STATISTICAL INFERENCE OF TYPE-I HYBRID CENSORED INVERSE LOMAX SAMPLES

by

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In this article, we adopted the classical and Bayesian approach to develop the problem of estimation and prediction of the inverse Lomax distribution under Type-I hybrid censored scheme. Firstly, we presented maximum likelihood estimators and Bayes estimators of the unknown parameters under consideration of squared error loss equation. In Bayesian approach, we used Markov chain Monte-Carlo method by applied importance sampling technique. Asymptotic confidence intervals and Bayes credible intervals are constructed. The estimators are tested by building simulation study. Secondly, For given Type-I hybrid censoring sample Bayesian prediction of future order statistics are formulated (two-sample case). Finally, the numerical computations are adopted on a real data set for illustrating purpose.

Key words: *inverse Lomax distribution, Type-I hybrid censoring scheme, maximum likelihood estimation, Bayes estimation, Bayes prediction*

Introduction

The reliability analysis is usually concerned with failures in the time domain which show the difference between the quality control and reliability engineering. In the literature of statistics, censoring was described as a condition in which the value of observation is only partially known. Different types of censoring schemes are available and the simplistic ones called by Type-I censoring and Type-II censoring schemes (CS). The two Type of CS can be used to saving time and money. The Type-I CS (time scheme) has fixed test time but, number of failures is random. The Type-II CS (numbered scheme) has a fixed number of failures but, the test time is random. The generalized form of Type-II CS is difened by Type-II progressive censoring scheme, Johnson [1]. For the cost and time lamination the experimenter may be need to run the experiment under joint case of Type-I and Type-II CS which is known by hybrid censoring scheme (HCS). The simplistic HCS in statistic are called by Type-I HCS and Type-II HCS. In both type of HCS, we propose the test time by τ and number of failure by m and the test terminated at $\min(\tau, T_m)$ in Type-I HCS and terminated at $\max(\tau, T_m)$ in Type-II HCS. Different authors discussed HCS, Gupta and Kundu [2], Kundu and Pradhan [3] Algarni *et al*. [4] and Tahani *et al*. [5]. In this article, we are adopted the Type-I HCS and hence, let a sample of size n is random selected from a population has probability density equation (PDF) given by $f(t)$ and cumulative density equation (CDF) given by $F(t)$. Under Type-I HCS, suppose $\underline{T} = \{T_{i;m,n}\}$, $i = 1, 2,..., r$ be identical independent random sample of size r where:

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$$r = \begin{cases} m, \min(\tau, t_m) = t_m \\ r < m, \min(\tau, t_m) = \tau \end{cases} \tag{1}$$

Also, the test terminated time $\eta = (\tau, T_m)$. Therfore, the joint likelihood equation of Type-I HCS, $\underline{T} = \{T_{i;m,n}\}$, $i = 1, 2, ..., r$ is given:

$$f_{1,2,,...,r}(\underline{t}) = \frac{n!}{(n-r)!}\left[S(\eta)\right]^{n-r}\prod_{i=1}^{r} f(t_i) \tag{2}$$

where $S(.) = 1 - F(.)$ is the survival equation.

The inverse Lomax (IL) distribution has different applications in medical fields, economics, geography and actuarial, Kleiber and Kotz [6]. The IL distribution was applieded in modelling stochastic of decreasing failure rate equation of life components, Kleiber and Kotz [6]. For the relationship bettween ordered statistics, Kleiber [7] and the the problem of analyzed order statistics generated from the mixture of two IL distributions, Rahman and Aslam [8]. The estimator of reliability equation of IL distribution was developed under Type-II censoring scheme, Singh *et al*. [9]. Bayes estimators was introduced of two-components mixture of IL distributions by Rahman and Aslam [10]. For the E-Bayesian approach descussed recently by Reyad and Othman [11] under Type-I censoring scheme. Also, Bayes estimate of the parameter of stress-strength reliability by Sing *et al*. [12].

In this article, we estimate the unknown parameters of IL distribution by two methods of estimation, MLE and Bayes estimation. Also, the reliability of the system and the corresponding system failure rate for given time values are formulated. The problem of comparison between two methods of estimation are constructed through Monte-Carlo simulation study and the resultts are measured under mean squared error (MSE) for point estimators and coverage percentage (CP), and mean interval length (MIL) for interval estimators. Another important problem in life-testing experiments namely the prediction of unknown observables belonging to a future sample. In this paper we consider the prediction problem in terms of the estimation of the posterior predictive density of a future observation for two-sample schemes. Therefore, for given Type-I HCS sample, we predictive interval for a future observation using Gibbs sampling procedure. The developed results are discussed through a set of real data.

The model and likelihood equation

The random variable T is called IL random variable if and only if has the CDF given:

$$F(t) = \left(1 + \frac{\alpha}{t}\right)^{-\beta} \tag{3}$$

where α, β, are the scale and shape parameters, respectively. Also, if the random variable has IL distribution then, the value $1/t$ is distributed by Lomax distribution. The corresponding PDF, and the survival equation $S(t)$ and hazrad failure rate equation $H(t)$ of IL distribution, respectively are given:

$$f(t) = \frac{\alpha\beta}{t^2}\left(1 + \frac{\alpha}{t}\right)^{-(\beta+1)} \tag{4}$$

$$S(t) = 1 - \left(1 + \frac{\alpha}{t}\right)^{-\beta} \tag{5}$$

and

$$H(t)=\frac{\alpha\beta\left(1+\dfrac{\alpha}{t}\right)^{-(\beta+1)}}{t^2\left[1-\left(1+\dfrac{\alpha}{t}\right)^{-\beta}\right]} \tag{6}$$

Suppose that, n independent units put under life testing experiment and the two values (τ, m) are prior proposed to present the test time and number of failures units. The observed failure time is record until the min(τ, t_m) is observed to present $\underline{t}=\{T_{i;m,n}\}$, $i = 1, 2,..., r$ where r is defined by (1). The joint likelihood equation given by eq. (2) under observed Type-I HCS sample $\underline{t}=\{T_{i;m,n}\}$ and IL distribution eqs. (3)-(6) is reduced:

$$L(\alpha,\beta\,|\,\underline{t})=\frac{n!}{(n-r)!}\alpha^r\beta^r\left[1-\left(1+\frac{\alpha}{\eta}\right)^{-\beta}\right]^{n-r}\prod_{i=1}^{r}\frac{1}{t_i^2}\left(1+\frac{\alpha}{t_i}\right)^{-(\beta+1)} \tag{7}$$

The corresponding log-likelihood equation is given:

$$\ell(\alpha,\beta\,|\,\underline{t})=\log\left[\frac{n!}{(n-r)!}\right]+r\log\alpha+r\log\beta+(n-r)\log\left[1-\left(1+\frac{\alpha}{\eta}\right)^{-\beta}\right]-$$
$$-2\log t_i-(\beta+1)\sum_{i=1}^{m}\log\left[1+\frac{\alpha}{t_i}\right] \tag{8}$$

Maximum likelihood estimation

Point maximum likelihood estimators

Parameters estimation under ML method need to calculate the absolutely maximum values of likelihood eq. (7) and hence, the maximum values of Log-likelihood eq. (8), Abd-El-mougod *et al*. [13]. Therefore, we compute the first partial derivatives of eq. (8) with respect to α and β and equating each to zero:

$$\frac{\partial\ell(\alpha,\beta\,|\,\underline{t})}{\partial\alpha}=\frac{r}{\alpha}+\frac{(n-r)\beta\left(1+\dfrac{\alpha}{\eta}\right)^{-(\beta+1)}}{\eta\left[1-\left(1+\dfrac{\alpha}{\eta}\right)^{-\beta}\right]}-(\beta+1)\sum_{i=1}^{r}\frac{1}{t_i\left(1+\dfrac{\alpha}{t_i}\right)}=0 \tag{9}$$

and

$$\frac{\partial\ell(\alpha,\beta\,|\,\underline{t})}{\partial\beta}=\frac{r}{\beta}+(n-r)\frac{\left(1+\dfrac{\alpha}{\eta}\right)^{-\beta}\log\left[1+\dfrac{\alpha}{\eta}\right]}{\left[1-\left(1+\dfrac{\alpha}{\eta}\right)^{-\beta}\right]}-\sum_{i=1}^{r}\log\left[1+\frac{\alpha}{t_i}\right]=0 \tag{10}$$

Hence, The ML estimate of the parameters α and β say, $\hat{\alpha}$ and $\hat{\beta}$ are obtained by solve the two non-linear eqs. (9) and (10) with any iteration method such as Newton Raphson. The corresponding estimate of the system relability and system hazard failure rate equation for given time t is given:

$$\hat{S}(t)=1-\left(1+\frac{\hat{\alpha}}{t}\right)^{-\hat{\beta}} \tag{11}$$

and

$$\hat{H}(t)=\frac{\hat{\alpha}\hat{\beta}\left(1+\frac{\hat{\alpha}}{t}\right)^{-1}}{t^2\left[\left(1+\frac{\hat{\alpha}}{t}\right)^{\hat{\beta}}-1\right]} \tag{12}$$

Asymptotic confidence intervals

The Fisher information matrix in statistical literature is defined as the expectation of the minus second partially derivatives of the log-likelihood equation. Therefore, under the last definition suppose, the Fisher information matrix of the model parameters is denoted by FIM(Θ) where Θ = {α, β}:

$$\text{FIM}(\Theta)=E\left[\frac{-\partial^2\ell(\alpha,\beta\,|\,\underline{t})}{\partial\Theta_i\partial\Theta_j}\right],\quad i,j=1,2 \tag{13}$$

The expectation of second partially derivative in several cases more serous to obtain therefore, we replace the Fisher information matrix by approximate information matrix which is denoted by $\text{FIM}_0(\Theta)$, where:

$$\text{FIM}_0(\Theta)=\left[\frac{-\partial^2\ell(\alpha,\beta\,|\,\underline{t})}{\partial\Theta_i\partial\Theta_j}\right]\Big|_{\hat{\Theta}=\{\hat{\alpha},\hat{\beta}\}},i,j=1,2 \tag{14}$$

Also, the second partially derivative with respected to the model parameters are given:

$$\frac{\partial^2\ell(\alpha,\beta\,|\,\underline{t})}{\partial\alpha^2}=\frac{-r_1}{\alpha^2}-\frac{(n-r)\beta\left[(\beta+1)\left(1+\frac{\alpha}{\eta}\right)^{-(\beta+2)}-\left(1+\frac{\alpha}{\eta}\right)^{-2(\beta+1)}\right]}{\eta^2\left[1-\left(1+\frac{\alpha}{\eta}\right)^{-\beta}\right]^2}+(\beta+1)\sum_{i=1}^{r}\frac{1}{t_i^2\left(1+\frac{\alpha}{t_i}\right)^2} \tag{15}$$

$$\frac{\partial^2\ell(\alpha,\beta\,|\,\underline{t})}{\partial\beta^2}=\frac{-r_2}{\beta^2}+(n-r)\frac{\left(1+\frac{\alpha}{\eta}\right)^{\beta}\log^2\left[1+\frac{\alpha}{\eta}\right]}{\left[\left(1+\frac{\alpha}{\eta}\right)^{\beta}-1\right]^2} \tag{16}$$

and

$$\frac{\partial^2\ell(\alpha,\beta\,|\,\underline{t})}{\partial\alpha\partial\beta}=\frac{\partial^2\ell(\alpha,\beta\,|\,\underline{t})}{\partial\beta\partial\alpha}=\frac{(n-r)}{\eta\left(1+\frac{\alpha}{\eta}\right)}\left[\frac{1}{\left(1+\frac{\alpha}{\eta}\right)^{\beta}-1}-\frac{\beta\left(1+\frac{\alpha}{\eta}\right)^{\beta}\log\left[1+\frac{\alpha}{\eta}\right]}{\left[\left(1+\frac{\alpha}{\eta}\right)^{\beta}-1\right]^2}\right]-\sum_{i=1}^{r}\frac{1}{t_i\left(1+\frac{\alpha}{t_i}\right)} \tag{17}$$

Under the normality property of the ML estimators with mean $\Theta = \{\alpha, \beta\}$ and variance obtained from the diagonal of variance-covariance matrix $\text{FIM}_0^{-1}(\Theta)$, where $\text{FIM}_0^{-1}(\Theta)$ is denoted to the inverse of approximate information matrix at ML estimate of the model parameters. The approximate interval estimators with $(1-\gamma)100\%$ confidence level of the model parameters $\Theta = \{\alpha, \beta\}$ is given:

$$\hat{\alpha} \mp z_{\gamma/2} e_1 \quad \text{and} \quad \hat{\beta} \mp z_{\gamma/2} e_2 \tag{18}$$

where the values e_1 and e_2 present the diagonal of approximate variance-covariance matrix $\text{FIM}_0^{-1}(\Theta)$ with percentile standard normal tabulated value $x/2$ at γ confidence level.

Bayesian approach

In Bayesian approach, information come from two sources, Type-I HCS sample $\underline{t}$ which is described by the likelihood equation $L(\Theta|\,\underline{t})$ and prior information about the unknown parameters which described by prior distribution. Under consideration that, the prior information presented as independent gamma distributions which is given:

$$\alpha \propto \alpha^{a-1} \exp\{-b\alpha\}, \;\; a,b > 0, \;\; \alpha > 0 \tag{19}$$

and

$$\beta \propto \beta^{c-1} \exp\{-d\beta\}, \;\; c,d > 0, \;\; \beta > 0 \tag{20}$$

The joint prior distribution is given:

$$\pi^*(\alpha,\beta) \propto \alpha^{a-1}\beta^{c-1} \exp\{-b\alpha - d\beta\} \tag{21}$$

Then, we formulate the joint posterior density equation of $\Theta = \{\alpha, \beta\}$ under consideration prior density eq. (15) and the likelihood eq. (7):

$$\pi(\alpha,\beta\,|\,\underline{t}) \propto \alpha^{r+a-1}\beta^{r+c-1} \exp\{-b\alpha - d\beta\}\left[1-\left(1+\frac{\alpha}{\eta}\right)^{-\beta}\right]^{n-r} \prod_{i=1}^{r}\left(1+\frac{\alpha}{t_i}\right)^{-(\beta+1)} \tag{22}$$

Hence, the Bayes estimators of any equation of the parameters $\Psi\{\alpha, \beta\}$ respected SEL equation is given:

$$\hat{\Psi}_{\text{B}}(\alpha,\beta) = \frac{\int_{\Theta} \Psi(\alpha,\beta)\pi^*(\alpha,\beta) L\big(\alpha,\beta\,|\,\underline{t}\big)\text{d}\alpha\text{d}\beta}{\int_{\Theta} \pi^*(\alpha,\beta) L\big(\alpha,\beta\,|\,\underline{t}\big)\text{d}\alpha\text{d}\beta} \tag{23}$$

The Bayes estimator (23) need to compute the ratio of two integrals which in general more complicated specially, in a high dimensional cases. Therefore, we search for the alternative methods other than the analytical solution. Different approximated methods can be used, such as numerical integration, Lendly approximations and MCMC method, Algarni *et al*. [14]. In this section, we adopted the MCMC methods as follows.

The joint posterior density eq. (22) can be written:

$$\pi(\alpha,\beta) \propto K_1(\alpha\,|\,\underline{t}) K_2(\beta\,|\,\alpha,\underline{t}) h(\alpha,\beta) \tag{24}$$

where $K_1(\alpha|\underline{t})$ is proper equation of α and its plot is similar to normal distribution is given:

$$K_1(\alpha|\underline{t}) = \frac{\alpha^{r+a-1}}{\left[d + _{i=1}^{r} \log\left[1+\frac{\alpha}{t_i}\right]\right]^{r+c}} \exp\left\{-b\alpha - \sum_{i=1}^{r} \log\left[1+\frac{\alpha}{t_i}\right]\right\} \tag{25}$$

The $K_2(\beta|\underline{t})$ is a gamma density equation with the shape and scale parameters as and:

$$r+c \quad \text{and} \quad d + \sum_{i=1}^{r} \log\left[1+\frac{\alpha}{t_i}\right]$$

respectively. Also, $h(\alpha, \beta)$ is given:

$$h(\alpha,\beta) = \left[1-\left(1+\frac{\alpha}{\eta}\right)^{-\beta}\right]^{n-r} \tag{26}$$

As given by Chen and Shao [15], Gibbs sampling is used to draw the MCMC samples under importance sampling technique. Also the corresponding HPD credible intervals are constructed. Therefore, Gibbs sampling under importance sampling technique is described as algorithms:

- Begin with initial values $\alpha^{(0)} = \hat{\alpha}$ and $\kappa = 1$.
- The value $\beta^{(\kappa)}$ is generated from gamma distribution K2$[\beta|\alpha^{(\kappa-1)}, \underline{t}]$.
- The value $\alpha^{(\kappa)}$ is generated from eq. (25) under MH algorithms with normal proposal distribution with mean $\alpha^{(\kappa-1)}$ and variance e_1, where e_1 is computed from approximate information matrix.
- Compute $h^{(\kappa)} = h(\alpha^{(\kappa)}, \beta^{(\kappa)})$ and put $\kappa = \kappa = \kappa + 1$.
- Repeat 2 and 4 N times.
- The Bayes point estimate of $\Psi(\alpha, \beta)$ under a SEL is given:

$$\hat{\Psi}_{\mathrm{B}}(\alpha,\beta) = \frac{\frac{1}{N-M}\sum_{i=M+1}^{N} \Psi(\alpha^{(i)},\beta^{(i)})h(\alpha^{(i)},\beta^{(i)})}{\frac{1}{N-M}\sum_{i=M+1}^{N} h(\alpha^{(i)},\beta^{(i)})} \tag{27}$$

and the posterior variance of $\Psi(\alpha, \beta)$ is given:

$$V(\Psi(\alpha,\beta)) = \frac{\frac{1}{N-M}\sum_{i=M+1}^{N} (\Psi(\alpha^{(i)},\beta^{(i)}) - \hat{\Psi}_{\mathrm{B}}(\alpha,\beta))^2 h(\alpha^{(i)},\beta^{(i)})}{\frac{1}{N-M}\sum_{i=M+1}^{N} h(\alpha^{(i)},\beta^{(i)})} \tag{28}$$

where M is the first number of iteration need to reach to stationary distribution of posterior distribution. Also, $\Psi(\alpha, \beta)$ is any equation of the parameter may be α, β reliability equation, hazard rate equation or others.

– The HPD credible intervals of $\Psi(\alpha, \beta)$ is computed with the help of the idea of Chen and Shao [15] as the following algorthms:

Step 1.Calculate $\Psi^{(i)} = \Psi(\alpha^{(i)}, \beta^{(i)})$, $i = M + 1, 2,..., N$.

Step 2. Put $\Psi^{(i)}$, $i = M + 1, 2,..., N$ in asdeing order to be $\Psi_{(i)}$, $i = 1, 2,..., N – M$.

Step 3. Calculate the weighted equation $w^{(i)}$:

$$w^{(i)} = \frac{h(\alpha^{(i)}, \beta^{(i)})}{\sum_{j=M+1}^{N} h(\alpha^{(i)}, \beta^{(i)})}, i = M + 1, 2, ..., N \tag{29}$$

Put $w^{(i)}$, $i = M + 1, 2,..., N$ in asdeing order to be $w_{(i)}$, $i = 1, 2,..., N – M$. Hence, for each value $\Psi_{(i)}$ corresponding value of $w_{(i)}$.

Step 4. The marginal posterior of Ψ can be estimated for γ the quantile:

$$\Psi = \begin{cases} g_{(1)}, \text{ if } \gamma = 0 \\ 1cg_{(j)}, \text{ if } \sum_{i=1}^{j-1} w_{(i)} < \gamma < \sum_{i=1}^{j} w_{(i)} \end{cases} \tag{30}$$

Step 5. The $(1 – \gamma\%)100$ credible intervals of Ψ:

$$\left(\Psi^{(L/N)}, \Psi^{(\{L+[(1-\alpha)N]\}/N)}\right) \tag{31}$$

where $L = 1, 2,..., N – [(1 – \gamma)N]$.

Step 6. The $100(1 – \gamma\alpha\%)$ HPD interval of Ψ is the one, with the smallest interval width among all credible intervals.

Monte-Carlo simulations

The estimators obtained under ML and Bayes methods are compared and assessed with Monte-Carlo simulation study. In our studying, we used two set of the parameter values as $(\alpha, \beta) = \{(0.5, 1.5), (2, 1.2)\}$. For Type-I HCS, we used different combination of (n, m, τ). The prior information are selected to be non-informative prior information P^0 (mean that $a = b = c = d = 0.0001$) and informative prior information P^1 (mean that, a, b, c, d are selected to satisfy the true parameter $\simeq$ prior shape/prior scale). Through the simulation problem, we generate 1000 different samples from IL distribution. For each sample, we compute MLE and the corresponding Bayes estimate under P^0 and P^1. Therefore, for the numerical result compute MSE and the result reported in tabs. 1 and 3. Also, for interval estimate coverage percentage (CP) and interval length are reported in tabs. 2 and 4.

It is clear from tabs. 1-4 some point reported:

– The proposed informative Bayes estimate perform very well for all choices of the parameter value and censoring scheme
– The results in non-informative Bayes estimate are closed to MLE.
– The value of MSE are decreasing as the proportion (m/n) is increasing.
– The value of MIL are decreasing and CP reduced to the confidence level as the proportion (m/n) is increasing.
– The results are more better for large value of τ than smaller values.

Table 1. The MSE of the different estimators when (α, β) = (0.5, 1.5)

(τ, n, m)	MLE		Bayes (P^0)		Bayes (P^1)	
	α	β	α	β	α	β
(0.1, 25, 15)	0.1254	0.5244	0.1214	0.5100	0.1014	0.3523
(0.1, 25, 20)	0.1211	0.4392	0.1187	0.4255	0.0989	0.2987
(0.1, 40, 20)	0.1201	0.4383	0.1175	0.4242	0.0969	0.2970
(0.1, 40, 30)	0.1150	0.4320	0.1122	0.4194	0.0912	0.2911
(0.1, 60, 40)	0.1138	0.4308	0.1109	0.4181	0.0900	0.2895
(0.1, 60, 50)	0.1138	0.4308	0.1109	0.4181	0.0900	0.2895
(0.5, 25, 15)	0.1130	0.4295	0.1100	0.4010	0.0921	0.2400
(0.5, 25, 20)	0.1076	0.4281	0.1065	0.4132	0.0862	0.2851
(0.5, 40, 20)	0.1077	0.4269	0.1051	0.4110	0.0841	0.2851
(0.5, 40, 30)	0.1018	0.4199	0.1001	0.4078	0.0800	0.2801
(0.5, 60, 40)	0.1115	0.4195	0.1002	0.4114	0.0782	0.2761
(0.5, 60, 50)	0.1014	0.4181	0.1000	0.4051	0.0788	0.2775
(2.0, 25, 15)	0.0832	0.3819	0.0811	0.3600	0.0670	0.2810
(2.0, 25, 20)	0.0780	0.3810	0.0800	0.3569	0.0655	0.2772
(2.0, 40, 20)	0.0771	0.3790	0.0775	0.3545	0.0635	0.2741
(2.0, 40, 30)	0.0715	0.3779	0.0749	0.3525	0.0618	0.2722
(2.0, 60, 40)	0.0701	0.3762	0.0728	0.3501	0.0601	0.2718
(2.0, 60, 50)	0.0699	0.3732	0.0702	0.3481	0.0582	0.2701

Table 2. The MIL(CP) of the different estimators when (α, β) = (0.5, 1.5)

(τ, n, m)	MLE		Bayes (P^0)		Bayes (P^1)	
	α	β	α	β	α	β
(0.1, 25, 15)	1.541(88)	3.889(89)	1.530(89)	3.878(89)	1.410(91)	3.655(90)
(0.1, 25, 20)	1.511(89)	3.858(90)	1.507(90)	3.858(91)	1.392(90)	3.632(90)
(0.1, 40, 20)	1.501(89)	3.845(89)	1.495(92)	3.841(93)	1.379(91)	3.611(93)
(0.1, 40, 30)	1.472(90)	3.817(90)	1.481(91)	3.828(92)	1.365(94)	3.600(93)
(0.1, 60, 40)	1.461(91)	3.800(92)	1.466(92)	3.814(91)	1.351(92)	3.585(92)
(0.1, 60, 50)	1.445(91)	3.469(91)	1.456(91)	3.801(93)	1.135(93)	3.571(92)
(0.5, 25, 15)	1.335(89)	3.450(90)	1.325(90)	3.431(90)	1.117(96)	3.351(94)
(0.5, 25, 20)	1.300(91)	3.435(92)	1.311(93)	3.415(94)	1.104(93)	3.324(93)
(0.5, 40, 20)	1.289(92)	3.419(91)	1.300(96)	3.401(93)	1.091(91)	3.311(93)
(0.5, 40, 30)	1.271(90)	3.401(93)	1.287(93)	3.391(92)	1.076(95)	3.300(92)
(0.5, 60, 40)	1.264(93)	3.382(94)	1.271(92)	3.379(95)	1.061(95)	3.284(92)
(0.5, 60, 50)	1.238(94)	3.788(93)	1.254(94)	3.361(96)	1.045(93)	3.271(95)
(2.0, 25, 15)	1.220(90)	3.241(90)	1.127(93)	3.214(91)	1.015(91)	3.148(90)
(2.0, 25, 20)	1.211(93)	3.222(94)	1.114(93)	3.200(94)	1.001(93)	3.128(93)
(2.0, 40, 20)	1.187(92)	3.201(91)	1.107(94)	3.187(92)	0.987(93)	3.114(94)
(2.0, 40, 30)	1.169(90)	3.182(92)	1.095(93)	3.171(92)	0.971(94)	3.101(92)
(2.0, 60, 40)	1.151(92)	3.179(94)	1.087(92)	3.165(94)	0.961(95)	3.089(95)
(2.0, 60, 50)	1.143(91)	3.151(95)	1.069(95)	3.144(96)	0.952(91)	3.073(94)

Table 3. The MSE of the different estimators when (α, β) = (2.0, 1.2)

(τ, n, m)	MLE		Bayes (P^0)		Bayes (P^1)	
	α	β	α	β	α	β
(0.5, 25, 15)	0.6451	0.3421	0.6432	0.3407	0.4436	0.2356
(0.5, 25, 20)	0.6402	0.3390	0.6381	0.3374	0.4400	0.2321
(0.5, 40, 20)	0.6390	0.3369	0.6374	0.3356	0.4387	0.2304
(0.5, 40, 30)	0.6344	0.3335	0.6339	0.3319	0.4351	0.2274
(0.5, 60, 40)	0.6312	0.3313	0.6304	0.3294	0.4311	0.2228
(0.5, 60, 50)	0.6281	0.3288	0.6271	0.3269	0.4281	0.2201
(2.0, 25, 15)	0.5654	0.2451	0.5641	0.2427	0.2370	0.1784
(2.0, 25, 20)	0.5619	0.2422	0.5612	0.2401	0.2339	0.1745
(2.0, 40, 20)	0.5591	0.2392	0.5584	0.2375	0.2302	0.1714
(2.0, 40, 30)	0.5564	0.2369	0.5554	0.2352	0.2274	0.1681
(2.0, 60, 40)	0.5531	0.2325	0.5522	0.2314	0.2236	0.1641
(2.0, 60, 50)	0.5502	0.2300	0.5491	0.2284	0.2203	0.1611
(4.0, 25, 15)	0.5002	0.1800	0.4952	0.1745	0.1854	0.1124
(4.0, 25, 20)	0.4965	0.1762	0.4925	0.1732	0.1818	0.1103
(4.0, 40, 20)	0.4938	0.1741	0.4914	0.1709	0.1800	0.1087
(4.0, 40, 30)	0.4911	0.1718	0.4901	0.1692	0.1750	0.1045
(4.0, 60, 40)	0.4871	0.1700	0.4862	0.1671	0.1718	0.1024
(4.0, 60, 50)	0.4847	0.1674	0.4824	0.1655	0.1700	0.1003

Table 4. The MIL(CP) of the different estimators when (α, β) = (2.0, 1.2)

(τ, n, m)	MLE		Bayes (P^0)		Bayes (P^1)	
	α	β	α	β	α	β
(0.5, 25, 15)	4.234(89)	2.542(88)	4.218(89)	2.528(90)	4.002(90)	3.403(90)
(0.5, 25, 20)	4.202(89)	2.515(90)	4.197(89)	2.501(91)	3.974(91)	3.361(92)
(0.5, 40, 20)	4.171(90)	2.487(91)	4.165(90)	2.472(91)	3.941(92)	3.325(93)
(0.5, 40, 30)	4.141(91)	2.452(93)	4.134(90)	2.441(93)	3.905(92)	3.300(92)
(0.5, 60, 40)	4.115(93)	2.427(91)	4.107(92)	2.418(93)	3.882(95)	3.269(96)
(0.5, 60, 50)	4.082(90)	2.401(91)	4.074(92)	2.389(92)	3.854(95)	3.241(95)
(2.0, 25, 15)	3.842(92)	2.245(94)	3.815(91)	2.219(92)	3.548(93)	3.024(94)
(2.0, 25, 20)	3.817(92)	2.218(94)	3.803(96)	2.198(96)	3.514(93)	3.000(92)
(2.0, 40, 20)	3.792(93)	2.189(92)	3.777(96)	2.168(94)	3.482(93)	2.969(93)
(2.0, 40, 30)	3.758(94)	2.147(94)	3.742(93)	2.139(90)	3.448(96)	2.931(95)
(2.0, 60, 40)	3.719(93)	2.112(91)	3.708(93)	2.101(96)	3.414(92)	2.911(94)
(2.0, 60, 50)	3.692(92)	2.079(94)	3.681(92)	2.059(93)	3.387(94)	2.875(93)
(4.0, 25, 15)	3.470(93)	1.874(91)	3.452(92)	1.850(91)	3.147(92)	2.663(94)
(4.0, 25, 20)	3.445(92)	1.852(92)	3.438(93)	1.844(93)	3.113(96)	2.624(94)
(4.0, 40, 20)	3.414(91)	1.822(93)	3.403(94)	1.812(92)	3.084(94)	2.582(92)
(4.0, 40, 30)	3.380(93)	1.800(94)	3.364(95)	1.795(96)	3.055(92)	2.547(96)
(4.0, 60, 40)	3.342(92)	1.769(93)	3.331(92)	1.744(94)	3.014(93)	2.517(93)
(4.0, 60, 50)	3.315(96)	1.727(93)	3.302(94)	1.714(93)	2.970(94)	2.480(95)

Bayes predicted interval of a future order statistics (two-sample case)

For given Type-I HCS $\underline{T} = \{T_{i;m,n}\}$, i = 1, 2,..., r suppose that, the future order statistics (Y_1, Y_2,..., Y_s) of size s which is independent of the informative sample. The predictive density equation of the future order statisti $Y_{(J)}$, given the informative sample $\underline{T}$ can be written:

$$g_J(y) = J\binom{s}{J}[F(y)]^{J-1}[1-F(y)]^{S-J} f(y) \tag{32}$$

Which is reduced:

$$g_J(y) = J\binom{s}{J}\sum_{i=0}^{S-J}(-1)^i\binom{S-J}{i}[F(y)]^{J+i-1} f(y) \tag{33}$$

By using eqs. (3) and (4) in eq. (33), we obtain:

$$g_J(y) = s\binom{r}{s}\sum_{i=0}^{S-J}\frac{(-1)^i\binom{S-J}{i}}{(J+i)} f\left[y,\alpha,\beta\left(J+i\right)\right] \tag{34}$$

and the corresponding CDF is defined by $G_J(y)$ where:

$$G_J(y) = s\binom{r}{s}\sum_{i=0}^{S-J}\frac{(-1)^i\binom{S-J}{i}}{(J+i)} F\left[y,\alpha,\beta\left(J+i\right)\right] \tag{35}$$

where $f(y)$ and $F(y)$ given by eqs. (3) and (4). Therefore, the predictive density of $Y_{(J)}$ is given:

$$g_J^*(y) = \int_0^\infty\int_0^\infty g_J(y)\pi(\alpha,\beta)\mathrm{d}\alpha\mathrm{d}\beta \tag{36}$$

and predictive distribution of $Y^{(J)}$ is given:

$$G_J^*(y) = \int_0^\infty\int_0^\infty G_J(y)\pi(\alpha,\beta)\mathrm{d}\alpha\mathrm{d}\beta \tag{37}$$

where $\pi(\alpha, \beta)$ is the joint posterior density of α and β given by eq. (22). The explicit solutions of eqs. (36) and (37) does not available to determine the prediction bounds of $Y_{(J)}$ then, we used MCMC samples $\{(\alpha_i, \beta_i), i = M + 1, M + 2,..., N\}$, a simulation consistent estimators of $g^*_J(y)$ and $G^*_J(y)$, can be obtained:

$$\hat{g}_J^*(y) = \sum_{i=M+1}^{N} g_J\left(y,\alpha^{(i)},\beta^{(i)}\right)w_i \tag{38}$$

and

$$\hat{G}_J^*(y) = \sum_{i=M+1}^{N} G_J\left(y,\alpha^{(i)},\beta^{(i)}\right)w_i \tag{39}$$

Hence, the two estimators $\hat{g}^*_J(y)$ and $\hat{G}^*_J(y)$ can be computed for each sample $\{(\alpha_i, \beta_i), i = M + 1, M + 2,..., N\}$. Moreover, The Bayesian predictive bounds of a two-sided equated $(1 - \gamma)100\%$ interval for $Y_{(J)}$ can be obtained by solving the following two equations for lower bound, L and upper bound, U:

$$P\left(Y_{(J)} > L \mid \underline{t}\right) = 1 - \hat{G}_J^*(L) = 1 - \frac{\gamma}{2}$$

hence

$$\hat{G}_J^*(L) = \frac{\gamma}{2} \tag{40}$$

and

$$P\left(Y_{(J)} > U \mid \underline{t}\right) = 1 - \hat{G}_J^*(U) = \frac{\gamma}{2}$$

hence

$$\hat{G}_J^*(U) = 1 - \frac{\alpha}{2} \tag{41}$$

The eqs. (40) and (41) does not solve analytically but, the numerical technique can be used to solve these non-linear equations such as Newton Raphson.

Data analysis

In this section, we are adopted the real data set which has presented the remission times of a 116 bladder cancer patients, Lee and Wang [16]. For $n = 16$, $m = 50$, and $\tau = 5$ the generated Type-I HC sample given by {0.08, 0.2, 0.4, 0.5, 0.51, 0.81, 0.9, 1.05, 1.19, 1.26, 1.35, 1.4, 1.46, 1.76, 2.02, 2.09, 2.23, 2.26, 2.46, 2.54, 2.62, 2.64, 2.69, 2.69, 2.75, 2.83, 2.87, 3.02, 3.25, 3.31, 3.48, 3.52, 3.57, 3.64, 3.7, 3.82, 3.88, 4.18, 4.23, 4.26, 4.33, 4.34, 4.4, 4.5, 4.51, 4.87, 4.98}. From the real Type-I HC data the value of $r = 47$ and $\eta = \tau$. For Bayesian approach, we adopted the non-informative information $a = b = c = d = 0.0001$. Also, MCMC approach run with 11000 chan terminated the first 1000 chan as burn-in. The simulated number generated from posterior distribution described by figs. 1-4 the results of the point estimate of ML and Bayes approach are reported in tab. 6. The results of interval estimate are reported in tab. 7. Also, the result of future order statistic and the corresponding predictive intervals are reported in tab. 8. Since the prediction of the future order statistic that is far a way from the last observed value has less accuracy than that of other future order statistics.

Table 5. The real data set represents the remission times of a 116 bladder cancer patients

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23	3.52	4.98
6.97	9.02	13.29	0.40	2.26	3.57	5.06	7.09	9.22	13.80	25.74	0.50
2.46	3.64	5.09	7.26	9.47	14.24	25.82	0.51	2.54	3.70	5.17	7.28
9.74	14.76	26.31	0.81	2.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64
3.88	5.32	7.39	10.34	14.83	34.26	0.90	2.69	4.18	5.34	7.59	10.66
15.96	36.66	1.05	2.69	4.23	5.41	7.62	10.75	16.62	43.01	1.19	2.75
4.26	5.41	7.63	17.12	46.12	1.26	2.83	4.33	5.49	7.66	11.25	17.14
79.05	1.35	2.87	5.62	7.87	11.64	17.36	1.40	3.02	4.34	5.71	7.93
11.79	18.10	1.46	4.40	5.85	8.26	11.98	19.13	1.76	3.25	4.50	6.25
8.37	12.02	2.02	3.31	4.51	6.54	8.53	12.0				

Table 6. Point MLE and Bayes estimate of parameters, reliability and hazard failure rate

	λ	β	$S(1.0)$	$H(1.0)$
MLE	3.82795	1.69025	0.930133	0.100666
Bayes	2.1296	1.9788	0.891766	0.161085

Table 7. The 95% interval MLE and Bayes estimate of parameters

	λ	β
MLE	(0.7452, 5.2456)	(0.7872, 3.5547)
Bayes	(0.3654, 3.6958)	(0.4215, 2.3245)

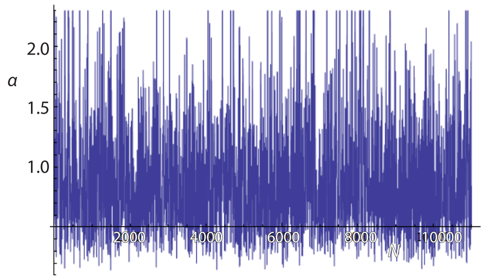

Figure 1. Simulation number of generated by important sample method of *α*

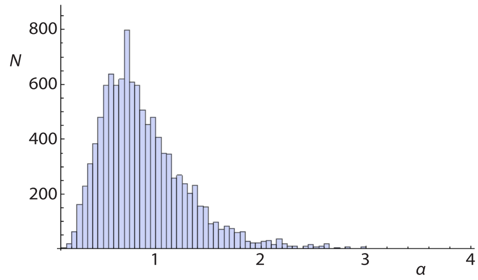

Figure 2. The histogram of the number generated by important sample method of *α*

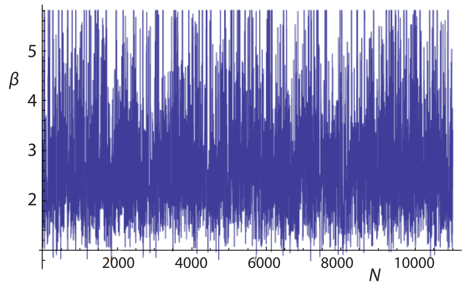

Figure 3. Simulation number of generated by important sample method of *β*

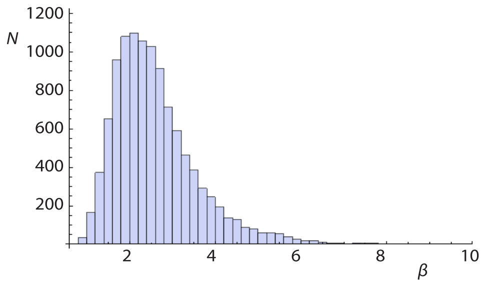

Figure 4. The histogram of the number generated by important sample method of *β*

Table 8. The 95% Bayesian prediction intervals of a future order statistic

$Y_{(s)}$	L	U
$Y_{(1)}$	0.0013	0.1124
$Y_{(2)}$	0.0155	0.4521
$Y_{(3)}$	0.1472	0.8217
$Y_{(4)}$	0.1777	0.8541
$Y_{(5)}$	0.2004	1.0026
$Y_{(6)}$	0.2874	1.3229
$Y_{(7)}$	0.3325	1.4219

Conclusion

In this paper, the problems of statistical inference and prediction of inverse Lomax distribution is discussed under Type-I HCS . This sampling plan is quite useful to practitioners, because they provide savings in resources and in total test time. Estimations under ML methods are computed with the help of Newton Raphson iteration and interval estimations are computed under the normality properties of ML estimators. The prior information of the model parameters in Baysian approach is represented by the independent gamma priors. The appropriate squared error loss equation is used. The Bayes estimation are computed with the help of MCMC methods. The same MCMC samples are used for two sample prediction problems. The details have been explained using a real life example. Also, the results from simulation studies illustrate that the performance of our proposed method is acceptable.

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