THE GENERALIZED TIME FRACTIONAL GARDNER EQUATION VIA NUMERICAL MESHLESS COLLOCATION METHOD

by

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In this study, the meshless collocation approach is used to determine the numerical solution the generalized time-fractional Gardner equation. The Crank-Nicolson technique is used to approximate space derivatives, whereas the Caputo derivative of fractional order is used to approximate the first order time fractional derivative. The numerical solutions, which show the method's efficacy and accuracy, are provided and discussed. The numerical solution shows that our method is effective in producing extremely accurate results.

Key words*: time-fractional Gardner equation, Crank-Nicolson scheme, radial basis functions, Caputo fractional derivative*

Introduction

The history of fractional calculus (FC) is as old as that of classical calculus (or integer order calculus). In the beginning, the idea of FC was developed slowly. However, it has gained the intensive attention of researchers in the past few decades due to its increasing applications in engineering and science. Fractional derivatives are commonly used to describe the various materials and processes with memory and hereditary properties. In 1695, when Newton and Leibniz had just been introduced classical calculus, L'Hospital and Leibniz discussed the meaning of the derivative of order 1/2. After that, many mathematicians worked on this question, which gave rise to the development of FC. Abel, Riesz, Liouville, Laplace, Grunwald, Erdelyi, Fourier, Letnikov, Riemann, Marchaud, and Levy worked on the fractional derivatives in the middle of last century [1].

In order to solve numerous physical models, fractional PDE are utilized to simulate problems that are functions of various variables. The fractional derivatives in these equations, however, prevent any approach from offering a closed-form solution for non-linear FPDEs. Consequently, there is growing demand for an effective and reliable numerical approach to solve these types of problems [2-12].

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The Gardner equation (GE), which is derived to explain the internal solitary waves in shallow water, is a related merging of the Korteweg-de Vries (KdV) and modified KdV equations. The GE is frequently used in many areas of physics, including fluid physics, plasma physics, and quantum field theory. It also discusses how wave phenomena spread in plasma and solid states. The non-linear propagation of ion-acoustic waves in an unmagnetized plasma composed of negative ions, non-thermal electrons, positive ions, and a negative-ion beam with the Tsallis distribution [13] is studied using time fractional GE (TFGE) in plasma physics. In this study, we take into account the generalized TFGE (GTFGE):

$$\frac{\partial^{\xi} w}{\partial t^{\xi}} + A_1 w w_x + A_2 w^2 w_x + A_3 w_{xxx} = \psi(x,t), \;\; t \in [0, T_0], \;\; 0 < \xi \le 1, \;\; x \in R \tag{1}$$

where $\psi(x, t)$ is the source term, ξ – the time fractional derivative order, $\partial^{\xi} w/\partial t^{\xi}$ is fractional derivative in the Caputo sense, A_1 and A_2 are the non-linear coefficients and A_3 is dispersion coefficient [13]. The system is affected by the time fractional derivative order ξ. An increase in the time fractional derivative order ξ value can reduce the system's non-linearity and the amplitudes of the solitary pulses [13]. It is also concluded that time fractional order introduces higher order non-linearity or dispersion relationships into the plasma system, which plays an important role in varying the amplitude of solitary waves.

Formulation of the numerical scheme

Time fractional derivative

The time fractional derivative $\partial^{\xi} w(x, t)/ \partial t^{\xi}$ in eq. (1), is the Caputo fractional derivative [8], which can be written:

$$\frac{\partial^{\xi} w(x,t)}{\partial t^{\xi}} = \begin{cases} \dfrac{1}{\Gamma(1-\xi)} \displaystyle\int_0^t \frac{\partial w(x,\delta)}{\partial \delta} \frac{1}{(t-\delta)^{\xi}} \mathrm{d}\delta, & 0 < \xi < 1 \\ \dfrac{\partial w(x,t)}{\partial t}, & \xi = 1 \end{cases} \tag{2}$$

where ξ is order of fractional derivative, $t_m = m\Delta t$, $m = 0, 1, 2,...,N$ and Δt – the time step. The finite difference scheme is used to discretize the classical derivative term:

$$\frac{\partial^{\xi} w(x,t_{m+1})}{\partial t^{\xi}} = \frac{1}{\Gamma(1-\xi)} \int_0^{t_{m+1}} \frac{\partial w(x,\delta)}{\partial \delta} (t_{m+1} - \delta)^{-\xi} \mathrm{d}\delta = \frac{1}{\Gamma(1-\xi)} \sum_{l=0m} \int_{l\Delta t}^{(l+1)\Delta t} \frac{\partial w(x,\delta_l)}{\partial \delta} (t_{m+1} - \delta)^{-\xi} \mathrm{d}\delta \tag{3}$$

the first order time derivative appearing in eq. (3) is approximated:

$$\frac{\partial w(x,\delta_l)}{\partial \delta} = \frac{w(x,\delta_{l+1}) - w(x,\delta_l)}{\delta} + O(\Delta t) \tag{4}$$

Then

$$\frac{\partial^{\xi} w(x,t_{m+1})}{\partial t^{\xi}} \approx \frac{1}{\Gamma(1-\xi)} \sum_{l=0}^{m} \int_{l\Delta t}^{(l+1)\Delta t} \left(\frac{w(x,\delta_{l+1}) - w(x,\delta_l)}{\delta} + O(\Delta t) \right) (t_{m+1} - \delta)^{-\xi} \mathrm{d}\delta$$

$$\frac{\partial^{\xi} w(x,t_{m+1})}{\partial t^{\xi}} = a_0 \sum_{l=0}^{m} b_l (w_{m-l+1} - w_{m-l}) + \left[O(\Delta t)^{2-\xi} \right] \tag{5}$$

where

$$a_0 = \frac{\Delta t^{-\xi}}{\Gamma(2-\xi)},\ b_l = (l+1)^{1-\xi} - l^{1-\xi},\ l = 0,1,2,\cdots,m$$

and $w(x, t = 0) = w_0(x)$ is initial condition (IC).

Finally, eq. (5) can be written in precise form as:

$$\frac{\partial^\xi w}{\partial t^\xi} = \begin{cases} a_0(w_{m+1} - w_m) + a_0 \sum_{l=1}^{m} b_l \left(w_{m-l+1} - w_{m-l}\right), & m \geq 1 \\ a_0(w_1 - w_0), & m = 0 \end{cases} \tag{6}$$

Space fractional derivative

In the next step, the meshless collocation method is applied and $w(x, t_{m+1})$ is collocated by the RBF. The solution is interpolated at M different collocation points $x_j|j = 1, 2,..., M$, where $x_j|j \in \Omega$ are interior points while x_1 and x_M are boundary points, Ω represents a bounded domain and $\partial\Omega$ is its boundary. The numerical solution of $w(x, t_{m+1})$ can be expressed in terms of the RBF:

$$w(x,t_{m+1}) = \sum_{j=1}^{M} \lambda_{j_{m+1}} \varphi\left(\| x_i - x_j \|\right) = \sum_{j=1}^{M} \lambda_{j_{m+1}} \varphi\left(d_{ij}\right) \tag{7}$$

where $i = 1, 2,..., M$, $\lambda_{j_{m+1}}$ are the unknown coefficients at the $(m + 1)^{\text{th}}$ time level, $\varphi(d_{ij})$ is the RBF, and $\|.\|$ is Euclidean norm and $d_{ij} = \|x_i - x_j\|$.

Equation (7) can be written in matrix form:

$$w_{m+1} = S\lambda_{m+1} \tag{8}$$

where

$$w_{m+1} = \left[w_{1_{m+1}}, w_{2_{m+1}}, \cdots, w_{M_{m+1}}\right]^T,\ \lambda_{m+1} = \left[\lambda_{1_{m+1}}, \lambda_{2_{m+1}}, \cdots, \lambda_{M_{m+1}}\right]^T$$

and the collocation matrix S^0 is given:

$$S^0 = \begin{bmatrix} \varphi_{11} & \cdots & \varphi_{1j} & \cdots & \varphi_{1M} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \varphi_{i1} & \cdots & \varphi_{ii} & \cdots & \varphi_{iM} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varphi_{M1} & \cdots & \varphi_{Mj} & \cdots & \varphi_{MM} \end{bmatrix}$$

$$S^1 = \left[\varphi'\left(d_{ij}\right)\right],\ i,j = 1,\cdots,M \tag{9}$$

$$S^2 = \left[\varphi''\left(d_{ij}\right)\right],\ i,j = 1,\cdots,M$$

$$S^3 = \left[\varphi'''\left(d_{ij}\right)\right],\ i,j = 1,\cdots,M$$

Numerical experiments

In this section, the proposed meshless collocation method is applied to solve the governing eq. (1). We have applied schemes over three problems including 1-D TFPDE. We have utilized the following error norms:

$$E_\infty = max\,|\,w_{EXACT}(i) - w_{APP}(i)\,|$$

where $i = 1, 2, \ldots, M$, w_{EXACT}, and w_{APP} represents exact solution and approximate solution, respectively.

Example 1. Consider eq. (1). In [13], linear dispersion relationship is given by the formula:

$$V = -\sqrt{\frac{\gamma\upsilon_1 + \beta\upsilon_2 + 1}{c_1\vartheta}}$$

where

$$c_1 = \varrho_1 + \frac{(k+1)}{2}$$

In eq. (1), the non-linear coefficients A_1, A_2 and dispersion coefficients A_3:

$$A_1 = \frac{-3\gamma\upsilon_1^2 - 3\beta\upsilon_2^2 + 1 - 2V^4c_2 + 2V^4c_2\beta + 2V^4c_2\gamma}{2V(\gamma\upsilon_1 + \beta\upsilon_2 + 1)}$$

$$A_2 = \frac{15(1 + \beta\upsilon_2^3 + \gamma)}{4V^3(\gamma\upsilon_1 + \beta\upsilon_2 + 1)} \text{ and } A_3 = \frac{V^3}{2(\gamma\upsilon_1 + \beta\upsilon_2 + 1)}$$

where

$$c_2 = \frac{((3-k)(k+1) + 4\varrho_1(k+1) + 8\varrho_2)}{8},\ \varrho_1 = \frac{-16k\alpha}{(3 - 14k + 15k^2 + 12\alpha)},\ \text{and } \varrho_2 = \frac{16(2k-1)k\alpha}{(3 - 14k + 15k^2 + 12\alpha)}$$

The rarefactive solitary wave solution of eq. (1) can be obtained by taking the initial value of the classical Gardner equation as the zeroth order approximation:

$$w(x,0) = \frac{24A_3\text{sech}^2(x)}{(A_1 + \sqrt{\sigma}) - (A_1 - \sqrt{\sigma})\tanh^2(x)} \tag{10}$$

Because of the absence of the exact solution, the convergence of the method is assessed by the well known double mesh principle and the results are mentioned in tab. 2. Table 1 shows the numerical solutions for the different values of the time step size Δt for t and x. In fig. 1, the numerical solutions at different values of ξ are plotted which demonstrates that by increasing the value of ξ the amplitude and steepness decreases. Figure 2 is devoted to plotting the numerical values at different values of t, which shows that the smoothness increases by increasing time, t. In fig. 3, the 3-D plots of the numerical solutions are plotted against the position x and time, t. Consider $c = 1500$, $\gamma = 0.1$, $\beta = 0.1$, $\alpha = 0.2$, $\vartheta = 0.7$, $\upsilon_1 = 0.5$, $\upsilon_1 = 0.8$, and $k = 1.8$ in Test *Problem 1*.

Table 1. Numerical values for different values of Δt at $\xi = 0.8$, $t = 0.1$, and $M = 81$ considered in Test *Problem 1*

$x/\Delta t$	0.0001	0.0005	0.001	0.005
1.1	–0.32472	–0.32471	–0.32470	–0.32461
1.2	–0.26689	–0.26689	–0.26688	–0.26683
1.3	–0.21914	–0.21913	–0.21913	–0.21910
1.4	–0.17982	–0.17982	–0.17981	–0.17979

Table 2. The E_∞ for different values of M and ξ for Test *Problem 1*

ξ/M	20	40	80	160
0.30	$3.4030 \cdot 10^{-1}$	$1.1500 \cdot 10^{-2}$	$2.8000 \cdot 10^{-2}$	$5.0500 \cdot 10^{-3}$
0.60	$2.0010 \cdot 10^{-1}$	$3.8900 \cdot 10^{-2}$	$2.3000 \cdot 10^{-3}$	$9.2000 \cdot 10^{-3}$
0.75	$1.4930 \cdot 10^{-1}$	$3.1500 \cdot 10^{-2}$	$1.9000 \cdot 10^{-3}$	$7.6000 \cdot 10^{-3}$

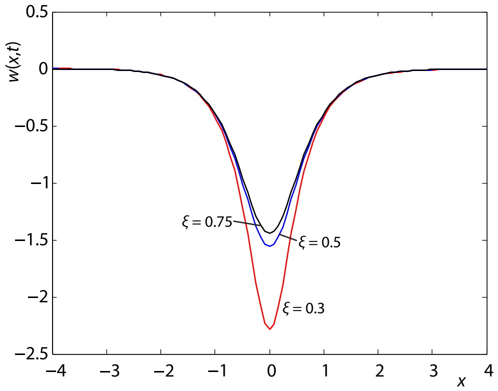

Figure 1. Numerical solution for different values of ξ at $t = 0.1$, $\Delta t = 0.005$, and $M = 101$ for Test *Problem 1*

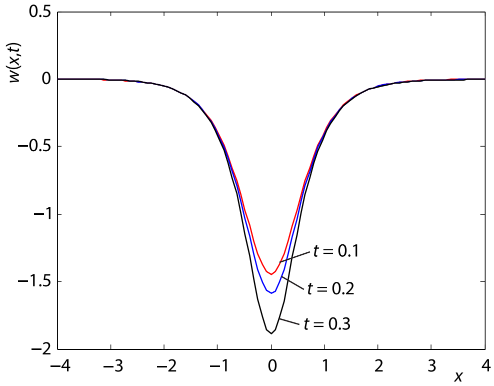

Figure 2. Numerical solution for different values of t at $\xi = 0.75$, $\Delta t = 0.005$, and $M = 101$ for Test *Problem 1*

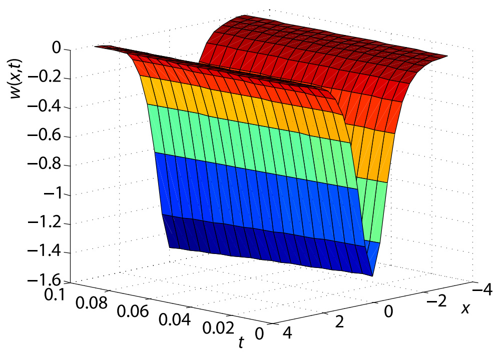

Figure 3. Time evolutions of rarefactive solitary wave solution of TFGE with $\xi = 0.5$, $t = 0.1$, $\Delta t = 0.005$, and $M = 21$ for Test *Problem 1*

Conclusion

In this work, the meshless collocation method is applied to find the numerical solution of GTFGE. The time derivative is considered in Caputo sense, and the scheme is derived for $0 < \xi < 1$. Different Test Problems are included to check the efficiency and accuracy of the scheme, and that the current method is basic and simple. The effect of the fractional order on the solution is discussed with the help of figures in all of the Test Problems, and it is concluded that the fractional order derivative plays a significant role in accuracy, as well as in variation in amplitude (either increasing or decreasing).

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