remove the first component to get *m*-orders of an $(n-p)$-rectangle or an $(n-q)$-rectangle, and this is the same as the number of all *m*-orders for an $(n-p)$-rectangle or an $(n-q)$-rectangle according to this first component being a *p* and a *q*, respectively. Then $s^d \mathcal{Y}_{p,q}(n-p) = \mathcal{Y}_p(n)$ or $t\mathcal{Y}_{p,q}(n-q) = \mathcal{Y}_q(n)$. Thus, $\mathcal{Y}_{p,q}(n)$ and $F^d_{s,t}(p,q;n)$ in eqs. (1) and (3) have the same recurrence relations and initial terms. From here we directly obtain the following theorem.

Theorem 4. Let $d,p,q,s,t \ge 1$ be integers. For $q > p$, $n \ge p$, $\mathcal{Y}_{p,q}(n) = F^d_{s,t}(p,q;n)$.

We now present another general formula that directly gives the *n*th term of sequences $\{F^d_{s,t}(p,q;n)\}_{n\ge 0}$.

Theorem 5: Let $d, p, q, s, t \geq 1$ be integers. For $q \geq p$, $n \geq 0$, then

$$F_{s,t}^{d}(p,q;n)=\sum_{j=0}^{\left\lfloor \frac{n}{q}\right\rfloor} s^{d\left\lfloor \frac{n-jq}{p}\right\rfloor} t^{j}\begin{pmatrix} j+\left\lfloor \dfrac{n-jq}{p}\right\rfloor \\ j\end{pmatrix}$$

Proof. Let $d, p, q, s, t \geq 1$, $q \geq p$ and $n \geq 0$ be integers. From *Theorem 4,* the number of all *m*-orders of the set $\mathcal{Y}$ is equal to $F_{s,t}^{d}(p,q;n)$ where there are s^d different colors for the components *p* and *t* different colors for the components *q*. Assume that there are i times the component *p* and *j* times the component *q* where $0 \leq i \leq \lfloor n/p \rfloor$ and $0 \leq j \leq \lfloor n/q \rfloor$. So, the number of ways of choosing of component *q* found *j* times in all *m*-orders consisting of $i+j$ components in the set $\mathcal{Y}$ is equal to

$$F_{p,q}^{d}(p,q;n)=\sum_{j=0}^{\lfloor n/q \rfloor} s^{di} t^{j}\binom{i+j}{j}.$$

Using *Definition 1,* $\lfloor n/q \rfloor = 0$ for $n \leq q-1$, and so

$$\Sigma_{j=0}^{\lfloor n/q \rfloor} s^{di} t^{j}\binom{i+j}{j}=s^{di} \ .$$

Suppose that $n \geq q$. In addition, since $ip + jq \leq n$, it is obtained as $i \leq \lfloor n-jq/p \rfloor$ for a fixed *j* and the desired

$$F_{p,q}^{d}(p,q;n)=\sum_{j=0}^{\lfloor n/q \rfloor} s^{di} t^{j}\binom{i+j}{j}=\sum_{j=0}^{\lfloor n/q \rfloor} s^{d\left\lfloor \frac{n-jq}{p}\right\rfloor} t^{j}\begin{pmatrix} j+\left\lfloor \dfrac{n-jq}{p}\right\rfloor \\ j\end{pmatrix}$$

is found. So, the proof is completed.

Theorem 6: Let $d, k, p, q, s, t \geq 1$ be integers. For $n \geq kq$, we have

$$F_{s,t}^{d}(p,q;n)=\sum_{i=0}^{k}\binom{k}{i} s^{d(k-i)} t^{i} F_{s,t}^{d}\left(p,q;n-(k-i)p-iq\right).$$

Proof. This result is obtained directly by induction from the eq. (1).

Theorems 5 and *6* generalize previously obtained general formulas, well-known combinatorial formulas, that allow us to calculate the *n*th Fibonacci number [8], the *n*th distance Fibonacci number [18], the *n*th (2, *k*)-distance Fibonacci number [19], the *n*th *k*-Fibonacci number and the *n*th generalized (*k*, *r*)-Fibonacci number [21], the *n*th Fibonacci *p*-number [28], other numbers in tab. 1 [8, 9, 11, 13-15].

Identities of the numbers $F_{s,t}^{d}(p,q;n)$

In this section we present the sums formulas for the numbers $F_{s,t}^{d}(p,q;n)$. These formulas give the sums of the odd and even terms of the sequences $\{F_{s,t}^{d}(p,q;n)\}_{n\geq 0}$ as well as the sums of the first *n* terms with special subscripts.

Theorem 7: Let $d, p, q, s, t \geq 1$ and $n \geq 0$ be integers. For $p \leq q$, we have

$$F_{s,t}^{d}(p,q;n+q) = s^{d\left\lfloor \frac{n+q}{p} \right\rfloor} + \sum_{i=0}^{\left\lceil \frac{n-p+1}{p} \right\rceil} s^{di} t F_{s,t}^{d}(p,q;n-ip)$$

Proof. By using *Theorem 3*, we get that the left-hand side of the identity, $F_{s,t}^{d}(p,q;n+q)$, counts the number of distinct ways to tile an $(n+q)$-rectangle with different colored *p*-rectangles and different colored *q*-rectangles. If we show that the right-hand side of the identity counts the same number tilings, the proof is completed. Let us consider an $(n+q)$-rectangle and place the cells, beginning to place the first rectangle from the leftmost cell. In tilings, either all cells are the *p*-rectangle, counting in $s^{d\lfloor n+q/p \rfloor}$ tilings, or there is at least a *q*-rectangle. To calculate other tilings, we consider the smallest length tilings containing only a *q*-rectangle, relative to the location of the first *q*-rectangle. In this case, the *q*-rectangle that can be selected with *t* different colors is either in the first cell, counting in $tF_{s,t}^{d}(p,q;n)$ tilings, or there are *i* times the *p*-rectangle that can be selected with s^d different colors before the *q*-rectangle, counting in $s^{di}tF_{s,t}^{d}(p,q;n-ip)$ tilings for $1 \leq i \leq \lceil n-p+1/p \rceil$. Summing all the tilings, we achieve the desired result.

When tiling the rectangles of lengths $np+q$ and $np+q+r$ for $0 \leq r < p$ we obtain the following theorem, and its proof is similar to that of the previous theorem.

Theorem 8: Let $d, p, q, s, t \geq 1$ and $n \geq 0$ be integers. For $p \leq q$ and $0 \leq r < p$, we have

i: $$F_{s,t}^{d}(p,q;np+q) = s^{dn+d\left\lfloor \frac{q}{p} \right\rfloor} + \sum_{i=0}^{n} s^{d(n-i)} t F_{s,t}^{d}(p,q;ip)$$

ii: $$F_{s,t}^{d}(p,q;np+q+r) = s^{dn+d\left\lfloor \frac{q+r}{p} \right\rfloor} + \sum_{i=0}^{n} s^{d(n-i)} t F_{s,t}^{d}(p,q;ip+r).$$

Theorem 9. Let $d, k, p, q, s, t \geq 1$ and $n \geq 0$ be integers. For $p \leq q$ and $n+p = qk+r$, $0 \leq r < q$, we have

$$F_{s,t}^{d}(p,q;n+p) = \begin{cases} \displaystyle\sum_{i=0}^{\left\lceil \frac{n-q+1}{q} \right\rceil} s^{d} t^{i} F_{s,t}^{d}(p,q;n-iq) & \text{if } r \geq p \\ t^{\left\lfloor \frac{n+p}{q} \right\rfloor} + \displaystyle\sum_{i=0}^{\left\lceil \frac{n-q+1}{q} \right\rceil} s^{d} t^{i} F_{s,t}^{d}(p,q;n-iq) & \text{if } r < p \end{cases}$$

Proof. By using *Theorem 3*, we get that the left-hand side of the identity, $F_{s,t}^{d}(p,q;n+p)$, counts the number of distinct ways to tile an $(n+p)$-rectangle with different colored *p*-rectangles and different colored *q*-rectangles. If we show that the right-hand side of the identity counts the same number tilings, the proof is completed. Let us consider an $(n+p)$-rectangle and place the cells, beginning to place the first rectangle from the leftmost cell. When $r < p$ there is exactly a tiling where all the cells are the *q*-rectangle, counting in $t^{\lfloor n+p/q \rfloor}$ tilings. In the other cases there is at least a *p*-rectangle. To calculate other tilings, we consider the smallest length tilings containing only a *p*-rectangle, relative to the location of the first *p*-rectangle. In this case, the *p*-rectangle that can be selected with s^d different colors is either in the first cell, counting in $s^d F_{s,t}^d(p,q;n)$ tilings, or there are i times the q-rectangle that can be selected with t different colors before the p-rectangle, counting in $t^i s^d F_{s,t}^d(p,q;n-iq)$ tilings for $1 \le i \le \lceil n-q+1/q \rceil$. Summing all the tilings, we achieve the desired result.

When tiling the rectangles of lengths $nq+p$ and $nq+p+r$ for $0 \le r < q$ we obtain the following theorem, and its proof is similar to that of the previous theorem.

Theorem 10: Let $d,k,p,q,s,t \ge 1$ and $n \ge 0$ be integers. For $p \le q$ and $nq+p = qk+r,\ 0 \le r < q,$ we have

i:
$$F_{s,t}^d(p,q;nq+p) = \begin{cases} \sum_{i=0}^{n} s^d t^{n-i} F_{s,t}^d(p,q;iq) & \text{if} \quad r \ge p \\ t^{n+\left\lfloor \frac{p}{q} \right\rfloor} + \sum_{i=0}^{n} s^d t^{n-i} F_{s,t}^d(p,q;iq) & \text{if} \quad r < p \end{cases}$$

ii:
$$F_{s,t}^d(p,q;nq+p+r) = \begin{cases} \sum_{i=0}^{n} s^d t^{n-i} F_{s,t}^d(p,q;iq+r) & \text{if} \quad 2r \ge p \\ t^{n+\left\lfloor \frac{p+r}{q} \right\rfloor} + \sum_{i=0}^{n} s^d t^{n-i} F_{s,t}^d(p,q;iq+r) & \text{if} \quad 2r < p \end{cases}$$

Theorems 7, 8, 9, and *10* generalize previously obtained identities for any special case of the generalized Fibonacci numbers given in tab. 1. For these identities, the parameters d,p,q,s,t given in tab. 1 can be used in *Theorems 7, 8, 9,* and *10*.

Conclusion

Generalizations of the Fibonacci numbers are related to an arbitrary integer p for initial terms, the coefficient of the added terms in the recurrence relation as well as and the distance between them. In the distance sense, while generalizing the Fibonacci numbers, one of the terms in the recurrence relation is taken as the exactly well-known Fibonacci number, for the other, an arbitrary integer k distance is considered.

In this study, for arbitrary integers $d,p,q,s,t \ge 1$ the nth generalized Fibonacci numbers are obtained by adding two previous terms which are the $(n-p)^{\text{th}}$ generalized Fibonacci number and the $(n-q)^{\text{th}}$ generalized Fibonacci number and adding parameters d,s,t to recurrence relation. Thus, we have generalized the well-known Fibonacci, Pell, Modified Pell, Jacobsthal, Narayana, Padovan, k-Fibonacci, k-Pell, Modified k-Pell, k-Jacobsthal numbers and the Fibonacci p-numbers, the distance Fibonacci numbers, the (2, k)-distance Fibonacci numbers, the generalized (k, r)-Fibonacci numbers in the distance sense given in [1-5, 13-16, 18, 19, 21]. These numbers are expressed by tiling, combinatorial and set decomposition interpretations which allow to derive properties, and generating functions, the general formulas used to calculate the nth term and to find the sums of the first n terms with special subscripts.

It would be interesting to study these numbers in matrix theory. More general formulas that allow us to calculate the nth Fibonacci number and relations like the well-known relations between the Fibonacci and Fibonacci type numbers can be explored.

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