ANALYTICAL FORMULA OF THE WEHRL ENTROPY AND WEHRL PHASE DISTRIBUTION OF THE FIELD IN GENERALIZED COHERENT SQUEEZED STATES

by

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Original scientific paper
https://doi.org/10.2298/TSCI22S1425A

In this framework, the effect of a Kerr-like medium and the coupling function dependent on the number of photons operator on the interaction between a two-level atom and a non-linear field is studied. A relation between the Kerr-like medium parameter and the field-atom coupling parameter is used to obtain a simplified formula for Rabi frequency. The wave function of the proposed model is obtained, followed by the derivation of the phase distribution and from which the wehrl entropy formula is calculated. The effect of the initial state and the non-linear function dependent on the number of photons operator and the Kerr-like medium on entanglement is calculated through the Wehrl entropy formula, wehrl distribution and the behaviour of photons by studying the correlation function. The entanglement decreases when the function dependent on the number of photons operator is taken into account, while the entanglement gradually improves when the squeezed state is considered, and the entanglement decreases significantly when considering the Kerr medium. An oscillatory distribution is formed between the classical and non-classical in the coherent state. The non-classical distribution disappears when considering the squeezed state and the Kerr-like medium.

Key words: *Kerr like medium, Wehrl entropy; correlation function, von Neumann*

Introduction

It has been thoroughly investigated the importance of Kerr medium on the dynamical properties of a quantum states of light. It has been defined that, the Schroedinger's cat states is a superposition of coherent states which exhibit macroscopic properties for a large value of the relevant complex amplitude in the phase space [1-5]. For a 3-D circuit quantum electrodynamic structure, it has developed a Kerr regime that allows the interaction strength between the photons to exceed the loss rate [6]. This makes it easier to construct and manipulate superpositions of coherent states, which in turn enables the production of multicomponent cat states in an experimental setting. *Tara et al*. [7] it has been discovered that the coherent field propagating through a Kerr medium in a coherent state lead to Schrodinger macroscopic superposition states in addition the production superpositions of squeezed coherent states.

However, since they require lowering the variations of a single quadrature variable below the ground state uncertainty, the compressed states of the harmonic oscillator have attracted a lot of experimental attention. These states are critical for enhancing interferometer

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sensitivity in quantum metrology [8] and are also crucial for the new gravitational wave detection by high power laser interferometers [9]. The continuous variable quantum key distribution techniques [10] likewise depend on the compressed states, and they impart [11] an improvement over their coherent counterparts. Additionally, they have recently been employed as sensitive detectors for single-photon level photon scattering recoil events [12]. Additionally, Giacomo has researched the uncertainty relations for the Wehrl entropy and its link with quantum memory [13]. It is shown that the sequence of appropriate quantum Gaussian states asymptotically achieves the smallest conditional Wehrl entropy can be obtained for all quantum states with a given conditional von Neumann entropy. The dynamical behavior of the Wehrl entropy and its phase distribution has been investigate in many quantum systems such as, single Cooper-pair box in a cavity field in the presence of dissipation [14], field-superconducting under the effect of decoherence [15], moving four-level atom [16] and recently, two atoms in a dissipative cavity field under Kerr medium effect [17].

Entanglement, entropic uncertainty relations, and Wehrl entropy have recently been connected for many quantum states [18]. A quantifiable perfect witness for pure state bipartite entanglement may be obtained using the Wehrl mutual information, which also lowers the entanglement entropy. However, the squeezed state is one of the non-classical electromagnetic field states, and as a result, some observables show fewer fluctuations than for the vacuum state [19-22]. It's called the squeezed operator on the coherent state [19]. Many facets of squeezed displaced Fock states (SDFS), including squeezing and photon statistics, have been explored and researched [20]. The SDFS are a particular case of squeezed coherent states, squeezed number states [21], and two-photon coherent states (squeezed coherent states).

Recently, experimental reports have described the formation of non-classical states of motion for trapped ions, including Fock states, coherent states, compressed states, and Schrodinger cat states [23-27]. Studies of certain superpositions of Glauber (ordinary) coherent states have demonstrated sub-Poissonian statistics and quadrature squeezing [28]. Many researchers have developed techniques for superimposing coherent states in experiments [29-38]. Joshi and Obada [39] has explored the superposition of two binomial states as well as two negative binomial states. The generating strategy and properties of the superposition of displaced Fock states have been addressed [40]. On the other hand, research has been done on the superposition of two SDFS with various coherent parameters [41].

In previous studies, it was found that the JCM is a valid and attractive source for its more realistic applications in the field of information technology [42]. Most of the generalizations are based on the expansion of the number of photons transmitted or the multiplicity of modes of the electromagnetic field [43, 44]. Non-linear geometric connections based on the number of photons operator which are field distortion in several different modes are included in the atom field coupling [45]. The non-linear formulas of the functions based on the photons operator were linked to the group of SU(1,1) or SU(2), which revived the process of generalizations with several advantages [46, 47]. The most important of which is the ease of obtaining the solution of the normalsize Schroedinger equation and the description of most of the previous models in a simple framework [48]. Also, a Kerr-like medium has been included in many models, because of its wide impact in measuring the amount of entanglement between quantum systems, whether they contain one or several atoms [43, 49]. The interactions between the atom and the field related to a single mode or several modes in the presence of stark shift have been studied [50]. The Stark shift has a direct impact on the amount of entanglement between the field and the atom [51].

The dynamics of the system

In the current section, the system Hamiltonian which describes the interaction between the two-level atom and the cavity field with single-mode in the presence of Kerr-like medium. The dynamics is described via Schrodinger equation:

$$i\frac{d|\psi\rangle}{dt} = H|\psi\rangle$$

where the Hamiltonian, H, of such a system takes the form:

$$H = \chi\hat{A}^{+2}\hat{A}^2 + g(\sqrt{\hat{A}^+\hat{A}}\,\hat{A}^j\,|+\rangle\langle-| + \sqrt{\hat{A}^+\hat{A}}\,\hat{A}^{+j}\,|-\rangle\langle+|), \tag{1}$$

where $\hat{A}^+$ and $\hat{A}$ are creation and annihilation operators, respectively. The letter χ is the Kerr-like medium and g is the coupling constant.

The exact solution of the proposed system takes the form:

$$|\psi\rangle = \sum\nolimits_n A_n\,|n,e\rangle + B_n\,|n+1,g\rangle$$

where A_n and B_n are the amplitude of the probabilities of the excited and ground states, respectively. By using the Schroedinger equation:

$$i\frac{\partial|\psi\rangle}{\partial t} = H|\psi\rangle$$

we can obtain the system of differential equations:

$$\begin{aligned} H|\psi\rangle &= (\omega_F nA_n + \chi n(n-1)A_n)\,|n,e\rangle + \\ &+(\omega_F(n+1)B_n + \chi n(n+1)B_n)\,|n+1,g\rangle + \\ &+g((n+1)B_n\,|n,e\rangle + (n+1)A_n\,|n+1,g\rangle) \end{aligned} \tag{2}$$

Now, equating coefficients in Schrodinger equation results:

$$i\frac{\mathrm{d}A_n}{\mathrm{d}t} = (n\omega_F + \chi n(n-1))A_n + g(n+1)B_n \tag{3}$$

Also, we have:

$$i\frac{\mathrm{d}B_n}{\mathrm{d}t} = ((n+1)\omega_F + \chi n(n+1))B_n + g(n+1)A_n \tag{4}$$

Setting

$$h_1 = n\omega_F + \chi n(n-1), h_2 = (n+1)\omega_F + \chi n(n+1) \ \text{ and } \ v = g(n+1)$$

eqs. (3) and (4) become:

$$\left(i\frac{\mathrm{d}}{\mathrm{d}t} - h_1\right)A_n = vB_n \tag{5}$$

$$\left(i\frac{\mathrm{d}}{\mathrm{d}t}\right) - h_2 B_n = vA_n \tag{6}$$

Together reduces to:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} - i(h_1 + h_2)\frac{\mathrm{d}}{\mathrm{d}t} + h_1 h_2 - v^2\right]A_n = 0 \tag{7}$$

Similarly, for B_n we have:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2}+i(h_1+h_2)\frac{\mathrm{d}}{\mathrm{d}t}-h_1h_2+v^2\right]B_n=0 \tag{8}$$

Therefore, when $\omega_F = 2\chi$ and

$$\mu=\sqrt{\chi^2+g^2}(n+1)$$

we have the solution:

$$A_n(t)=q_n(\cos\mu t-ih\sin\mu t) \tag{9}$$

where

$$h=\frac{\chi}{\sqrt{\chi^2+g^2}} \text{ and } q_n=e^{-\frac{1}{2}|\alpha|^2}\frac{e^{i\alpha}\,|\,\alpha\,|^n}{\sqrt{n!}}$$

Substituting in eq. (5) we obtain the solution for B_n:

$$B_n(t)=-ikq_n\sin\mu t \tag{10}$$

where

$$k=\frac{g}{\sqrt{g^2+\chi^2}}$$

The density matrix:

$$\rho_F=\sum_n\sum_m(A_nA_m^*+B_nB_m^*)\,|\,n\rangle\langle m\,|$$

So,

$$\langle\beta\,|\,\rho_F\,|\,\beta\rangle=|\,\langle\beta\,|\,A_n\,|\,n\rangle\,|^2+|\,\langle\beta\,|\,B_n\,|\,n\rangle\,|^2 \tag{11}$$

Now, for

$$\langle\beta\,|=\sum\nolimits_{k=0}^{\infty}\frac{e^{-\frac{1}{2}|\beta|^2}}{\sqrt{k!}}\beta^k e^{-ik\theta}\langle k\,| \text{ we have}$$

$$\langle\beta\,|\,A_n\,|\,n\rangle=\sum_{n=0}^{\infty}\frac{e^{-\frac{1}{2}|\beta|^2}}{\sqrt{n!}}\beta^n e^{-in\theta}A_n,\ \langle\beta\,|\,B_n\,|\,n\rangle=\sum_{n=0}^{\infty}\frac{e^{-\frac{1}{2}|\beta|^2}}{\sqrt{n!}}\beta^n e^{-in\theta}B_n$$

Thus

$$\langle\beta\,|\,A_n\,|\,n\rangle=e^{-\frac{1}{2}\left(|\beta|^2+|\alpha|^2\right)}\sum_{n=0}^{\infty}e^{-in\theta}\frac{(|\,\beta\,||\,\alpha\,|)^n}{n!}(\cos\mu t-ih\sin\mu t)=$$

$$=e^{-\frac{1}{2}\left(|\beta|^2+|\alpha|^2\right)}\left(\sum_{n=0}^{\infty}e^{-in\theta}\frac{(|\,\beta\,||\,\alpha\,|)^n}{n!}\times\frac{1}{2}\left(e^{ir(n+1)t}+e^{-ir(n+1)t}\right)-\right.$$

$$-ih\sum_{n=0}^{\infty}e^{-in\theta}\frac{(|\beta||\alpha|)^n}{n!}\times\frac{1}{2i}\left(e^{ir(n+1)t}-e^{-ir(n+1)t}\right)\Bigg)=$$

$$=\frac{1}{2}e^{-\frac{1}{2}\left(|\beta|^2+|\alpha|^2\right)}\left(e^{irt}\sum_{n=0}^{\infty}\frac{(|\alpha||\beta|e^{-i(\theta-rt)})^n}{n!}+e^{-irt}\sum_{n=0}^{\infty}\frac{(|\alpha||\beta|e^{-i(\theta+rt)})^n}{n!}-\right.$$

$$\left.-he^{irt}\sum_{n=0}^{\infty}\frac{\left(|\alpha||\beta|e^{-i(\theta-rt)}\right)^n}{n!}+he^{-irt}\sum_{n=0}^{\infty}\frac{\left(|\alpha||\beta|e^{-i(\theta+rt)}\right)^n}{n!}\right)=$$

$$=\frac{1}{2}e^{-\frac{1}{2}\left(|\beta|^2+|\alpha|^2\right)}\left((1-h)e^{irt}\sum_{n=0}^{\infty}\frac{\left(|\alpha||\beta|e^{-i(\theta-rt)}\right)^n}{n!}+(1+h)e^{-irt}\sum_{n=0}^{\infty}\frac{\left(|\alpha||\beta|e^{-i(\theta+rt)}\right)^n}{n!}\right)=$$

$$=\frac{1}{2}e^{-\frac{1}{2}\left(|\beta|^2+|\alpha|^2\right)}\left((1-h)e^{irt}e^{|\alpha||\beta|e^{-i(\theta-rt)}}+(1+h)e^{-irt}e^{|\alpha||\beta|e^{-i(\theta+rt)}}\right)$$

Similarly, we have

$$\langle\beta|B_n|n\rangle=e^{-\frac{1}{2}\left(|\beta|^2+|\alpha|^2\right)}\sum_{n=0}^{\infty}e^{-in\theta}\frac{(|\beta||\alpha|)^n}{n!}\frac{-ig}{\sqrt{g^2+\chi^2}}\sin\mu t=$$

$$=\frac{-k}{2}e^{-\frac{1}{2}\left(|\beta|^2+|\alpha|^2\right)}\sum_{n=0}^{\infty}e^{-in\theta}\frac{(|\beta||\alpha|)^n}{n!}\left(e^{ir(n+1)t}-e^{-ir(n+1)t}\right)=$$

$$=\frac{-k}{2}e^{-\frac{1}{2}\left(|\beta|^2+|\alpha|^2\right)}\left(e^{irt}\sum_{n=0}^{\infty}e^{-in\theta}\frac{(|\beta||\alpha|)^n}{n!}e^{irnt}-e^{-irt}\sum_{n=0}^{\infty}e^{-in\theta}\frac{(|\beta||\alpha|)^n}{n!}e^{-irnt}\right)=$$

$$=\frac{-k}{2}e^{-\frac{1}{2}\left(|\beta|^2+|\alpha|^2\right)}\left(e^{irt}e^{|\alpha||\beta|e^{-i(\theta-rt)}}-e^{-irt}e^{|\alpha||\beta|e^{-i(\theta+rt)}}\right)$$

Therefore:

$$\langle\beta|\rho_F|\beta\rangle=|\langle\beta|A_n|n\rangle|^2+|\langle\beta|B_n|n\rangle|^2=$$
$$=\frac{e^{-\left(|\alpha|^2+|\beta|^2\right)}}{2}\left((1-h)e^{2|\alpha||\beta|\cos(\theta-rt)}+(1+h)e^{2|\alpha||\beta|\cos(\theta+rt)}\right)$$

Hence, the Husimi function reads:

$$Q(\beta)=\frac{\langle\beta|\rho_F|\beta\rangle}{\pi}=\frac{e^{-\left(|\alpha|^2+|\beta|^2\right)}}{2\pi}\left((1-h)e^{2|\alpha||\beta|\cos(\theta-rt)}+(1+h)e^{2|\alpha||\beta|\cos(\theta+rt)}\right)\tag{12}$$

where

$$r=\sqrt{\chi^2+g^2}$$

The phase distribution

The Wehrl phase distribution is given:

$$S_\theta = -\int_0^\infty Q(\beta)\ln Q(\beta)\,|\beta|\,\mathrm{d}|\beta| \tag{13}$$

So, for Husimi function (12) when $t = 0$ we have:

$$S_\theta = -\int_0^\infty \frac{e^{-\left(|\alpha|^2+|\beta|^2\right)+2|\alpha||\beta|\cos\theta}}{\pi}\{-|\alpha|^2-|\beta|^2-\ln 2\pi+2|\alpha||\beta|\cos\theta\}|\beta|\,\mathrm{d}|\beta| =$$

$$= -\frac{e^{-|\alpha|^2}}{2\pi}(-\sqrt{\pi}|\alpha|\left(|\alpha|^2+\ln\pi+\frac{1}{2}\right)(1+\mathrm{erf}(\alpha\cos\theta))e^{|\alpha|^2\cos^2\theta}\cos\theta + \tag{14}$$

$$+|\alpha|^2\cos^2\theta+\sqrt{\pi}|\alpha|^3(1+\mathrm{erf}(\alpha\cos\theta))e^{|\alpha|^2\cos^2\theta}\cos^3\theta-\left(|\alpha|^2+\ln\pi+1)\right)$$

In this case, the Wehrl entropy is:

$$S_w = \int_0^{2\pi} S_\theta \mathrm{d}\theta = -\frac{e^{-|\alpha|^2}}{2\pi}\left(\int_0^{2\pi} -\sqrt{\pi}|\alpha|(|\alpha|^2+\ln\pi+\frac{1}{2}\right)\cos\theta(1+\mathrm{erf}(\alpha\cos\theta))e^{|\alpha|^2\cos^2\theta}\mathrm{d}\theta +$$

$$+\int_0^{2\pi}|\alpha|^2\cos^2\theta\mathrm{d}\theta+\int_0^{2\pi}\sqrt{\pi}|\alpha|^3\left(1+\mathrm{erf}(\alpha\cos\theta)\right)e^{|\alpha|^2\cos^2\theta}\cos^3\theta\mathrm{d}\theta-\int_0^{2\pi}\left(|\alpha|^2+\ln\pi+1)\mathrm{d}\theta\right) \tag{15}$$

Expressing the factor

$$e^{|\alpha|^2\cos^2\theta} \quad \text{as} \quad \sum_{j=0}^{\infty}\frac{|\alpha|^{2j}\cos^{2j}\theta}{j!}$$

and the error function:

$$\mathrm{erf}(\alpha\cos\theta) = \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k|\alpha|^{2k+1}\cos^{2k+1}\theta}{k!(2k+1)} \tag{16}$$

we can subsequently use the result:

$$\int_0^{2\pi}\cos^n\theta\mathrm{d}\theta = \frac{\left[1-(-1)^{n+1}\right]\pi}{2^n(n+1)\beta\left(\frac{n+2}{2},\frac{n+2}{2}\right)}$$

where β is the beta function. So, for $j + k = r$ we get:

$$S_w = -\frac{e^{-|\alpha|^2}}{2\pi}\left(-\pi\left(|\alpha|^2+2\ln\pi+1\right)+\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{\pi(-1)^k|\alpha|^{2(r+1)}}{k!j!(2k+1)4^r}\cdot\right.$$

$$\left.\cdot\frac{-|\alpha|^2(2r+5)-2\left(\ln\pi+\frac{1}{2}\right)\left[2r^2+9r+10\right]}{2\left[4(r+2)r+15\right](r+2)\beta(r+2,r+2)}\right) \tag{17}$$

Finally, the second order correlation function is used to detect the non-classical behavior of the field during the interaction with a two-level atom under the effect of Kerr medium and intensity dependent coupling. The second order correlation function can be expressed on terms of the two quantities $\langle\hat{A}^{+}\hat{A}\rangle$, and $\langle\hat{A}^{+2}\hat{A}^{2}\rangle$ as [52]:

$$G^2(t) = \frac{\left\langle \hat{A}^{+2}\hat{A}^2 \right\rangle}{\left(\left\langle \hat{A}^{+}\hat{A} \right\rangle\right)^2} \tag{18}$$

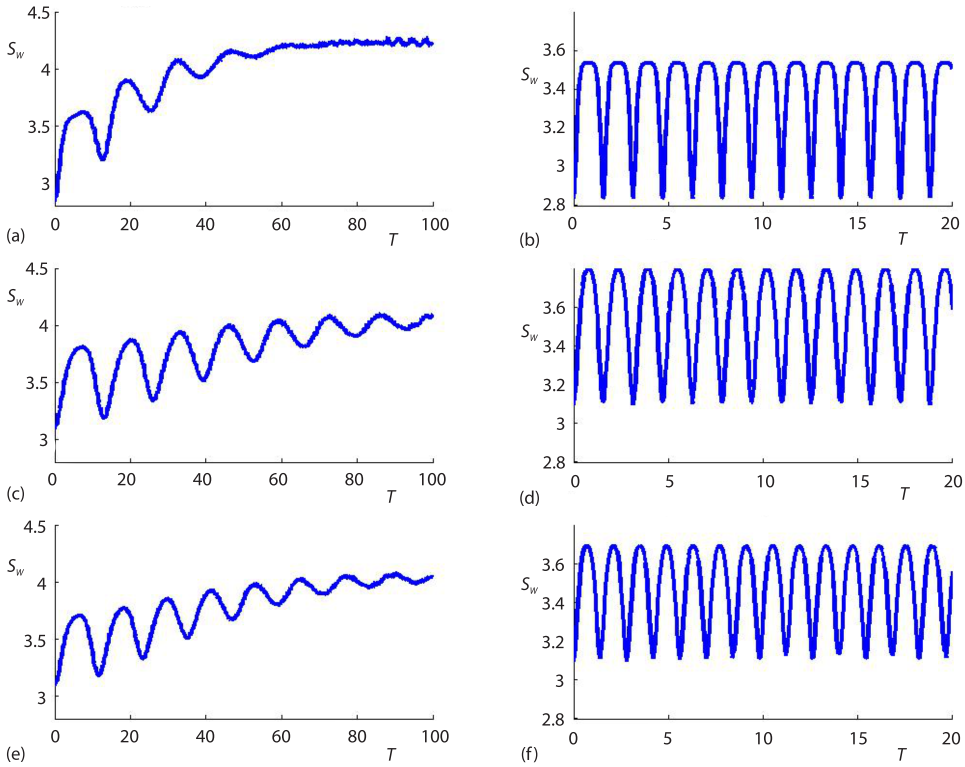

Figure 1. The Wehrl entropy as function of time, where (a) $\chi = 0, f(\hat{n}) = \hat{I}, r = 0.001$, (b) $\chi = 0, f(\hat{n}) = (\hat{n}+1)^{1/2}, r = 0.001$, (c) $\chi = 0, f(\hat{n}) = \hat{I}, r = 0.75$, (d) $\chi = 0, f(\hat{n}) = (\hat{n}+1)^{1/2}$, $r = 0.75$, (e) $\chi = 0.75, f(\hat{n}) = \hat{I}, r = 0.001$, and (f) $\chi = 0.75, f(\hat{n}) = (\hat{n}+1)^{1/2}, r = 0.7$

The effect of the Kerr medium, the dependence on the number of photons in the coupling and the initial state of the field on the entanglement between the field and the atom are studied here. Figure 1(a) represents the $S_w(t)$ relation in the absence of the effect of both the Kerr medium and the function of the dependence on the number of photons and the consideration of the coherent state approximately ($\chi = 0, f(\hat{n}) = \hat{I}, r = 0.001$). The interaction begins with a weak entanglement, followed by an entanglement that gradually increases in strength with increasing time. The coherence stabilizes after a period of time and the amplitude of oscillations decreases when the function $S_w(t)$ reaches its maximum values. The situation is quite different when the dependence on the number of photons is taken into account $f(\hat{n}) = (\hat{n}+1)^{1/2}$, as shown in fig. 1(b). An oscillating of entanglement function between maximum and minimum values is formed in a recursive manner. Therefore, adding a function of dependence on the number of photons leads to a weak entanglement between the field and the atom. Fig.1c shows the activation of the role of the initial state of the field ($\chi = 0, f(\hat{n}) = \hat{I}, r = 0.75$). The interaction begins almost from the state of separation, followed by the formation of the entanglement and gradually grows. The function $S_w(t)$ needs a large period of time to reach the maximum values compared to the previous case. The entanglement improves significantly after taking into account the photon dependence function $f(\hat{n}) = (\hat{n} + 1)^{1/2}$. Moreover, the function $S_w(t)$ reaches both minimum and maximum values repeatedly, as can be seen from fig.1(d). Figure 1(e) shows the effect of the Kerr medium on the entanglement between the field and the atom by setting ($\chi = 0.75, f(\hat{n}) = \hat{I}, r = 0.5$). The inclusion of the Kerr medium into the interaction cavity leads to more weak entanglement, therefore, the function $S_w(t)$ needs a large period of time to reach the maximum state. The position changes completely after activating the role of the photon dependence function. It is clear fig. 1(f) that a fluctuating entanglement between the minimum and maximum values is clearly generated. Moreover, activating the role of the function $S_w(t)$ leads to more order and repetition of the function periodically.

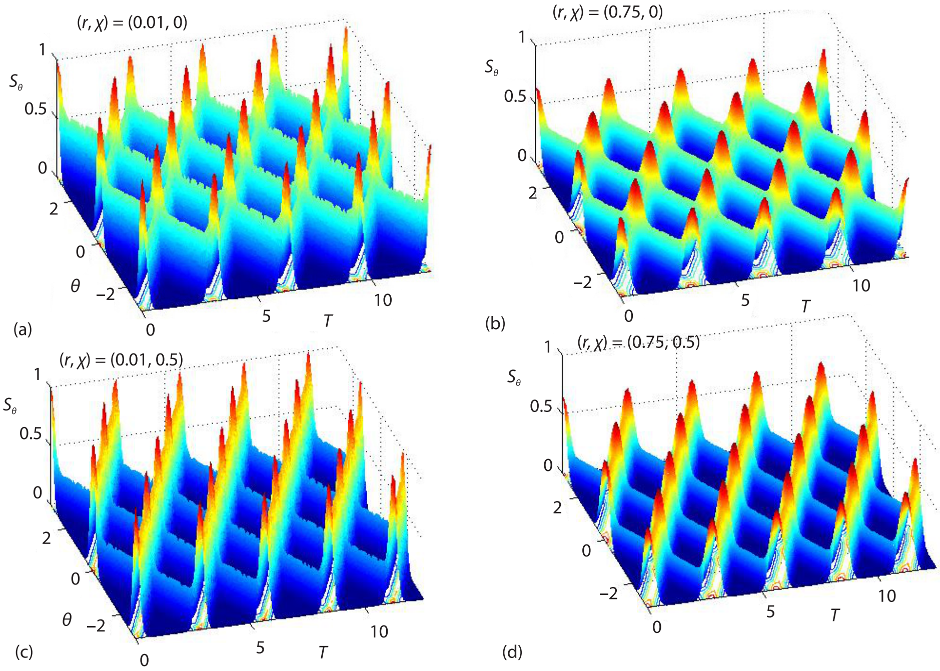

Figure 2. The Wehrl distribution as function of the angle θ and the time t, where (a) $\chi = 0$, $f(\hat{n}) = \hat{I}$, $r = 0.01$, (b) $\chi = 0$, $f(\hat{n}) = (\hat{n} + 1)^{1/2}$, $r = 0.001$, (c) $\chi = 0$, $f(\hat{n}) = \hat{I}$, $r = 0.75$, (d) $\chi = 0$, $f(\hat{n}) = (\hat{n} + 1)^{1/2}$, $r = 0.75$

Here, the effect of Kerr-like medium, initial state, and the function of the photon operator on Wehrl distribution are studied. The field is initially in the coherent state and the atom in the excited state, excluding the influence of a Kerr-like medium and the function of photon number operator ($\chi = 0, f(\hat{n}) = \hat{I}, r = 0.01$). The distribution begins with a symmetrical peak about the $\theta = 0$. With increasing time, the peak is divided into two branches in the direction of the sides, with a decrease in the maximum values. The peak reaches its highest value at $\theta = \pm\pi/2$. Continuously increasing the time, the peak begins to return again, in the opposite direction, with a decrease in the maximum values. Finally, the two peaks meet at $\theta = \pm\pi$. Thus, the previous shape is repeated with increasing time and produces similar shapes as shown in fig. 2(a). Figure 2(b) shows the effect of the squeezed state, an isomorph is generated with the previous case with a decrease in the maximum values of the Wehrl distribution. It is known that the Wehrl distribution is a probability distribution, that is, the area under the Wehrl curve is constant. Therefore, when the extreme values decrease, the thickness of the distribution increases, as is clear from the shape of the distribution. When the Kerr-like medium is taken into account, this is illustrated by fig. 2(c). The extreme values decrease on one side of the distribution while they increase on the other side. This is due to the constancy of the area under a Wehrl distributed curve, so it increases in one area and decreases in the other. The maximum values decrease and the thickness of the Wehrl distribution increases after considering the squeezed state, as shown in fig. 2(d). From the results, it is clear that Kerr medium breaks the symmetry of Wehrl distribution around the $\theta = 0$-axis. While considering the squeezed state reduces the maximum values and increases the thickness of the distribution.

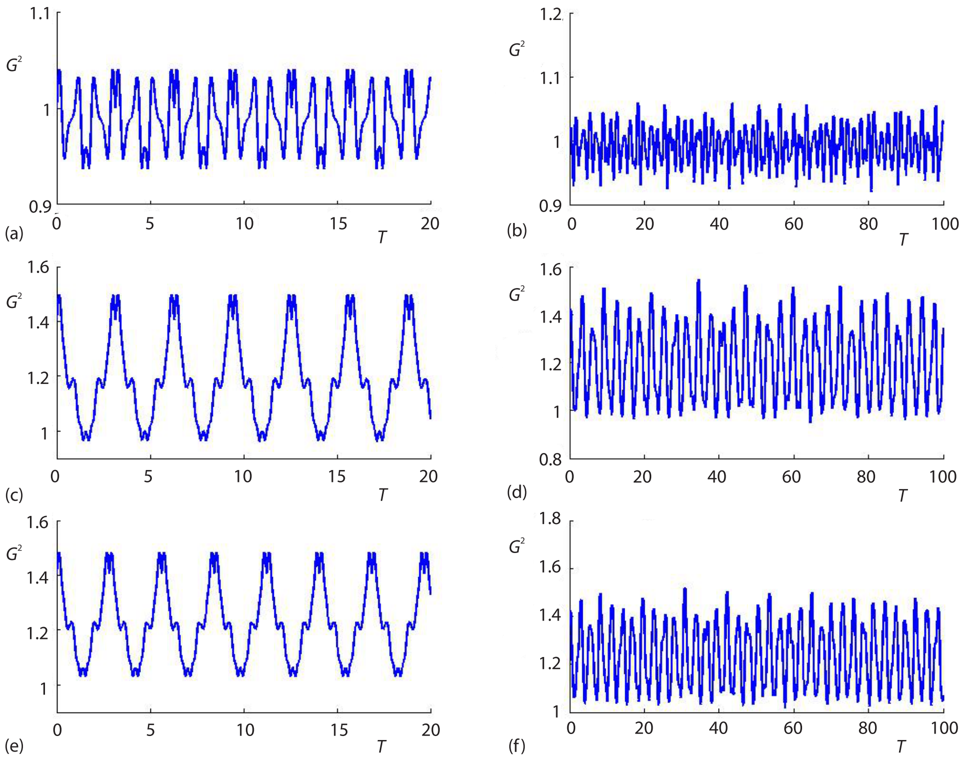

Figure 3. The correlation function as function of time, where (a) $\chi = 0, f(\hat{n}) = \hat{I}$, $r = 0.001$, (b) $\chi = 0$, $f(\hat{n}) = (\hat{n} + 1)^{1/2}$, $r = 0.001$, (c) $\chi = 0, f(\hat{n}) = \hat{I}$, $r = 0.75$, (d) $\chi = 0, f(\hat{n}) = (\hat{n} + 1)^{1/2}$, $r = 0.75$, (e) $\chi = 0.75$, $f(\hat{n}) = (\hat{I})^{1/2}$, $r = 0.001$, and (f) $\chi = 0.75, f(\hat{n}) = (\hat{n} + 1)^{1/2}$, $r = 0.7$

Correlation behavior between photons through the correlation function is studied here, under the same aforementioned conditions in entanglement. Figure 3(a) shows the correlation behavior of photons in the coherent state after excluding both the non-linear function and the Kerr-like medium ($\chi = 0, f(\hat{n}) = \hat{I}$, $r = 0.01$). The distribution fluctuates between classical and non-classical behavior, passing through the Poisson distribution frequently. The distribution tends to be non-classical most of the time after the inclusion of the function $f(\hat{n}) = (\hat{n} + 1)^{1/2}$ of the interaction cavity as shown in the fig. 3(b). The classical distribution improves when the squeezed state is considered, as shown in fig. 3(c). The correlation function appears more regular than the coherent case. The correlation function reaches the non-classical distribution at short intervals compared to the periods of the classical distribution. The correlation function appears irregularly and the non-classical distribution disappears completely after the inclusion of the non-linear function dependent on the number of photons operator into the interaction cavity as shown in fig. 3(d). Figure 3(e) illustrates the activation of the Kerr-like medium, showing the classical distribution more clearly than the squeezed state, and completely erasing the non-classical distribution when the Kerr-like medium is included for the interaction cavity. The case of the classical distribution increases further after taking the function of the photon operator into account, and the randomness of the distribution increases as seen in fig. 3(f).

Conclusion

In the previous sections, a model was proposed that contains a two-level atom that interacts with a single-mode field, taking into account a function that depends on the number of photons operator in the atom-field coupling. The wave function of this model was calculated under the condition of a relation between the Kerr-like medium and the field-atom coupling parameters. The effect of the initial state, the function dependent on the number of photons operator, and the Kerr-like medium on the entanglement between the parts of the proposed system, the Wehrl distribution, and the behaviour of correlation function were studied. Strong entanglement is generated when the effect of the photon-dependent function is neglected, and the entanglement increases with increasing time. While weak entanglement is generated in the case of considering the function that depends on the number of photons, it reaches at some times to the state of purity (the state of separation between the parts of the system). While the entanglement improves significantly when considering the squeezed state and decreases when considering the Kerr-like medium. It is clear that Kerr medium breaks the symmetry of Wehrl distribution around the $\theta = 0$-axis. While considering the squeezed state reduces the maximum values and increases the thickness of the distribution. The correlation function has both classical and non-classical behavior in the coherent state. Adding a function that depends on the number of photons reduces the non-classical distribution. Setting the field in the squeezed state generates a classical distribution. Adding a Kerr-like mean erases the non-classical distribution completely and the correlation function appears randomly.

Funding

This research did not receive any external funding other than that provided by Taif University, Taif, Saudi Arabia.

Acknowledgmen

Research support offered by the Deanship of Scientific Research, Taif University, Saudi Arabia [1-442-71] is acknowledged.

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Paper submitted: October 3, 2022
Paper revised: November 12, 2022
Paper accepted: November 18, 2022