ANALYTICAL STUDY OF MHD COUPLE STRESS CASSON NANOFLUID-FLOW OVER STRETCHING SURFACE

by

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The impact of coupled stress casson nanofluid-flow over a stretching surface will be examined in this paper. The fundamental controlling PDE are transformed using the stated similarity transformation into a pair of coupled, non-linear ODE, one for velocity and the other for temperature distribution. The modeled flow problem's approximate analytical solution was discovered using the approximate analytical approach. Graphs are used to illustrate the effects of different factors. A table illustrating the relationship between the Nusselt number and skin friction is provided.

Key words: *nanofluid-flow, computational analysis, MHD, couple stress, stretching surface*

Introduction

The application of non-Newtonian fluid is more as compare to the Newtonian fluids in industries and engineering sector due to the high heat transfer ratio. The sub class of non-Newtonian fluid is Casson nanofluid. Casson [1] for the first Casson familiarized casson fluid model which symbolizes a shear type thinning fluid. Attia *et al*. from [2] used parallel plates to study couette flow by using Casson fluid model. Megahe *et al*. [3] investigated Casson fluid along with flexible heat flux. Abolbashariet *et al*. [4] examined Casson fluid with entropy generation by using nanoparticles. Vijaya *et al*. [5] inspected Casson fluid film along with temperature dependency and viscous dissipated heat source. Saffman [6], worked on dusty fluid-flow for dusty gas for the first time. Miller and Michael [7] they used a stretching cylinder to study dusty fluid. Chakrabarti *et al*. [8] they investigate dusty gas-flow with in boundary-layer surface. Mishra and Datta [9] used flow semi-infinite plate to study dusty fluid. Nayfeh and Vajravelu [10] used to move surface to discuss the flow of hydrodynamic. Gireesha *et al*. [11] used numerical method to study dusty nanofluids.

Ramesh *et al.* [12] they used inclined plane sheet to study MHD boundary-layer flow. Gireesha *et al.* [13] used exponentially stretching sheet to study thermal radiation effect on MHD

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boundary-layer flow. Gupta and Gupta [14] and Attia [15], used open channel flow to study dusty fluid-flow that is Maxwell fluid. Nadeem *et al.* [16] they used exponentially stretching exterior to study Jeffery fluid-flow. Akbar *et al.* [17] studied Casson fluid model of stagnation point flow towards a decreasing sheet with MHD. Bhatti and Rashidi [18] used numerical method to study entropy generation with non-linear thermal radiation under the presence of MHD.

Currently in the field of science and engineering, majority of mathematical problems are so composite in their nature that their exact solution is impossible, for the solution of such problems, two approaches are used one is numerical and other is analytical. The HAM technique [19] is one of the main and will know method for solution of non-linear equation. The aim of this paper is to increase the heat transfer ratio by using non-Newtonian couple stress Casson fluid model with the help of approximate analytical method.

Mathematical formulation

Assume time dependent incompressible viscous 2-D MHD boundary-layer flow of Casson nanofluid over a porous stretching surface. It is assumed that the velocity of the stretching surface along with stretching velocity:

$$U_w(x,t)=\frac{U_0}{(1-\mathrm{d}t)}\exp^{\zeta x/\mathcal{L}} \tag{1}$$

The x-axis of the given flow problem is along the sheet and y-axis is perpendicular to the sheet. The continuity, momentum and temperature equation for the given flow problem:

$$u_x+u_y=0 \tag{2}$$

$$\rho_{\text{nf}}\left(u_t+uu_x+\upsilon u_y\right)=\nu_{\text{nf}}\left(1+\frac{1}{\beta}\right)u_{yy}+-\sigma_{\text{nf}}B^2u-\frac{\nu_{\text{nf}}}{\rho_{\text{nf}}}Ku_{yyyyy} \tag{3}$$

$$\left(\rho c_p\right)_{\text{nf}}\left(T_t+uT_t+\upsilon T_t\right)=k_{\text{nf}}T_{tt}-q_{ry}+\frac{\mu_{\text{nf}}}{\rho_{\text{nf}}c_p}\left(u_y\right)^2 \tag{4}$$

The defined boundary conditions for given model:

$$\begin{gathered}u=U_w(x,t),\ \upsilon=V_w(x,t),\ T=T_w(x,t),\quad \text{where } y=0\\ y\to 0,\quad T\to T_\infty\ \text{ as }\ y\to\infty\end{gathered} \tag{5}$$

The velocity of given fluid model along the x and y are denoted by (u, υ) and (u_p, υ_p). The μ_{nf} shows the variable viscosity and ρ_{nf} shows density of the nanofluid:

$$U_w(x,t)=\frac{U_0}{(1-\mathrm{d}t)}\mathrm{e}^{\zeta x/2\mathcal{L}}$$

denotes stretching velocity and d shows constant which measure un-steadiness:

$$V_w(x,t)=-q\sqrt{\frac{U_0\nu}{3\mathcal{L}(1-\mathrm{d}t)}}\mathrm{e}^{\zeta x/3\mathcal{L}}$$

when q is positive it represent suction velocity and when q is negative it shows injection velocity . Also the nanofluid constants are defined in [16-19]:

$$\begin{gathered}\left(\rho c_p\right)_{\text{nf}}=(1-\phi)^{2.5}\left(\rho c_p\right)_{\text{nf}}+\phi\left(\rho c_p\right)_{\text{nf}},\ \mu_{\text{nf}}=\mu(1-\phi)^{-2.5}\\ \frac{\kappa_{\text{nf}}}{\kappa_f}=\frac{\left[\kappa_{\text{nf}}+(n-1)\kappa_{\text{nf}}\right]-\phi\left(\kappa_{\text{nf}}-\kappa_s\right)}{\left[\kappa_s+(n-1)\kappa_{\text{nf}}\right]+\phi\left(\kappa_{\text{nf}}-\kappa_s\right)}\end{gathered} \tag{6}$$

The connected, non-dimensional form of a PDE is transformed using the defined similarity transformations into a non-linear, dimensionless form of two ODE, one for velocity and the other for temperature:

$$u=\frac{U_0}{1-\mathrm{d}t}\exp^{\varsigma x/\mathcal{L}}\frac{\mathrm{d}f}{\mathrm{d}\eta},\ \upsilon=-E\left(\frac{\upsilon_f U_0}{3\mathcal{L}(1-\mathrm{d}t)}\right)^{1/\mathcal{L}}\exp^{\varsigma x/2\mathcal{L}}\left\{f(\eta)+\eta\frac{\mathrm{d}f}{\mathrm{d}\eta}\right\}$$
$$\theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty},\ \eta=\left(\frac{U_0}{3\mathcal{L}(1-\mathrm{d}t)}\right)^{1/2}\exp^{\varsigma x/3\mathcal{L}}y,\ T-T_\infty=\frac{T_0}{(1-\mathrm{d}t)}\exp^{\varsigma x/3\mathcal{L}}\theta(\eta) \tag{7}$$

Using eq. (7), in eqs. (2)-(4), we get:

$$\left(1+\frac{1}{\beta}\right)\frac{\mathrm{d}^3 f}{\mathrm{d}\eta^3}-(1-\phi_d)^{2.5}\left[1-\phi+\phi\left(\frac{\rho_s}{\rho_{\mathrm{nf}}}\right)\right]S\left(2\frac{\mathrm{d}f}{\mathrm{d}\eta}+\eta\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2}\right)-\mathrm{M}\frac{\mathrm{d}f}{\mathrm{d}\eta}-K\frac{\mathrm{d}^5 f}{\mathrm{d}\eta^5}=0 \tag{8}$$

$$\frac{1}{\mathrm{Pr}\left\{1-\phi+\phi\left[(\rho c_p)_s(\rho c_p)_{\mathrm{nf}}\right]\right\}}\left[\frac{k_{\mathrm{nf}}}{k_f}+\frac{4Rd}{3}\right]\frac{\mathrm{d}^2\theta}{\mathrm{d}\eta^2}-S\left(\eta\frac{\mathrm{d}\theta}{\mathrm{d}\eta}+4\theta\right)+2\,\mathrm{Pr}\,\gamma\lambda\mathrm{Ec}\left(\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2}\right)^2=0 \tag{9}$$

The transform boundary cnditions for the flow problem, we have:

$$f(\eta)=0,\ \ \frac{\mathrm{d}f}{\mathrm{d}\eta}=1,\ \ \theta(\eta)=1\ \ at\ \ \eta=0$$
$$\frac{\mathrm{d}f}{\mathrm{d}\eta}=0,\ \ \theta(\eta)=0\ \ \text{as}\ \ \eta\to\infty$$

where ς represents exponential parameter and A represents unsteadiness parameter it is defined as $A=\mathrm{d}\mathcal{L}/U_0$, the concentration of a dust particle is denoted by γ is defined as $\gamma=Nm/\rho_f$, the fluid particle interaction parameter is denoted by λ, the magnetic field is denoted by M and is defined:

$$M=\frac{2\sigma_{\mathrm{nf}}B_0{}^2\mathcal{L}}{\rho_{\mathrm{nf}}u_0}$$

the relaxation time parameter is denoted by $\tau_v=m/K$ and $\mathrm{Ec}=U_0^2/(c_p)_{\mathrm{nf}}T_0$ represent Eckert number. For the flow model c_f (friction factor) and Nusselt number, are:

$$C_f\,\mathrm{Re}_x^{-1/2}=(1-\phi)^{-2.5}\left(1+\frac{1}{\beta}\right)\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2} \tag{10}$$

$$\mathrm{Nu}_x\,\mathrm{Re}_x^{1/2}=-\left(\frac{k_{\mathrm{nf}}}{k_f}\right)\frac{\mathrm{d}\theta}{\mathrm{d}\eta} \tag{11}$$

Homotopy analysis method

Here, we will investigate the solution of the recommended problem through HAM. This is used to solve the both linear and non-linear problem both strong and weaker non-linear problem, we presented the homotopy for this method which is a continuous mapping:

$$H(\chi(x;r);r)=(1-r)L(\chi(x;r)-v_0(x))-bH(x)rN(\chi(x;r)) \tag{12}$$

where $H(x)$ is the auxiliary function, $N(\chi(x;r))$ is the non-linear operator, b show the auxiliary parameter also called convergence control parameter, $r\in[0,1]$ represent embedding parameter, and $\phi(x;q)$ is the approximate analytical solution of the problem. From eq. (12) we see that the analytical solution by this method depend on $v_0(x)$ initial approximation, the linear operator L, $H(x)$ the auxiliary function, and the auxiliary parameter b when $r = 0$. From eq. (22) we find zero order deformation equation of the form:

$$L\left(\chi\left(x;0\right)-v_0\left(x\right)\right)=0 \tag{13}$$

The zero deformation equation in the view of linear operator L is of the form:

$$\chi\left(x;0\right)=v_0\left(x\right) \tag{14}$$

when $r = 1$, eq. (12) takes the form:

$$N\left(\chi\left(x,1\right)\right)=0 \tag{15}$$

Equation (12) is the same as:

$$\chi\left(x;1\right)=v\left(x\right) \tag{16}$$

This show that by changing embedding parameter r from 0 to 1 $\chi(x;\, r)$ from initial guess $v_0(x)$ to the exact solution $v_0(x)$.

Results and discussion

The influence of skin friction coefficient and Nusselt number are presented in tab. 1, for couple stress Casson nanofluids. From these two table we observe that enhancing the M along with enhancing ϕ_d, decreases both the friction parameter with temperature transfer rate. Also by enhancing the magnitude of interaction parameter and time dependent parameter reduction the value of C_f although enhancing the Nusselt number. Furthermore, Energy parameter

Table 1. Influence of $C_f = \mathrm{d}^2 f/\mathrm{d}^2(0)$ and Nu = $-\mathrm{d}\theta/\mathrm{d}\eta^2(0)$

K	Rd	ϕ	ϕ_d	γ_1	S	q_1	C_f [18]	C_f	Nu [18]	Nu
1.5							–1.5618	–1.7715	1.2541	3.6340
2.5							–1.8399	–1.7744	0.8082	3.6333
3.5							–2.0811	–1.7773	0.5396	3.6331
	1.5						–1.5618	–1.7715	1.2541	3.6340
	2.5						–1.5618	–1.7715	0.5427	2.5815
	3.5						–1.5618	–1.7715	0.1579	2.2656
		0.15					–1.5617	–1.9058	1.3523	3.4088
		0.25					–1.3659	–1.8370	1.5671	3.4098
		0.35					–1.2659	–1.7347	1.7666	3.4111
			0.15				–1.5618	–1.7687	1.2541	3.6303
			0.25				–1.6491	–1.7697	1.0925	3.6270
			0.35				–1.7540	–1.7707	0.9263	3.6242
				0.15			–1.1235	–1.7707	0.1209	3.6303
				0.55			–1.5618	–1.7810	1.2541	3.6305
				0.95			–1.9189	–1.7914	3.2040	3.6307
					0.55		–1.5618	–1.7707	1.2541	1.8731
					0.75		–1.7475	–1.8145	2.2470	2.5761
					0.95		–1.9189	–1.8581	3.2040	3.2784
						1.5	–1.6983	–1.9022	0.6751	0.4088
						0.0	–1.6225	–1.8323	0.8930	0.4609

has no consequence on C_f although decreases Nusselt number. Also by enhancing in the pours parameter and volume fraction of the nanofluid particles enhancing temperature field and friction parameter. Tables 2 and 3 show the convergences of velocity and temperature equation. The obtained results of present problem is presented in figs. 1-5. Figures 1-3 show the impact of different parameter on $f'(\eta)$. Figures 4 and 5 show the influence of different parameter on temperature profile, from fig. 1 we see that by increasing couple stress parameter the velocity field is decreasing due to the viscous forces the same impact is observed for fig. 2 as will, fig. 3, show the influence of magnetic field parameter on velocity field profile, the increasing value of magnetic field parameter decrease the velocity due to the resistance forces. Figures 4 and 5, show the influence of Prandtl number and Eckert number on temperature profile, from fig. 4 we see that by increasing Eckert number the temperature field is also increasing, but this impact is different in the case of Prandtl number, that is the increasing value of Prandtl number decrease the temperature field.

Table 2. Show Convergence of HAM for velocity equation

5	$0.4313 \cdot 10^{-1}$	$0.5217 \cdot 10^{-2}$
10	$0.7219 \cdot 10^{-3}$	$0.5573 \cdot 10^{-4}$
15	$0.0143 \cdot 10^{-4}$	$0.4319 \cdot 10^{-5}$
20	$0.9529 \cdot 10^{-7}$	$0.3213 \cdot 10^{-7}$
25	$0.1008 \cdot 10^{-9}$	$0.3173 \cdot 10^{-9}$

Table 3. Show the convergence of HAM for temperature equation

m		
5	$0.1099 \cdot 10^{-2}$	$0.2314 \cdot 10^{-1}$
10	$0.2376 \cdot 10^{-3}$	$0.00051 \cdot 10^{-2}$
15	$0.2138 \cdot 10^{-4}$	$0.109 \cdot 10^{-3}$
20	$0.1113 \cdot 10^{-5}$	$0.3076 \cdot 10^{-4}$
25	$0.1005 \cdot 10^{-7}$	$0.5615 \cdot 10^{-6}$

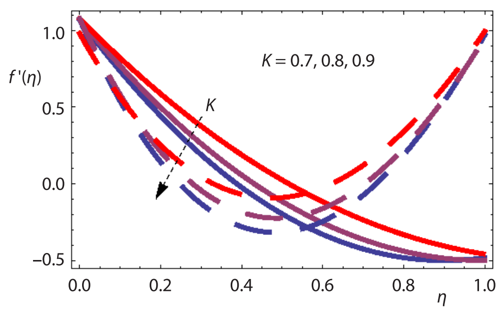

Figure 1. Show the influence of Couple stress parameter via velocity field

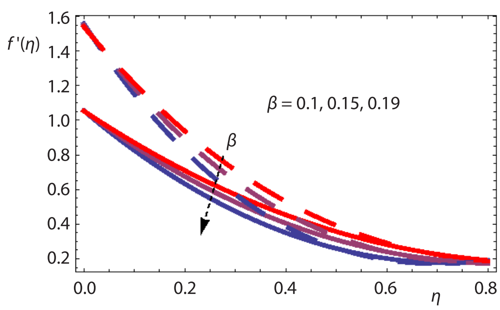

Figure 2. Show the influence of Casson parameter via velocity field

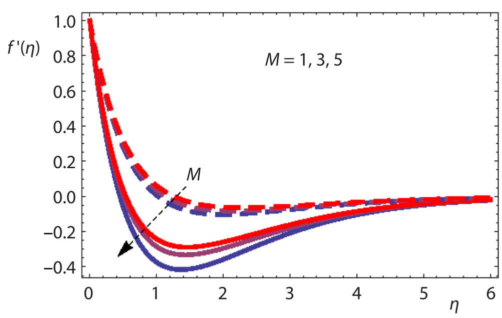

Figure 3. The influence of magnetic filed parameter via velocity field

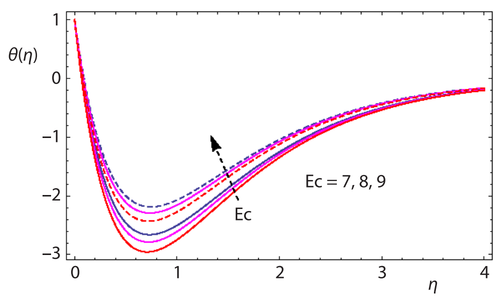

Figure 4. Show the influence of Eckert number via temperature profile

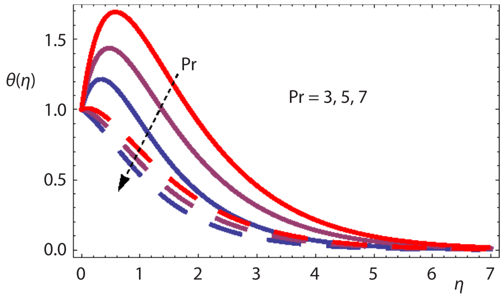

Figure 5. Show the influence of Prandtl number via temperature profile

Conclusions

This paper investigates couple stress MHD radiative flow and heat transfer characteristics of a non-Newtonian Casson fluid on exponentially porous extending surface. The key objective of this research work is to increase the heat transfer ration solve the energy feeding problem. To find the approximate solution we used HAM for the non-Newtonian casson fluid model. The Impacts of different parameter which are obtained from velocity and temperature equations are presented in the form of graphs. The influences of skin friction C_f is discus in the form of table. The principal determined points are

- By enhancing magnetic field velocity is decreasing.
- By enhancing casson parameter velocity field is decreasing
- By enhancing Eckert number temperature field is increasing.
- By enhancing Prandtl number temperature field is decreasing.
- By enhancing couple stress parameter increases the velocity profile.

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