STATISTICAL INFERENCE FOR STRESS-STRENGTH RELIABILITY USING INVERSE LOMAX LIFETIME DISTRIBUTION WITH MECHANICAL ENGINEERING APPLICATIONS

by

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The inverse Lomax distribution has been extensively used in many disciplines, including stochastic modelling, economics, actuarial sciences, and life testing. It is among the most recognizable lifetime models. The purpose of this research is to look into a new and important aspect of the inverse Lomax distribution: the calculation of the fuzzy stress-strength reliability parameter $R_F = P(Y < X)$*, assuming X and Y are random independent variables that follow the inverse Lomax probability distribution. The properties of structural for the proposed reliability model are studied along with the Bayesian estimation methods, maximum product of the spacing and maximum likelihood. Extensive simulation studies are achieved to explore the performance of the various estimates. Subsequently, two sets of real data are considered to highlight the practicability of the model.*

Key words: *inverse Lomax distribution, fuzzy reliability, real data, maximum likelihood, Bayes estimation, simulation*

Introduction

In medical, industrial, reliability and technical implementations, special-forces units or systems may be exposed to arbitrary ecological stresses including moisture, pressure, and temperature. In these disciplines, the stress-strength-based dependability model is significant. In fact, the survival of the system depends on the strength and efficiency of its applications. In the last century, in the WW2, a number of technologies were discovered. It is including sensors and communications systems, were unsuccessful when used in settings other than those for which they were intended. For that purpose, scientists have begun to evaluate the reliability of equipment while looking at the influence of environmental conditions. System reliability is the chance that the system is robust enough to overcome stress. A receiver operating characteristic (ROC) region for diagnostic testing can be explained in the same way as traditional reliability, Bamber [1].

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Research on the traditional model of reliability for stress-strength on the evaluation, calculation, and estimation of the reliability of different stress and strength probabilities is numerous. Coolen and Newby [2] provided a thorough examination of the classical reliability of the stress-strength system. Raqab *et al*. [3], the classic stress-strength reliability parameter (SSRP) was calculated by considering the generalised three-parameter exponential distribution. Similarly, the estimate of the SSRP for generalised Pareto distribution was described by [4]. Although the El-Sagheer *et al*. [5] discussed the inferences for the SSRP when its strength variable is subjected to a partially accelerated life test. The analyze of the SSRP for the generalised logistic distribution has been investigated by Asgharzadeh *et al*. [6]. Akgul and Senogglu [7], the Lindley distribution was considered, and an estimation work was performed. Al-Omary *et a*l. [8] discussed the SSRP estimate when the random variables follow the Pareto distribution, and are independently exponentiated when selected samples using certain classification sample designs. With the maximum product of the spacing (MPS) estimation approach, Lu *et al*. [9] determined the traditional SSRP. Also, Mohamed *et al.* [10] introduced an application of type II half logistic Weibull model inference of the reliability analysis for bladder cancer data.

On the other hand, Huang [11] looked at fuzzy reliability estimates in which the random data, say X and Y, are not identically distributed but are independent in the distribution. It is explained that the goal of fuzzy reliability is to enable researchers to conduct delicate and precise assessments of the underlying systems of life reliability. If the difference $X - Y$ is larger, the system is more stable and reliable. Character and randomness in reliability engineering are features of the fuzzy reliability model for stress-strength on the classical reliability system. More lately, Bayesian reliability for fuzzy lifetime data has been introduced and discussed in Huang *et al*. [12]. Wu [13] offered a Bayesian technique for fuzzy reliability estimation. Moreover, Wu [14] discussed the evaluation of the reliability of fuzzy Bayesian systems using the exponential distribution. Recent, inferential works on fuzzy reliability include those of [6], Buckley [15] and Chem and Pham [16]. In addition, the bootstrap confidence intervals (CI) for reliability functions has been evaluated and analyzed in Lee *et al*. [17].

The characteristics of the developed non-parametric estimation of the reliability function that are used in many reliability problems are studied in Zardasht *et al*. [18]. Neamah *et al*. [19] developed a fuzzy reliability estimate through the Frechet probability using a simulation method. Sabry *et al*. [20] presented a fuzzy approach to the reliability system for the inverse Rayleigh distribution.

In the same as the preceding references, this paper contributes to the inference of the fuzzy SSRP defined by $R_F = P(Y < X)$, where X and Y represent random independent variables with the inverse Lomax (IL) distribution. We recall that the IL distribution has been extensively used in many disciplines, including stochastic modelling, economics, actuarial sciences, and life testing. It is among the most recognizable lifetime models, but the estimation of R_F in this setting remain unexplored, and motivates this study. To this aim, the product of the spacing approach is utilised in conjunction with a number of other strategies to estimate the dependability of the fuzzy stress strength. The MPS estimation technique and asymptotic and bootstrap CI for maximum likelihood (ML) estimation technique are obtained. As an alternative, the highest posterior density credible intervals for the Bayesian estimates (BE) are displayed. The effectiveness of the various estimates is also assessed and compared using a Markov chain Monte-Carlo (MCMC) simulation. The estimated functions for the reliability parameter are tested and shown using two real-data applications. Based on referenced criteria, the obtained results are very convining, validating the importance of the findings.

Some basics on the inverse Lomax distribution

The IL distribution is a specific example of a second-type generalised beta distribution. In statistical applications, this is one of the more noticeable lifetime models. From the mathematical, probability density function (pdf), the cumulative distribution function (cdf), survival rate function and hazard rate function of the IL distribution are defined, respectively:

$$f(x;\alpha,\beta)=\alpha\beta x^{-2}\left(1+\frac{\beta}{x}\right)^{-(1+\alpha)} \tag{1}$$

$$F(x;\alpha,\beta)=\left(1+\frac{\beta}{x}\right)^{-\alpha} \tag{2}$$

$$S(x;\alpha,\beta)=1-\left(1+\frac{\beta}{x}\right)^{-\alpha} \tag{3}$$

and

$$h(x;\alpha,\beta)=\frac{\alpha\beta x^{-2}(1+\beta x^{-1})^{-(1+\alpha)}}{1-(1+\beta x^{-1})^{-\alpha}} \tag{4}$$

where $x > 0$, $\alpha > 0$, and $\beta > 0$ denote to the shape and the scale parameters, respectively. It is understood that, for

$$x\le 0,\ F(x;\alpha,\beta)=f(x;\alpha,\beta)=h(x;\alpha,\beta)=0 \ \text{ and } \ S(x;\alpha,\beta)=1$$

To specify these parameters, the IL distribution is also denoted as IL (α, β). The key reference for the Il distribution is [21], which studied it in various fields, including stochastic system, economics, and life-testing. Furthermore, the IL distribution has been utilised by [22] in order to get Lorenz to order relations between ordered data. This lifetime distribution was applied in [23] to geophysical data, especially on the sizes of ground fibres in the USA of California.

The stress strength parameter

The simple SSRP is based on strength variable, denoted by X, and a stress variable, denoted by Y. It models a system that functions properly if X exceeds Y. Therefore, $R = P(Y < X)$ is a probability measure of the reliability of the system. Several authors are exploring several distinct types of such system. In this manuscript, we focus on the IL distribution, see references and motivation developed in section *Introduction*. That is, suppose X and Y are two random variables that follow the IL distribution with two different shape parameters α_1 and α_2 and the scale parameter β, respectively. Therefore, the expression of R can be written as the integral:

$$R=\int_{-\infty}^{\infty}P(Y<x\mid X=x)f(x;\alpha_1,\beta)\mathrm{d}x=\int_{-\infty}^{\infty}F(x;\alpha_2,\beta)f(x;\alpha_1,\beta)\mathrm{d}x \tag{5}$$

Elementary integral developments give:

$$R=\int_0^{\infty}\left(1+\frac{\beta}{x}\right)^{-\alpha_2}\alpha_1\beta x^{-2}\left(1+\frac{\beta}{x}\right)^{-(\alpha_1+1)}\mathrm{d}x=\int_0^{\infty}\beta\alpha_1 x^{-2}\left(1+\frac{\beta}{x}\right)^{-(\alpha_2+\alpha_1+1)}\mathrm{d}x=\frac{\alpha_1}{\alpha_1+\alpha_2} \tag{6}$$

It can be noted that the obtained expression is simple, and independent of β.

Fuzzy stress-strength reliability parameter

The fuzzy SSRP $R_F = P(Y < X)$ is calculated:

$$R_F = \int\int_{y<x} \mu_{A(y)}(x) f(y;\alpha_2,\beta) f(x;\alpha_1,\beta) \mathrm{d}x\mathrm{d}y \tag{7}$$

where $\mu_{A(y)}(x)$ is supposed to be increasing on the difference $(x - y)$. The [20], for example, used the definition of the fuzzy SSRP where X and Y are two independent random variables that follow the Rayleigh distribution. In this paper, we consider a simple membership function:

$$\mu_{A(y)}(x) = 1 - \frac{k}{y^2}$$

where $x > y$ and $k \geq 0$. Therefore, after integral calculations, the fuzzy SSRP $R_F = P(Y < X)$ is given:

$$\begin{aligned} R_F &= \int_0^\infty\int_y^\infty \left(1-\frac{k}{y^2}\right)\alpha_1\beta x^{-2}\left(1+\frac{\beta}{x}\right)^{-(1+\alpha_1)} \alpha_2\beta y^{-2}\left(1+\frac{\beta}{y}\right)^{-(1+\alpha_2)} \mathrm{d}x\mathrm{d}y = \\ &= \left(1-\frac{2k}{\beta^2(-2+\alpha_1+\alpha_2)(-1+\alpha_1+\alpha_2)(\alpha_1+\alpha_2)}\right)\left(\frac{\alpha_1}{\alpha_1+\alpha_2}\right) = \\ &= \left(1-\frac{2k}{\beta^2(-2+\alpha_1+\alpha_2)(-1+\alpha_1+\alpha_2)(\alpha_1+\alpha_2)}\right)R \end{aligned} \tag{8}$$

under the constraint that $\alpha_1 + \alpha_2 > 2$. It is worth noting that, when $k = 0$, $R_F = R$.

Inference of stress-strength model

In this section, the estimation of the fuzzy reliability parameter R_F is calculated using two approaches (ML and MPS). Let $x_1, x_2,..., x_n$ be a random sample of strength, and $y_1, y_2,..., y_m$ be a random sample of stress from the IL(α_1, β) and IL(α_2, β) distributions, respectively.

Maximum likelihood estimation

The construction of the maximum likelihood function is given:

$$\begin{aligned} L(\alpha_1,\alpha_2,\beta\,|\,\text{data}) &= \prod_{i=1}^n f(x_i;\alpha_1,\beta)\prod_{j=1}^m f(y_j;\alpha_2,\beta) = \\ &= \prod_{i=1}^n \alpha_1\beta x_i^{-2}\left(1+\frac{\beta}{x_i}\right)^{-(1+\alpha_1)} \prod_{j=1}^m \alpha_2\beta y_j^{-2}\left(1+\frac{\beta}{y_j}\right)^{-(1+\alpha_2)} \end{aligned} \tag{9}$$

$$L(\alpha_1,\alpha_2,\beta\,|\,\text{data}) = \alpha_1^n\alpha_2^m\beta^{n+m}\prod_{i=1}^n x_i^{-2}\prod_{j=1}^m y_j^{-2} = \prod_{i=1}^n\left(1+\frac{\beta}{x_i}\right)^{-(1+\alpha_1)}\prod_{j=1}^m\left(1+\frac{\beta}{y_j}\right)^{-(1+\alpha_2)} \tag{10}$$

After taking the logarithm of the sides, the previous equation becomes:

$$\ell(\alpha_1,\alpha_2,\beta\,|\,\text{data}) = n\log\alpha_1 + m\log\alpha_2 + (n+m)\log\beta - (1+\alpha_1)\sum_{i=1}^{n}\log\left(1+\frac{\beta}{x_i}\right) -$$

$$-2\sum_{j=1}^{m}\log y_j - 2\sum_{i=1}^{n}\log x_i - (1+\alpha_2)\sum_{j=1}^{n}\log\left(1+\frac{\beta}{y_j}\right) \tag{11}$$

The IL distribution parameters α_1, α_2, and β are estimated by the values of α_1, α_2, and β maximizing $\ell(\alpha_1$, α_2, and $\beta|\text{data})$. A system of MLE is obtained by differentiating with respect to the three parameters α_1, α_2, and β:

$$\frac{\partial\ell}{\partial\alpha_1} = \frac{n}{\alpha_1} - \sum_{i=1}^{n}\log\left(1+\frac{\beta}{x_i}\right) \tag{12}$$

$$\frac{\partial\ell}{\partial\alpha_2} = \frac{m}{\alpha_2} - \sum_{j=1}^{m}\log\left(1+\frac{\beta}{y_j}\right) \tag{13}$$

and

$$\frac{\partial\ell}{\partial\beta} = \frac{n+m}{\beta} - (1+\alpha_1)\sum_{i=1}^{n}\frac{x_i^{-1}}{1+\dfrac{\beta}{x_i}} - (1+\alpha_2)\sum_{j=1}^{m}\frac{y_j^{-1}}{1+\dfrac{\beta}{y_j}} \tag{14}$$

and find the solutions in α_1, α_2, and β of the system of equations defined by these derivatives equal to 0. Then the estimates of α_1 and α_2 are given:

$$\hat{\alpha}_1 = \frac{n}{\displaystyle\sum_{i=1}^{n}\log\left(1+\frac{\hat{\beta}}{x_i}\right)} \tag{15}$$

and

$$\hat{\alpha}_2 = \frac{m}{\displaystyle\sum_{j=1}^{m}\log\left(1+\frac{\hat{\beta}}{y_j}\right)} \tag{16}$$

The SSRP of the IL distribution on traditional reliability R fuzzy SSRP R_F are calculated using the ML invariance property:

$$\hat{R} = \frac{\hat{\alpha}_1}{\hat{\alpha}_1+\hat{\alpha}_2} \quad \text{and} \quad \hat{R}_F = \left(1 - \frac{2k}{\hat{\beta}^2(-2+\hat{\alpha}_1+\hat{\alpha}_2)(-1+\hat{\alpha}_1+\hat{\alpha}_2)(\hat{\alpha}_1+\hat{\alpha}_2)}\right)\hat{R} \tag{17}$$

Asymptotic confidence intervals

The usual asymptotic normality of $\hat{\alpha}_1$, $\hat{\alpha}_2$, and $\hat{\beta}$, by using the inverse of the computed Fisher information matrix, that is, the inverse of the matrix of the second derivatives of the log-probability function locally at the hat, can be used to calculate the $100(1-\gamma)\%$ CI for α_1, α_2, and β. The calculation of the second derivatives of with respect to the parameters are given:

$$\frac{\partial^2\ell}{\partial\alpha_1^2} = -\frac{n}{\alpha_1^2}, \quad \frac{\partial^2\ell}{\partial\alpha_2^2} = -\frac{m}{\alpha_2^2}$$

$$\frac{\partial^2 \ell}{\partial \alpha_1 \partial \beta} = -\sum_{i=1}^{n} \frac{x_i^{-1}}{1+\frac{\beta}{x_i}}, \quad \frac{\partial^2 \ell}{\partial \alpha_2 \partial \beta} = -\sum_{j=1}^{m} \frac{y_j^{-1}}{1+\frac{\beta}{y_j}}$$

$$\frac{\partial^2 \ell}{\partial \alpha_1 \partial \alpha_2} = \frac{\partial^2 \ell}{\partial \alpha_2 \partial \alpha_1} = 0$$

$$\frac{\partial^2 \ell}{\partial \beta^2} = -\frac{(n+m)}{\beta^2} - (1+\alpha_1)\sum_{i=1}^{n} \log\left(1+\frac{\beta}{x_i}\right) - (1+\alpha_2)\sum_{j=1}^{m} \log\left(1+\frac{\beta}{y_j}\right)$$

$$\frac{\partial^2 \ell}{\partial \beta \partial \alpha_1} = -\sum_{i=1}^{n} \frac{x_i^{-1}}{1+\frac{\beta}{x_i}}$$

and

$$\frac{\partial^2 \ell}{\partial \beta \partial \alpha_2} = -\sum_{j=1}^{m} \frac{y_j^{-1}}{1+\frac{\beta}{y_j}}$$

Thus, the calculated matrix of Fisher information $\hat{I}_{ij} = E[-\partial^2 \ell / \partial \phi_i \partial \phi_j]$, where $i, j = 1,2,3$ and $\phi = (\phi_1, \phi_2, \phi_3) = (\alpha_1, \alpha_2, \beta)$, which is given through removed the expectation operator E, will be used to evaluate the CI. Thus, the calculated information matrix and its inverse are given:

$$\hat{I} = \begin{pmatrix} -\frac{\partial^2 \ell}{\partial \alpha_1^2} & -\frac{\partial^2 \ell}{\partial \alpha_1 \partial \alpha_2} & -\frac{\partial^2 \ell}{\partial \alpha_1 \partial \beta} \\ -\frac{\partial^2 \ell}{\partial \alpha_2 \partial \alpha_1} & -\frac{\partial^2 \ell}{\partial \alpha_2^2} & -\frac{\partial^2 \ell}{\partial \alpha_2 \partial \beta} \\ -\frac{\partial^2 \ell}{\partial \beta \partial \alpha_1} & -\frac{\partial^2 \ell}{\partial \beta \partial \alpha_2} & -\frac{\partial^2 \ell}{\partial \beta^2} \end{pmatrix}_{(\alpha_1, \alpha_2, \beta) = (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta})}$$

$$\hat{I}^{-1} = \begin{pmatrix} \widehat{Var}(\hat{\alpha}_1) & \widehat{Cov}(\hat{\alpha}_1, \hat{\alpha}_2) & \widehat{Cov}(\hat{\alpha}_1, \hat{\beta}) \\ \widehat{Cov}(\hat{\alpha}_2, \hat{\alpha}_1) & \widehat{Var}(\hat{\alpha}_2) & \widehat{Cov}(\hat{\alpha}_2, \hat{\beta}) \\ \widehat{Cov}(\hat{\beta}, \hat{\alpha}_1) & \widehat{Cov}(\hat{\beta}, \hat{\alpha}_2) & \widehat{Var}(\hat{\beta}) \end{pmatrix}$$

Thus, the $100(1-\gamma)\%$ normal approximate CI for $(\alpha_1, \alpha_2, \beta)$:

$$\hat{\alpha}_1 \pm Z_{\gamma/2}\sqrt{\widehat{Var}(\hat{\alpha}_1)}, \ \ \hat{\alpha}_2 \pm Z_{\gamma/2}\sqrt{\widehat{Var}(\hat{\alpha}_2)} \ \text{and} \hat{\beta} \pm Z_{\gamma/2}\sqrt{Var(\hat{\beta})} \tag{18}$$

where $Z_{\gamma/2}$ is the percentile of the standard normal distribution with right-tail probability $\gamma/2$. Furthermore, with respect to construct the approximate CI of the hazard functions and reliability, we must determine their variance. The delta method [24] is used to find the asymptotic estimates of the variance of $\hat{R}_F$. The delta method is a standard approach for calculating CI for MLE functions. It takes a function that is too complex to compute the variance analytically, produces a linear approximation of it, and then calculates the variance of the simpler linear function that can be employed for large sample inference, [25]. We define:

$$G' = \left(\frac{\partial R_F}{\partial \alpha_1}, \frac{\partial R_F}{\partial \alpha_2}, \frac{\partial R_F}{\partial \beta} \right)$$

where

$$\frac{\partial R_F}{\partial \alpha_1} = \left(-\frac{k}{(\alpha_1 + k)^2} \right) \left(\frac{\alpha_2}{\alpha_1 + \alpha_2} \right) - \left(\frac{\alpha_2}{(\alpha_1 + \alpha_2)^2} \right) \left(\frac{k}{\alpha_1 + k} \right)$$

$$\frac{\partial R_F}{\partial \alpha_2} = \left(\frac{\alpha_1}{(\alpha_1 + \alpha_2)^2} \right) \left(\frac{k}{\alpha_1 + k} \right), \ \frac{\partial R_F}{\partial \beta} = 0$$

Then the approximate estimate of $Var(\hat{R}_F)$ is given:

$$\widehat{Var}(\hat{R}_F) = \left[G' \hat{I}^{-1} G \right]_{(\alpha_1, \alpha_2, \beta) = (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta})}$$

and the approximate CI for R_F:

$$\hat{R}_F \pm Z_{\gamma/2} \sqrt{\widehat{Var}(\hat{R}_F)} \tag{19}$$

Maximum product of spacing estimation

Suppose that the data are ordered in an increasing manner. Then, the maximum product spacing for the SSRP is denoted:

$$Gs(\alpha_1, \alpha_2, \beta \mid \text{data}) = \left(\prod_{i=1}^{n+1} D_l(x_i, \alpha_1, \beta) \right)^{\frac{1}{n+1}} \left(\prod_{j=1}^{m+1} D_l(y_j, \alpha_2, \beta) \right)^{\frac{1}{m+1}} \tag{20}$$

where $D_l(x_i, \alpha, \beta)) = F(x_i; \alpha, \beta) - F(x_{i-1}; \alpha, \beta)$ and $D_l(y_i, \alpha, \beta)) = F(y_i; \alpha, \beta) - F(y_{i-1}; \alpha, \beta)$

$$\begin{aligned} Gs(\alpha_1, \alpha_2, \beta \mid \text{data}) = & \left[\left(1 + \frac{\beta}{x_1} \right)^{-\alpha_1} \left(1 - \left(1 + \frac{\beta}{x_n} \right)^{-\alpha_1} \right) \prod_{i=2}^{n} \left[\left(1 + \frac{\beta}{x_i} \right)^{-\alpha_1} - \left(1 + \frac{\beta}{x_{i-1}} \right)^{-\alpha_1} \right] \right]^{\frac{1}{n+1}} \cdot \\ & \cdot \left[\left(1 + \frac{\beta}{y_1} \right)^{-\alpha_2} \left(1 - \left(1 + \frac{\beta}{y_m} \right)^{-\alpha_2} \right) \prod_{j=2}^{m} \left[\left(1 + \frac{\beta}{y_j} \right)^{-\alpha_2} - \left(1 + \frac{\beta}{y_{j-1}} \right)^{-\alpha_2} \right] \right]^{\frac{1}{m+1}} \end{aligned} \tag{21}$$

After taking the natural logarithm of both sides, the following expression is obtained:

$$\begin{aligned} & \log Gs(\alpha_1, \alpha_2, \beta \mid \text{data}) = \\ & = \frac{1}{n+1} \left[-\alpha_1 \log\left(1 + \frac{\beta}{x_1} \right) + \log\left(1 - \left(1 + \frac{\beta}{x_n} \right)^{-\alpha_1} \right) + \sum_{i=2}^{n} \log\left(\left(1 + \frac{\beta}{x_i} \right)^{-\alpha_1} - \left(1 + \frac{\beta}{x_{i-1}} \right)^{-\alpha_1} \right) \right] + \\ & + \frac{1}{m+1} \left[-\alpha_2 \log\left(1 + \frac{\beta}{y_1} \right) + \log\left(1 - \left(1 + \frac{\beta}{y_m} \right)^{-\alpha_2} \right) + \sum_{j=2}^{m} \log\left(\left(1 + \frac{\beta}{y_j} \right)^{-\alpha_2} - \left(1 + \frac{\beta}{y_{j-1}} \right)^{-\alpha_2} \right) \right] \end{aligned} \tag{22}$$

In order to derive the normal equations for the unknown parameters, eq. (22) for parameters α_1, α_2, and β are partially differentiated and zero equated. The estimates can be differentiated by solving:

$$\frac{\partial \log Gs}{\partial \alpha_1} = \frac{-1}{n+1}\log\left(1+\frac{\beta}{x_1}\right) + \frac{1}{n+1}\frac{\left(1+\frac{\beta}{x_n}\right)^{-\alpha_1}\log\left(1+\frac{\beta}{x_n}\right)}{1-(1+\frac{\beta}{x_n})^{-\alpha_1}} \cdot$$
$$\cdot \frac{1}{n+1}\sum_{i=2}^{n}\frac{\left(1+\frac{\beta}{x_{i-1}}\right)^{-\alpha_1}\log\left(1+\frac{\beta}{x_{i-1}}\right) - \left(1+\frac{\beta}{x_i}\right)^{-\alpha_1}\log\left(1+\frac{\beta}{x_i}\right)}{\left(1+\frac{\beta}{x_i}\right)^{-\alpha_1} - \left(1+\frac{\beta}{x_{i-1}}\right)^{-\alpha_1}} \tag{23}$$

$$\frac{\partial \log Gs}{\partial \alpha_2} = \frac{-1}{m+1}\log\left(1+\frac{\beta}{y_1}\right) + \frac{1}{m+1}\frac{\left(1+\frac{\beta}{y_m}\right)^{-\alpha_2}\log\left(1+\frac{\beta}{y_m}\right)}{1-\left(1+\frac{\beta}{y_m}\right)^{-\alpha_2}} \cdot$$
$$\cdot \frac{1}{m+1}\sum_{j=2}^{m}\frac{\left(1+\frac{\beta}{y_{j-1}}\right)^{-\alpha_2}\log\left(1+\frac{\beta}{y_{j-1}}\right) - \left(1+\frac{\beta}{y_j}\right)^{-\alpha_2}\log\left(1+\frac{\beta}{y_j}\right)}{\left(1+\frac{\beta}{y_j}\right)^{-\alpha_2} - \left(1+\frac{\beta}{y_{j-1}}\right)^{-\alpha_2}} \tag{24}$$

and

$$\frac{\partial \log Gs}{\partial \beta} = \frac{1}{n+1}\cdot\left(\frac{-\alpha_1 x_1^{-1}}{1+\frac{\beta}{x_1}} + \frac{-\alpha_1 x_n^{-1}\left(1+\frac{\beta}{x_n}\right)^{-\alpha_1-1}}{1-\left(1+\frac{\beta}{x_n}\right)^{-\alpha_1}} + \sum_{i=2}^{n}\frac{-\alpha_1 x_i^{-1}\left(1+\frac{\beta}{x_n}\right)^{-\alpha_1-1} + \alpha_1 x_{i-1}^{-1}\left(1+\frac{\beta}{x_{i-1}}\right)^{-\alpha_1-1}}{\left(1+\frac{\beta}{x_i}\right)^{-\alpha_1} - \left(1+\frac{\beta}{x_{i-1}}\right)^{-\alpha_1}}\right) +$$
$$+\frac{1}{m+1}\left(\frac{-\alpha_2 y_1^{-1}}{1+\frac{\beta}{y_1}} + \frac{-\alpha_2 y_m^{-1}\left(1+\frac{\beta}{y_m}\right)^{-\alpha_2-1}}{1-\left(1+\frac{\beta}{y_m}\right)^{-\alpha_2}} + \sum_{j=2}^{n}\frac{-\alpha_2 y_j^{-1}\left(1+\frac{\beta}{y_m}\right)^{-\alpha_2-1} + \alpha_2 y_{j-1}^{-1}\left(1+\frac{\beta}{y_{j-1}}\right)^{-\alpha_2-1}}{\left(1+\frac{\beta}{y_j}\right)^{-\alpha_2} - \left(1+\frac{\beta}{y_{j-1}}\right)^{-\alpha_2}}\right) \tag{25}$$

To find the estimates $\hat{\alpha}_1$, $\hat{\alpha}_2$, and $\hat{\beta}$ of the unknown parameters α_1, α_2, and β, the following non-linear equations cannot be solved analytically. The estimates of α_1, α_2, and β are computed using optimization techniques such as conjugate-gradient or Newton-Raphson optimization methods. Classical reliability R and fuzzy reliability R_F for IL probability for the SSRP can be approximated using the invariance property of the MPSE, which has been explained by [2, 8, 26-28] have concluded that it is the same as that of MLE.

Bootstrap confidence intervals

In this section, we propose using bootstrap CI instead of asymptotic CI for R. We generated parametric bootstrap samples for this purpose and found two distinct bootstrap CI. First, we used [29] percentile bootstrap approach (boot-p). The bootstrap-t approach (boot-t) was proposed, based on the concept of [30]. See [26, 27, 31, 32] for details on how these bootstrap CI techniques.

– Boot-p method

Step 1: Generate random samples $y_1, y_2,...,y_m$ from $F(y)$ and $x_1, x_2,...,x_n$ from $F(x)$.

Step 2: Generate independent bootstrap samples $x^*_1, x^*_2,..., x^*_n$ and $y^*_1, y^*_2,..., y^*_m$ from $F(x)$ and $F(y)$ on $\hat{\alpha}_1$, $\hat{\alpha}_2$, and $\hat{\beta}$, respectively, after computing the MLE of all parameters $\hat{\alpha}_1$, $\hat{\alpha}_2$, and $\hat{\beta}$.

Then, using the bootstrap samples to compute the MLE of all parameters, denoted by $\hat{\alpha}^*_1$, $\hat{\alpha}^*_2$, and $\hat{\beta}^*$.

Step 3: Replace the parameters in eq. (8) with their bootstrap estimates to obtain the bootstrap estimate of R_F, which is denoted by R^*_F.

Step 4: Repeat Step 2, B times and get a set of bootstrap estimates of R_F.

Step5: Let $\hat{F}_1(x) = P(R^*_F \le x)$ denotes the cdf of R^*_F. Define $R_{F(\text{Boot}-\text{p})}(x) = \hat{F}_1^{-1}(x)$.

Thus, the asymptotic 100(1 – p)% CI for R_F is given:

$$\left[R_{F(\text{Boot}-p)}\left(\frac{P}{2}\right), R_{F(\text{Boot}-p)}\left(1-\frac{P}{2}\right)\right]$$

– Boot-*t* method

Step 1: Same as the Boot-*p* method.

Step 2: Replace the parameters in eq. (8) with their bootstrap estimates to compute the bootstrap estimate of R_F, which is denoted by R^*_F and the statistics:

$$T^* = \frac{R^*_F - R_F}{\sqrt{\hat{V}(R^*_F)}}$$

Step 3: Repeat Step 2, B times.

Step 4: Now, let

$$R_{F(\text{Boot}-t)}(x) = R_F + \sqrt{\hat{V}(R^*_F)}\hat{F}_2^{-1}(x) \quad \text{where} \quad \hat{F}_2(x) = P(R^*_F \le x)$$

denotes the cdf of R^*_F. The asymptotic100(1 – p)% CI for R_F is then given:

$$\left[R_{F(\text{Boot}-t)}\left(\frac{P}{2}\right), R_{F(\text{Boot}-t)}\left(1-\frac{P}{2}\right)\right]$$

Bayesian estimates

The BE of the unknown parameters α_1, α_2, and β under the squared error loss function are provided in this section. The α_1, α_2, and β represent the unknown parameters to be independent from the exponential distribution:

$$\begin{aligned} \pi_1(\alpha_1) &\propto \mathrm{e}^{-b_1\alpha_1}, \quad \alpha_1 > 0, \quad b_1 > 0 \\ \pi_2(\alpha_2) &\propto \mathrm{e}^{-b_2\alpha_2}, \quad \alpha_2 > 0, \quad b_2 > 0 \\ \pi_3(\beta) &\propto \mathrm{e}^{-b_3\beta}, \quad \beta > 0, \quad b_3 > 0 \end{aligned} \tag{26}$$

The joint posterior of α_1, α_2, and β parameters symbolized by $\pi(\alpha_1, \alpha_2, \beta|\text{data})$ up to can be realized by combining the likelihood eq. (26) by Bayesian inference and may be expressed:

$$\pi^*(\alpha_1,\alpha_2,\beta\,|\,\text{data}) = \frac{\pi_1(\alpha_1)\,\pi_2(\alpha_2)\,\pi_3(\beta)\,L(\alpha_1,\alpha_2,\beta\,|\,\text{data})}{\int_0^\infty\int_0^\infty\int_0^\infty \pi_1(\alpha_1)\,\pi_2(\alpha_2)\,\pi_3(\beta)\,L(\alpha_1,\alpha_2,\beta\,|\,\text{data})\,\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}\beta} \tag{27}$$

The joint - posterior may be expressed as in eq. (28):

$$\pi^*(\alpha_1,\alpha_2,\beta\,|\,\text{data}) \propto \alpha_1^n\alpha_2^m\beta^{n+m}\mathrm{e}^{-b_1\alpha_1-b_2\alpha_2-b_3\beta}\prod_{i=1}^n x_i^{-2}\left(1+\frac{\beta}{x_i}\right)^{-(1+\alpha_1)}\prod_{j=1}^m y_j^{-2}\left(1+\frac{\beta}{y_j}\right)^{-(1+\alpha_2)} \tag{28}$$

The complete conditionals for α_1, α_2, and β can be written, up to proportionality:

$$\pi_1^*(\alpha_1\,|\,\alpha_2,\beta,\text{data}) \propto \alpha_1^n\mathrm{e}^{-b_1\alpha_1}\prod_{i=1}^n\left(1+\frac{\beta}{x_i}\right)^{-(1+\alpha_1)}$$
$$\pi_2^*(\alpha_2\,|\,\alpha_1,\beta,\text{data}) \propto \alpha_2^m\mathrm{e}^{-b_2\alpha_2}\prod_{j=1}^m\left(1+\frac{\beta}{y_j}\right)^{-(1+\alpha_2)} \tag{29}$$

and

$$\pi_3^*(\beta\,|\,\alpha_1,\alpha_2,\text{data}) \propto \beta^{n+m}\mathrm{e}^{-b_3\beta}\prod_{i=1}^n\left(1+\frac{\beta}{x_i}\right)^{-(1+\alpha_1)}\prod_{j=1}^m\left(1+\frac{\beta}{y_j}\right)^{-(1+\alpha_2)} \tag{30}$$

The joint – posterior of α_1 and α_2:

$$\pi^*(\alpha_1,\alpha_2\,|\,\beta,\text{data}) = \alpha_1^n\alpha_2^m\mathrm{e}^{-b_1\alpha_1-b_2\alpha_2}\prod_{i=1}^n\left(1+\frac{\beta}{x_i}\right)^{-(1+\alpha_1)}\prod_{j=1}^m\left(1+\frac{\beta}{y_j}\right)^{-(1+\alpha_2)} \tag{31}$$

The insufficiency of difference-based loss functions in recent statistical literature, such as the squared error loss. Various other loss functions have been proposed, the most well-known of which in [33] normalised squared loss function. The posterior mean for the squared error loss (SEL) function (symmetric) is used as the parameter estimate. As a result, when compared to the loss function, the Bayes estimates α_1, α_2, β, and R_F are obtained as, respectively:

$$\widehat{\alpha}_1 = E(\alpha_1\,|\,\alpha_2,\beta,\text{data}) = \int_0^\infty \alpha_1^{n+1}\mathrm{e}^{-b_1\alpha_1}\prod_{i=1}^n\left(1+\frac{\beta}{x_i}\right)^{-(1+\alpha_1)}\mathrm{d}\alpha_1 \tag{32}$$

$$\widehat{\alpha}_2 = E(\alpha_2\,|\,\alpha_1,\beta,\text{data}) = \int_0^\infty \alpha_2^{m+1}\mathrm{e}^{-b_2\alpha_2}\prod_{j=1}^m\left(1+\frac{\beta}{y_j}\right)^{-(1+\alpha_2)}\mathrm{d}\alpha_2 \tag{33}$$

$$\widehat{\beta} = E(\beta\,|\,\alpha_1,\alpha_2,\text{data}) = \int_0^\infty \beta^{n+m+1}\mathrm{e}^{-b_3\beta}\prod_{i=1}^n\left(1+\frac{\beta}{x_i}\right)^{-(1+\alpha_1)}\prod_{j=1}^m\left(1+\frac{\beta}{y_j}\right)^{-(1+\alpha_2)}\mathrm{d}\beta \tag{34}$$

and

$$\widehat{R}_F = E\left(R_F \mid \beta, \text{data}\right) = \int_0^{\infty}\int_0^{\infty}\left(1-\frac{2k}{\beta^2(-2+\alpha_1+\alpha_2)(-1+\alpha_1+\alpha_2)(\alpha_1+\alpha_2)}\right)\cdot$$

$$\cdot\alpha_1^n\alpha_2^m \mathrm{e}^{-b_1\alpha_1-b_2\alpha_2}\prod_{i=1}^{n}\left(1+\frac{\beta}{x_i}\right)^{-(1+\alpha_1)}\prod_{j=1}^{m}\left(1+\frac{\beta}{y_j}\right)^{-(1+\alpha_2)}\mathrm{d}\alpha_1\mathrm{d}\alpha_2 \tag{35}$$

Because the aforementioned integrals are difficult to obtain, the Bayes estimate is evaluated numerically. The BE are obtained using the MCMC method. Gibbs' sampling and more general Metropolis within Gibbs samplers are sub-classes of MCMC approaches, as discussed in [20, 27, 34, 35]. The conditional posterior of α_1, α_2, β, and R_F are used to produce random samples using Metropolis Hastings algorithm (MHA). The highest posterior density (HPD) intervals and the Bayesian credible intervals (BCI) will be computed. The [36] proposed PHD algorithm may be expressed as:

– Organize the sample obser vations produced by MHA $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, β, and R_F as

$$\left(\tilde{\alpha}_1^{[1]} \le \tilde{\alpha}_1^{[2]} \le \ldots \le \tilde{\alpha}_1^{[A]}\right),\ \left(\tilde{\alpha}_2^{[1]} \le \tilde{\alpha}_2^{[2]} \le \ldots \le \tilde{\alpha}_2^{[A]}\right)$$

$$\left(\tilde{\beta}^{[1]} \le \tilde{\beta}^{[2]} \le \ldots \le \tilde{\beta}^{[A]}\right),\ \text{and}\ \left(\tilde{R}_F^{[1]} \le \tilde{R}_F^{[2]} \le \ldots \le \tilde{R}_F^{[A]}\right)$$

where the length of the generated of MHA is A.

– The $100(1-\varphi)$%, BCI for α_1, α_2, β, and R_F, are given

$$\left(\tilde{\alpha}_1^{[(\varphi/2)A]}, \tilde{\alpha}_1^{[(1-\varphi/2)A]}\right),\ \left(\tilde{\alpha}_2^{[(\varphi/2)A]}, \tilde{\alpha}_2^{[(1-\varphi/2)A]}\right)$$

$$\left(\tilde{\beta}^{[(\varphi/2)A]}, \tilde{\beta}^{[(1-\varphi/2)A]}\right),\ \text{and}\ \left(\tilde{R}_F^{[(\varphi/2)A]}, \tilde{R}_F^{[(1-\varphi/2)A]}\right)$$

Simulation

In this section, we will perform a simulation examine how each estimate of the vector parameters α_1, α_2, and β behaves numerically in terms of bias, mean-squared-error (MSE), and CI length for each technique (L.CI). The simulation algorithm is constructed using the techniques:

– The values of the stress-strength of IL distribution parameters α_1, α_2, and β are: in tab. 1 shows the constant $\alpha_1= 0.15$, $\beta = 0.5$, and the modifications in α_2 to 0.5 and 2. In tab. 2 explains the constant $\alpha_1= 0.75$, $\alpha_2 = 2$, and the modifications in β to 0.5 and 3. In tab. 3 explain the constant $\alpha_2= 3$, $\beta = 1.5$, and the modifications in α_1 to 1.5 and 3.
– The sample size of strength n, and the sample size of stress, m are determined. The sample sizes of n = 30, 40, 90, and 100, and m = 30, 50, 80, and 110 are being considered.
– The number of simulation replications is represented, L = 5000.
– Equation (2) of the IL distribution function, we generate random samples of size n from a uniform distribution U_1 over the interval (0, 1) and change them into samples of strength with an IL distribution with the parameters α_1 and β:

$$x_i = \beta\left(u_{1i}^{\frac{-1}{\alpha_1}} - 1\right)^{-1},\quad i = 1,\ldots,n$$

To generate random samples of size m using the IL distribution function in eq. (2) from a uniform distribution U_2 with (0,1), we change them into samples of stress with an IL distribution with the parameters α_2, and β:

$$u_j = \beta\left(u_{2j}^{\frac{-1}{\alpha_2}} - 1\right) - 1, \quad j = 1, ..., m$$

– Estimate the reliability stress-strength of IL model for each estimation method.
– Estimate the traditional reliability stress-strength for each estimation method.
– Determine the parameter of membership function give the fuzzy reliability stress-strength for each estimation method as: $k = 0.02$ is R_{F1} and $k = 0.15$ is R_{F2}.
– Calculate various performance measures such as the bias and MSE for each method.
– Calculate L.CI for each method where the length of asymptotic CI (L.ACI), length of bootstrap-p (L.BP), and length of bootstrap-t (L.BT) for MLE while for MPS, we used length of L.CI as asymptotic CI and while for Bayesian, we used length of credible CI (L.CCI).

Table 1. The MLE and Bayesian of the parameters and reliability stress-strength of IL distribution when $\alpha_1 = 0.15$ and $\beta = 0.5$

$\alpha_1 = 1.6, \beta_2 = 0.5$			MLE		MPS		Bayesian		MLE			MPS	Bayesian
α_2	n,m		Bias	MSE	Bias	MSE	Bias	MSE	L.ACI	L.BP	L.BT	L.CI	L.CCI
*0.5	6*30,30	α_1	0.2716	0.4588	–0.1544	0.2815	0.0010	0.0204	2.4335	0.272574	0.074712	1.9907	0.5441
		α_2	0.0586	0.0266	–0.0310	0.0160	0.0134	0.0060	0.5973	0.058588	0.003454	0.4811	0.2830
		β	–0.0066	0.0587	0.1963	0.1527	0.0044	0.0093	0.9500	0.006657	0.000106	1.3250	0.3643
		R	0.0006	0.0028	–0.0151	0.0029	–0.0048	0.0010	0.2065	0.000560	0.000003	0.2043	0.1157
		R_{F1}	0.0473	0.0082	0.0387	0.0078	0.0212	0.0106	0.3031	0.047275	0.002241	0.3122	0.3832
		R_{F2}	0.3807	0.2002	0.4072	0.2356	0.1435	0.0539	0.9223	0.380670	0.144968	1.0365	0.5855
	6*40,50	α_1	0.1890	0.2999	–0.1254	0.2059	–0.0110	0.0105	2.0159	0.189630	0.036230	1.7103	0.3965
		α_2	0.0333	0.0139	–0.0294	0.0098	0.0072	0.0035	0.4435	0.033279	0.001120	0.3713	0.2159
		β	–0.0051	0.0365	0.1386	0.0835	0.0040	0.0060	0.7487	0.004929	0.000061	0.9947	0.3035
		R	0.0027	0.0017	–0.0102	0.0018	–0.0038	0.0006	0.1600	0.002726	0.000009	0.1618	0.0908
		R_{F1}	0.0484	0.0071	0.0429	0.0060	0.0067	0.0107	0.2705	0.048342	0.002342	0.2541	0.4172
		R_{F2}	0.3543	0.1864	0.3916	0.2255	0.1059	0.0376	0.9675	0.354355	0.125627	1.0537	0.5719
	6*90,80	α_1	0.0977	0.1360	–0.1015	0.0997	0.0008	0.0047	1.3946	0.097858	0.009701	1.1724	0.2712
		α_2	0.0156	0.0061	–0.0246	0.0052	0.0042	0.0020	0.3003	0.015592	0.000249	0.2649	0.1674
		β	–0.0007	0.0180	0.0849	0.0312	0.0019	0.0028	0.5265	0.000929	0.000019	0.6069	0.2016
		R	0.0020	0.0009	–0.0056	0.0009	–0.0014	0.0003	0.1204	0.001970	0.000005	0.1182	0.0673
		R_{F1}	0.0437	0.0071	0.0468	0.0081	–0.0100	0.0091	0.3144	0.043774	0.001923	0.3022	0.4154
		R_{F2}	0.2819	0.1410	0.3197	0.1886	0.0859	0.0288	0.9730	0.282016	0.079598	1.1529	0.5340
	6*100,110	α_1	0.0841	0.1103	–0.0866	0.0839	–0.0006	0.0012	1.2602	0.084369	0.007220	1.0842	0.1381
		α_2	0.0139	0.0044	–0.0194	0.0038	0.0005	0.0009	0.2552	0.013874	0.000197	0.2296	0.1148
		β	–0.0035	0.0166	0.0677	0.0257	0.0002	0.0011	0.5049	0.003537	0.000030	0.5698	0.1260
		R	0.0013	0.0008	–0.0057	0.0008	–0.0001	0.0001	0.1130	0.001330	0.000003	0.1115	0.0443
		R_{F1}	0.0389	0.0078	0.0425	0.0075	–0.0187	0.0047	0.3118	0.038886	0.001518	0.2960	0.1725
		R_{F2}	0.2812	0.1394	0.2788	0.1583	0.0473	0.0166	0.9632	0.281103	0.079079	1.1132	0.4765
*2	6*30,30	α_1	0.2625	0.4013	–0.1886	0.2516	0.0041	0.0198	2.2613	0.262255	0.069106	1.8227	0.5395
		α_2	0.3559	0.7011	–0.2388	0.4389	0.0028	0.0197	2.9725	0.355985	0.127316	2.4238	0.5490
		β	–0.0137	0.0569	0.2198	0.1761	0.0086	0.0082	0.9337	0.013722	0.000247	1.4022	0.3444
		R	–0.0006	0.0042	0.0032	0.0037	0.0001	0.0007	0.2545	0.000726	0.000005	0.2385	0.1061

→

		R_{F1}	–0.0022	0.0044	–0.0018	0.0057	–0.0003	0.0007	0.2593	0.002261	0.000009	0.2968	0.1060
		R_{F2}	–0.0055	0.0054	–0.0223	0.0145	–0.0037	0.0011	0.2884	0.005497	0.000036	0.4639	0.1290
	6*40,50	α_1	0.1457	0.2184	–0.1779	0.1878	–0.0068	0.0101	1.7415	0.146069	0.021536	1.5499	0.3805
		α_2	0.1852	0.3559	–0.2274	0.3175	–0.0028	0.0118	2.2242	0.184688	0.034451	2.0218	0.4169
		β	–0.0035	0.0315	0.1639	0.0903	0.0045	0.0044	0.6958	0.003710	0.000043	0.9874	0.2536
		R	0.0006	0.0029	0.0021	0.0026	–0.0008	0.0004	0.2095	0.000656	0.000003	0.1995	0.0758
		R_{F1}	0.0006	0.0030	–0.0039	0.0034	–0.0010	0.0004	0.2163	0.000560	0.000003	0.2289	0.0765
		R_{F2}	–0.0055	0.0038	–0.0405	0.0112	–0.0036	0.0006	0.2409	0.005511	0.000034	0.3844	0.0911
	6*90,80	α_1	0.0873	0.1302	–0.1308	0.1037	–0.0032	0.0050	1.3731	0.087056	0.007696	1.1541	0.2760
		α_2	0.1198	0.2410	–0.1718	0.1898	0.0033	0.0052	1.8673	0.119741	0.014569	1.5704	0.2780
		β	0.0029	0.0217	0.1057	0.0428	0.0031	0.0027	0.5781	0.003005	0.000031	0.6970	0.1975
		R	–0.0001	0.0013	0.0024	0.0012	–0.0009	0.0002	0.1405	0.000052	0.000001	0.1360	0.0529
		R_{F1}	–0.0009	0.0013	–0.0014	0.0012	–0.0011	0.0002	0.1417	0.000898	0.000002	0.1354	0.0517
		R_{F2}	–0.0057	0.0019	–0.0248	0.0043	–0.0021	0.0003	0.1709	0.005696	0.000034	0.2369	0.0637
	6*100,110	α_1	0.0929	0.1218	–0.1085	0.0916	0.0006	0.0012	1.3195	0.092822	0.008728	1.1078	0.1384
		α_2	0.1334	0.2045	–0.1326	0.1502	–0.0004	0.0013	1.6946	0.133520	0.018021	1.4284	0.1383
		β	–0.0069	0.0169	0.0824	0.0294	–0.0004	0.0010	0.5091	0.007053	0.000066	0.5894	0.1209
		R	–0.0015	0.0012	0.0001	0.0011	0.0001	0.0000	0.1345	0.001558	0.000004	0.1302	0.0269
		R_{F1}	–0.0018	0.0011	–0.0023	0.0011	0.0001	0.0000	0.1328	0.001734	0.000004	0.1292	0.0267
		R_{F2}	–0.0038	0.0015	–0.0210	0.0028	–0.0006	0.0001	0.1519	0.003828	0.000016	0.1891	0.0365

Table 2. The MLE and Bayesian of the parameters and reliability stress-strength of IL distribution when $\alpha_1 = 0.75$ and $\alpha_2 = 2$

$\alpha_1 = 0.75$ and $\alpha_2 = 0.5$			MLE		MPS		Bayesian		MLE			MPS	Bayesian
β	n,m		Bias	MSE	Bias	MSE	Bias	MSE	L.ACI	L.BP	L.BT	L.CI	L.CCI
*0.5	6*30,30	α_1	0.0855	0.0695	–0.0733	0.0425	0.0040	0.0118	0.9782	0.085543	0.007377	0.7559	0.4082
		α_2	0.3403	0.7614	–0.2314	0.4552	0.0025	0.0189	3.1513	0.340731	0.116749	2.4856	0.5347
		β	0.0012	0.0552	0.2198	0.1652	0.0103	0.0091	0.9214	0.001164	0.000052	1.3406	0.3502
		R	0.0027	0.0030	–0.0119	0.0030	–0.0004	0.0010	0.2153	0.002603	0.000010	0.2111	0.1187
		R_{F1}	0.0029	0.0066	–0.0019	0.0082	–0.0027	0.0010	0.3187	0.002945	0.000015	0.3556	0.1211
		R_{F2}	0.0252	0.0225	0.0107	0.0304	–0.0228	0.0095	0.5797	0.025043	0.000649	0.6826	0.3385
	6*40,50	α_1	0.0681	0.0451	–0.0518	0.0308	0.0002	0.0079	0.7892	0.068395	0.004717	0.6573	0.3398
		α_2	0.2319	0.4281	–0.1804	0.3151	–0.0062	0.0116	2.3995	0.231542	0.053968	2.0847	0.4201
		β	–0.0069	0.0377	0.1447	0.0876	0.0032	0.0052	0.7614	0.006762	0.000084	1.0122	0.2798
		R	0.0005	0.0020	–0.0094	0.0020	–0.0003	0.0007	0.1746	0.000437	0.000002	0.1726	0.0995
		R_{F1}	–0.0019	0.0045	–0.0142	0.0094	–0.0023	0.0006	0.2625	0.001950	0.000008	0.3769	0.0976
		R_{F2}	0.0029	0.0195	–0.0211	0.0265	–0.0198	0.0053	0.5475	0.002885	0.000028	0.6334	0.2580
	6*90,80	α_1	0.0378	0.0182	–0.0353	0.0137	0.0005	0.0033	0.5072	0.037743	0.001441	0.4370	0.2216
		α_2	0.1252	0.2263	–0.1451	0.1721	–0.0003	0.0059	1.7997	0.125617	0.015986	1.5243	0.3007
		β	–0.0051	0.0194	0.0840	0.0341	0.0008	0.0028	0.5465	0.005413	0.000048	0.6448	0.2014
		R	–0.0008	0.0011	–0.0085	0.0012	–0.0001	0.0003	0.1302	0.000766	0.000002	0.1293	0.0637
		R_{F1}	–0.0074	0.0036	–0.0209	0.0073	–0.0010	0.0003	0.2321	0.007403	0.000058	0.3247	0.0647
		R_{F2}	–0.0074	0.0150	–0.0570	0.0238	–0.0093	0.0025	0.4799	0.007379	0.000069	0.5627	0.1864
	6*100,110	α_1	0.0332	0.0173	–0.0317	0.0133	0.0014	0.0011	0.4984	0.033228	0.001121	0.4354	0.1250

→

		α_2	0.1158	0.1763	–0.1203	0.1343	0.0003	0.0013	1.5827	0.115924	0.013599	1.3576	0.1432
		β	0.0002	0.0172	0.0773	0.0282	–0.0008	0.0010	0.5148	0.000190	0.000016	0.5851	0.1233
		R	0.0006	0.0008	–0.0057	0.0008	–0.0003	0.0001	0.1119	0.000548	0.000001	0.1103	0.0357
		R_{F1}	–0.0043	0.0029	–0.0190	0.0063	–0.0006	0.0001	0.2100	0.004295	0.000021	0.3015	0.0346
		R_{F2}	0.0066	0.0138	–0.0597	0.0223	–0.0034	0.0007	0.4599	0.006631	0.000058	0.5362	0.1006
*3	6*30,30	α_1	0.1215	0.1093	–0.0504	0.0364	0.0105	0.0107	1.2056	0.121769	0.014930	0.7219	0.3989
		α_2	0.4867	1.3777	–0.1614	0.3865	–0.0065	0.0188	4.1891	0.486131	0.237447	2.3547	0.5084
		β	–0.1150	1.6421	0.7548	2.0859	–0.0020	0.0196	5.0055	0.115139	0.014848	4.8293	0.5463
		R	0.0042	0.0031	–0.0094	0.0029	–0.0031	0.0009	0.2176	0.004157	0.000020	0.2079	0.1171
		R_{F1}	0.0040	0.0031	–0.0097	0.0032	–0.0031	0.0009	0.2192	0.003984	0.000019	0.2182	0.1169
		R_{F2}	0.0028	0.0039	–0.0081	0.0039	–0.0037	0.0009	0.2454	0.002899	0.000012	0.2419	0.1143
	6*40,50	α_1	0.0947	0.0664	–0.0366	0.0268	0.0018	0.0069	0.9397	0.095214	0.009125	0.6263	0.3168
		α_2	0.3495	0.7896	–0.1268	0.2710	–0.0035	0.0113	3.2041	0.349197	0.122638	1.9804	0.4076
		β	–0.1783	1.0533	0.5183	1.2633	–0.0014	0.0115	3.9640	0.178265	0.032825	3.9115	0.4049
		R	0.0027	0.0020	–0.0070	0.0018	–0.0006	0.0006	0.1732	0.002653	0.000009	0.1663	0.0941
		R_{F1}	0.0027	0.0020	–0.0062	0.0022	–0.0006	0.0006	0.1755	0.002692	0.000009	0.1810	0.0939
		R_{F2}	0.0036	0.0024	–0.0110	0.0035	–0.0009	0.0006	0.1905	0.003642	0.000016	0.2653	0.0923
	6*90,80	α_1	0.0491	0.0239	–0.0240	0.0140	0.0009	0.0034	0.5753	0.048780	0.002403	0.4541	0.2198
		α_2	0.1987	0.3314	–0.0830	0.1675	0.0012	0.0047	2.1191	0.198644	0.039733	1.5718	0.2639
		β	–0.0961	0.6673	0.3389	0.7123	–0.0003	0.0053	3.1815	0.097313	0.010156	3.0315	0.2771
		R	0.0025	0.0011	–0.0048	0.0011	0.0000	0.0003	0.1309	0.002567	0.000008	0.1270	0.0658
		R_{F1}	0.0024	0.0011	–0.0055	0.0012	0.0000	0.0003	0.1320	0.002355	0.000007	0.1364	0.0656
		R_{F2}	0.0013	0.0017	–0.0100	0.0028	–0.0001	0.0003	0.1593	0.001340	0.000003	0.2030	0.0647
	6*100,110	α_1	0.0343	0.0184	–0.0281	0.0118	0.0000	0.0012	0.5141	0.034191	0.001186	0.4124	0.1354
		α_2	0.1183	0.2113	–0.1108	0.1242	0.0023	0.0013	1.7423	0.118149	0.014157	1.3124	0.1355
		β	–0.0413	0.5347	0.3408	0.5956	–0.0001	0.0013	2.8633	0.041497	0.002243	2.7156	0.1364
		R	0.0001	0.0009	–0.0056	0.0009	0.0003	0.0001	0.1169	0.000033	0.000001	0.1136	0.0372
		R_{F1}	0.0000	0.0009	–0.0059	0.0009	0.0003	0.0001	0.1176	0.000037	0.000001	0.1148	0.0371
		R_{F2}	–0.0012	0.0011	–0.0101	0.0016	0.0003	0.0001	0.1287	0.001142	0.000002	0.1534	0.0360

Table 3. The MLE and Bayesian of the parameters and reliability stress-strength of IL distribution when $\alpha_2 = 3$, $\beta = 1.5$

$\alpha_1 = 3, \beta_2 = 1.5$			MLE		MPS		Bayesian		MLE			MPS	Bayesian
α_1	n,m		Bias	MSE	Bias	MSE	Bias	MSE	L.ACI	L.BP	L.BT	L.CI	L.CCI
*1.5	6*30,30	α_1	0.3234	0.5652	–0.1878	0.2204	–0.0102	0.0184	2.6619	0.323819	0.105305	1.6875	0.5229
		α_2	0.7575	2.6333	–0.4378	1.0253	0.0015	0.0223	5.6284	0.757685	0.576278	3.5809	0.5804
		β	–0.0290	0.6362	0.6786	1.3918	0.0049	0.0188	3.1261	0.029599	0.001512	3.7849	0.5267
		R	0.0021	0.0032	–0.0099	0.0030	0.0019	0.0005	0.2234	0.002100	0.000008	0.2108	0.0896
		R_{F1}	0.0021	0.0033	–0.0109	0.0034	0.0019	0.0005	0.2237	0.002220	0.000008	0.2260	0.0895
		R_{F2}	0.0020	0.0033	–0.0125	0.0036	0.0018	0.0005	0.2254	0.002081	0.000008	0.2309	0.0892
	6*40,50	α_1	0.2626	0.3871	–0.1362	0.1664	0.0009	0.0104	2.2122	0.261799	0.068877	1.5082	0.3993
		α_2	0.5366	1.6670	–0.3646	0.7776	–0.0063	0.0111	4.6057	0.536548	0.289347	3.1491	0.3990
		β	–0.0646	0.4131	0.4640	0.7886	–0.0071	0.0115	2.5080	0.065362	0.004685	2.9696	0.4039

→

		R	–0.0026	0.0024	–0.0109	0.0023	–0.0004	0.0003	0.1925	0.002559	0.000009	0.1833	0.0676
		R_{F1}	–0.0026	0.0024	–0.0112	0.0023	–0.0004	0.0003	0.1926	0.002615	0.000009	0.1836	0.0676
		R_{F2}	–0.0027	0.0024	–0.0129	0.0024	–0.0005	0.0003	0.1934	0.002713	0.000010	0.1864	0.0668
	6*90,80	α_1	0.1206	0.1659	–0.1184	0.0905	0.0026	0.0048	1.5259	0.120858	0.014748	1.0848	0.2721
		α_2	0.2891	0.9852	–0.2896	0.5153	0.0016	0.0053	3.7240	0.290114	0.085107	2.5761	0.2814
		β	–0.0179	0.2113	0.3110	0.3503	–0.0003	0.0046	1.8016	0.017944	0.000546	1.9749	0.2551
		R	–0.0002	0.0013	–0.0072	0.0013	–0.0002	0.0001	0.1430	0.000277	0.000001	0.1382	0.0445
		R_{F1}	–0.0002	0.0013	-0.0073	0.0013	–0.0002	0.0001	0.1431	0.000263	0.000001	0.1384	0.0444
		R_{F2}	–0.0003	0.0014	-0.0080	0.0013	–0.0002	0.0001	0.1443	0.000310	0.000001	0.1401	0.0441
	6*100,110	α_1	0.0909	0.1489	–0.1219	0.0865	0.0006	0.0013	1.4706	0.091711	0.008546	1.0497	0.1431
		α_2	0.2135	0.8096	–0.2943	0.4412	0.0007	0.0014	3.4281	0.213238	0.046195	2.3356	0.1508
		β	0.0132	0.2139	0.3053	0.3303	–0.0021	0.0013	1.8131	0.013675	0.000424	1.9098	0.1390
		R	–0.0005	0.0011	–0.0061	0.0010	0.0000	0.0000	0.1298	0.000476	0.000001	0.1248	0.0244
		R_{F1}	–0.0005	0.0011	–0.0062	0.0011	0.0000	0.0000	0.1300	0.000552	0.000001	0.1249	0.0244
		R_{F2}	–0.0006	0.0011	–0.0069	0.0011	0.0000	0.0000	0.1307	0.000578	0.000001	0.1259	0.0241
*3	6*30,30	α_1	0.7504	2.7398	–0.4920	0.9600	0.0054	0.0186	5.7863	0.750176	0.565001	3.3231	0.5401
		α_2	0.7480	2.7824	–0.4971	0.9547	0.0015	0.0204	5.8474	0.750173	0.565112	3.2990	0.5352
		β	0.0230	0.8414	0.7260	1.5332	–0.0030	0.0182	3.5963	0.023703	0.001398	3.9339	0.5217
		R	–0.0003	0.0040	–0.0004	0.0035	–0.0003	0.0003	0.2479	0.000402	0.000004	0.2325	0.0644
		R_{F1}	–0.0003	0.0040	–0.0005	0.0035	–0.0004	0.0003	0.2478	0.000359	0.000004	0.2325	0.0644
		R_{F2}	–0.0003	0.0040	–0.0009	0.0035	–0.0004	0.0003	0.2477	0.000394	0.000004	0.2322	0.0643
	6*40,50	α_1	0.5504	1.7949	–0.4026	0.7333	0.0021	0.0116	4.7905	0.550614	0.304686	2.9642	0.4147
		α_2	0.5014	1.7487	–0.4227	0.7614	–0.0081	0.0120	4.7992	0.499479	0.251045	2.9938	0.4258
		β	0.0047	0.5719	0.5404	0.9298	–0.0018	0.0100	2.9658	0.004366	0.000576	3.1321	0.3909
		R	–0.0035	0.0030	–0.0020	0.0027	–0.0009	0.0002	0.2144	0.003547	0.000015	0.2033	0.0492
		R_{F1}	–0.0035	0.0030	–0.0020	0.0027	–0.0009	0.0002	0.2144	0.003481	0.000015	0.2033	0.0491
		R_{F2}	–0.0035	0.0030	–0.0023	0.0027	–0.0009	0.0002	0.2142	0.003451	0.000015	0.2029	0.0489
	6*90,80	α_1	0.3011	0.9829	–0.3068	0.4395	0.0016	0.0051	3.7046	0.302277	0.092301	2.3047	0.2717
		α_2	0.3037	1.0016	–0.3106	0.4426	0.0027	0.0050	3.7399	0.303082	0.092757	2.3073	0.2795
		β	0.0089	0.2930	0.3423	0.4075	–0.0042	0.0047	2.1228	0.009187	0.000352	2.1133	0.2652
		R	0.0001	0.0014	–0.0004	0.0013	0.0001	0.0001	0.1457	0.000081	0.000001	0.1409	0.0327
		R_{F1}	0.0001	0.0014	–0.0004	0.0013	0.0001	0.0001	0.1457	0.000081	0.000001	0.1409	0.0327
		R_{F2}	0.0001	0.0014	–0.0005	0.0013	0.0001	0.0001	0.1456	0.000097	0.000001	0.1408	0.0327
	6*100,110	α_1	0.2699	0.8529	–0.2787	0.3682	–0.0011	0.0013	3.4638	0.272185	0.074877	2.1139	0.1380
		α_2	0.2481	0.8010	–0.2922	0.3669	–0.0003	0.0013	3.3725	0.247919	0.062165	2.0810	0.1388
		β	0.0000	0.2353	0.2899	0.2999	–0.0006	0.0013	1.9026	0.000338	0.000236	1.8220	0.1385
		R	–0.0015	0.0012	–0.0012	0.0012	0.0001	0.0000	0.1374	0.001410	0.000003	0.1330	0.0163
		R_{F1}	–0.0015	0.0012	–0.0012	0.0012	0.0001	0.0000	0.1374	0.001534	0.000004	0.1330	0.0163
		R_{F2}	–0.0015	0.0012	–0.0013	0.0011	0.0001	0.0000	0.1373	0.001507	0.000003	0.1328	0.0162

From tabs. 1-3, the simulation results are concluding:

- It is observed that MSE (MLE) > MSE (MPS), bias (MLE) > bias (MPS), and L.CI (MLE) > L.CI (MPS) in most parameters *i.e.* MPS performs better than MLE in the sense of bias, MSE, and L.CI.
- When the k value is increased, the fuzzy reliability stress-strength values tend to the conventional reliability stress-strength values, suggesting that the variability goes away.
- As predicted, when the sample sizes, n and m, are increased, the bias and MSE values for each parameter decrease.
- When $k = 6$, the fewest bias values for all calculations, as well as the smallest MSE values for fuzzy and traditional reliability, are found.
- When α_1 is increased, the largest MSE values for the parameters are obtained.
- When α_2 is increased, the smallest MSE values for the parameters are obtained
- In comparison MLE and MPS based on minimum bias, and MSE, Bayesian estimators perform better.
- When $\alpha_1 < \alpha_2$, reliability stress-strength value increases.
- The length bootstrap-*t* (L.BT) is the smallest length of CI.

Applications of real data

Following are two real data sets that were covered in this section.

First real data

We present a data analysis utilizing real data from [37, 38] to compare two distinct strategies for predicting unit capacity factors dubbed SC16 and P3. This data has been used for a different model of stress-strength such as in [39]. The information is SC16 data: x is (0.853, 0.759, 0.866, 0.809, 0.717, 0.544, 0.492, 0.403, 0.344, 0.213, 0.116, 0.116, 0.092, 0.07, 0.059, 0.048, 0.036, 0.029, 0.021, 0.014, 0.011, 0.008, 0.006) and P_3 data: y is (0.853, 0.759, 0.874, 0.8, 0.716, 0.557, 0.503, 0.399, 0.334, 0.207, 0.118, 0.118, 0.097, 0.078, 0.067, 0.056, 0.044, 0.036, 0.026, 0.019, 0.014, 0.01).

Using the Kolmogrov-Smirnov (KS) test, we conclude that the IL distribution with parameter $\hat{\alpha} = 1.00772$ and $\hat{\beta} = 0.1151$ can be fitted on strength variable, and $\hat{\alpha} = 1.5314$ and $\hat{\beta} = 0.07559$ can be fitted on stress variable, which are shown in tab. 4. Also figs. 1 and 2 confirmed fitting of theses data.

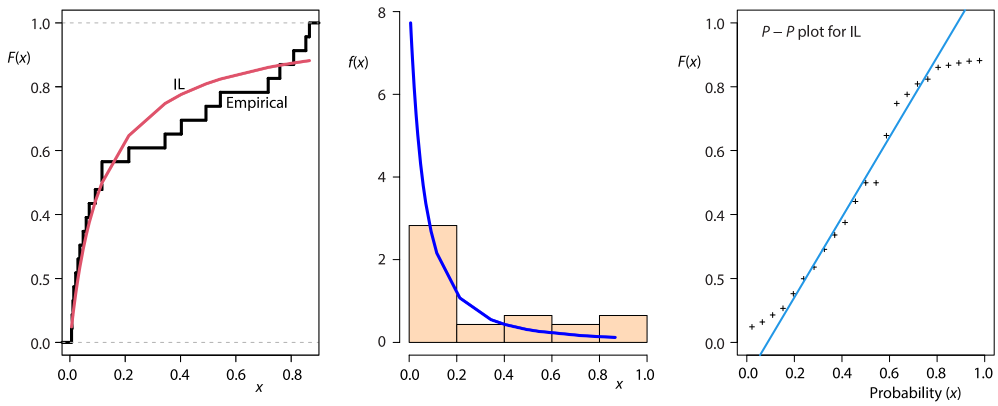

Figure 1. Estimated cdf, pdf and pp-plot for strength variable

Table 4. The KS test for the strength and stress variable for first data

	x	y
DKS	0.1389	0.1408
PVKS	0.7665	0.7761

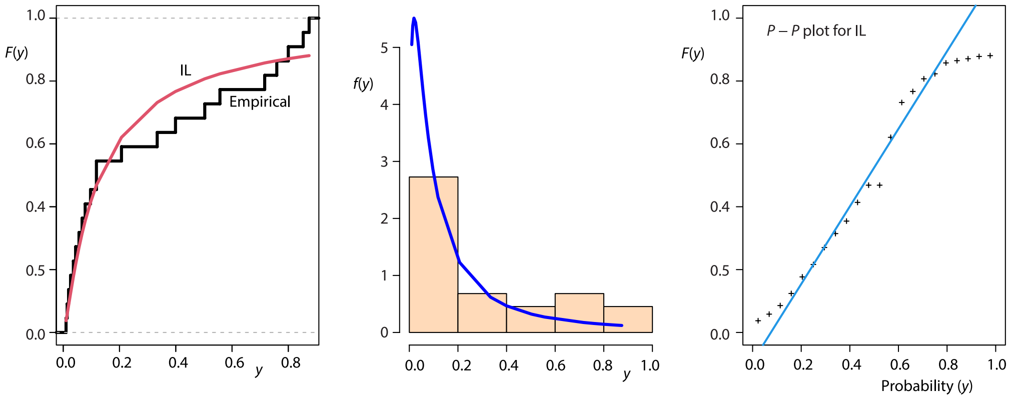

Figure 2. Estimated cdf, pdf and pp-plot for stress variable

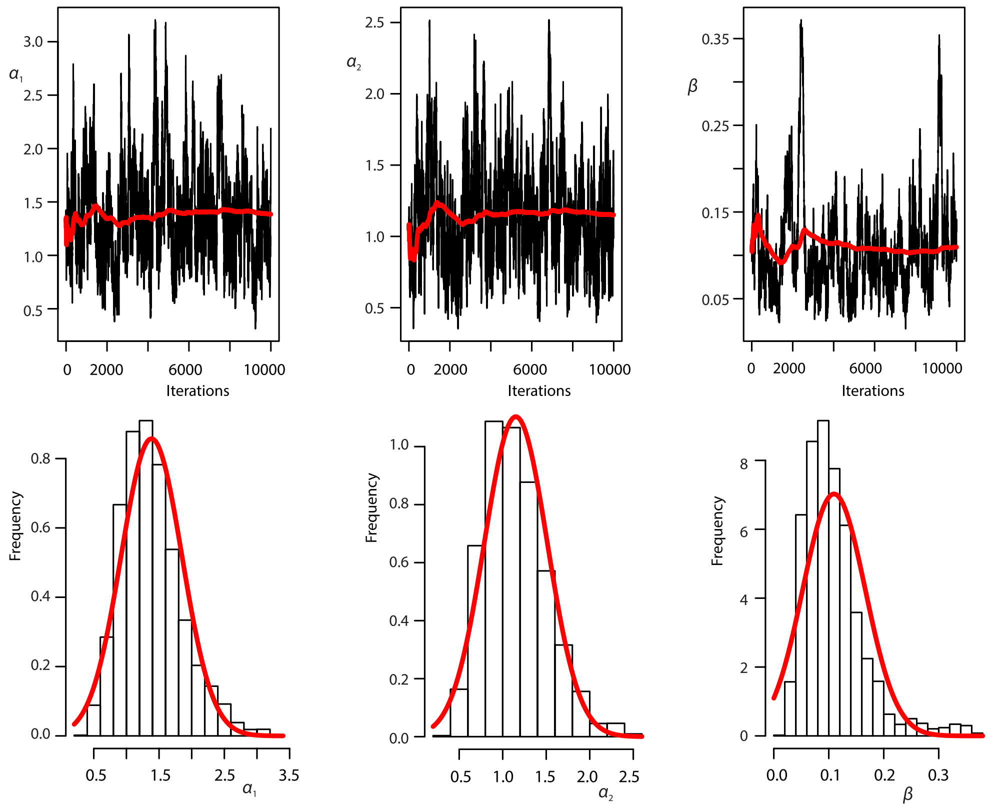

Figure 3. Iterations and convergence of MCMC results for first data of stress-strength model

Second real data

We use the successive failure times (in hours) of the air cooling system of jet jets, which were first reported by [40], for real-world application. We give two jet air-plane data sets for empirical analysis, each of which contains the following observations: x is (97, 51, 11, 4, 141, 18, 142, 68, 77, 80, 1, 16, 106, 206, 82, 54, 31, 216, 46, 111, 39, 63, 18, 191, 18, 163, 24) y is (90, 10, 60, 186, 61, 49, 14, 24, 56, 20, 79, 84, 44, 59, 29, 118, 25, 156, 310, 76, 26, 44, 23, 62, 130, 208, 70, 101, 208). This data used based on stress strength model of inverse Chen distribution by [35]. Using the KS test, we conclude that the IL distribution with parameter $\hat{\alpha} = 1.5152$ and $\hat{\beta} = 30.070$ can be fitted on strength variable, and $\hat{\alpha} = 536.02702$ and $\hat{\beta} = 0.07928$ can be fitted on stress variable, which are shown in tab. 6. Also Figures 5 and 6 confirmed fitting of theses data.

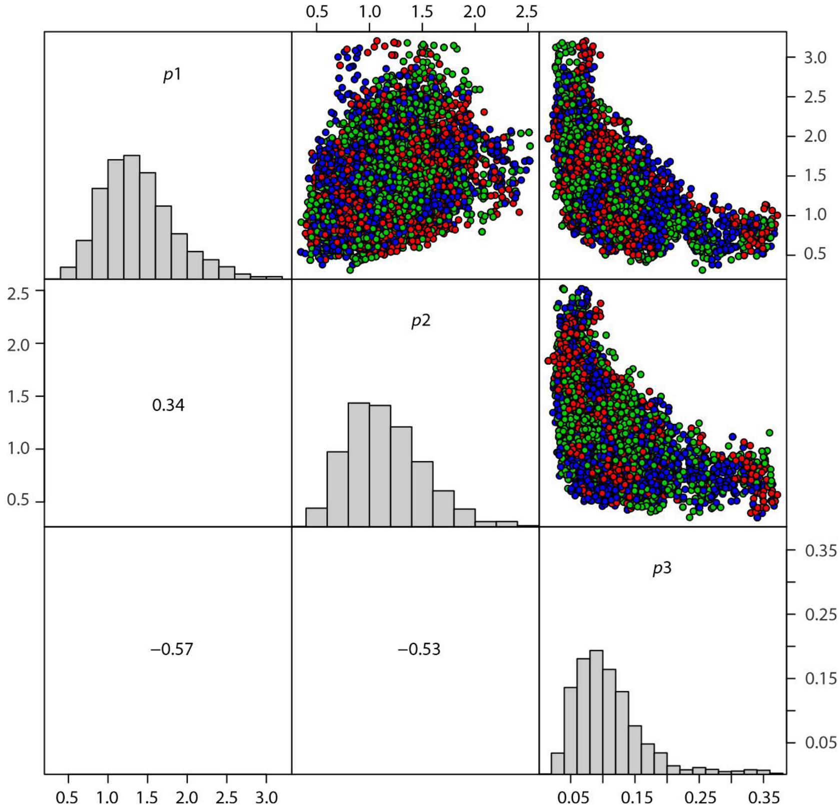

Figure 4. Scatter-plot matrices of MCMC results for first data of stress-strength model

Table 5. Classical and Bayesian estimation based on stress-strength model for first real data

	MLE		MPS		Bayeisna	
	Estimates	SE	Estimates	SE	Estimates	SE
α_1	1.3341	0.5609	1.0523	1.9566	1.3883	0.4736
α_2	1.1091	0.4289	0.8898	1.5421	1.1787	0.3791
β	0.0954	0.0598	0.1201	0.3413	0.1021	0.0443
R	0.5461		0.5418		0.5408	

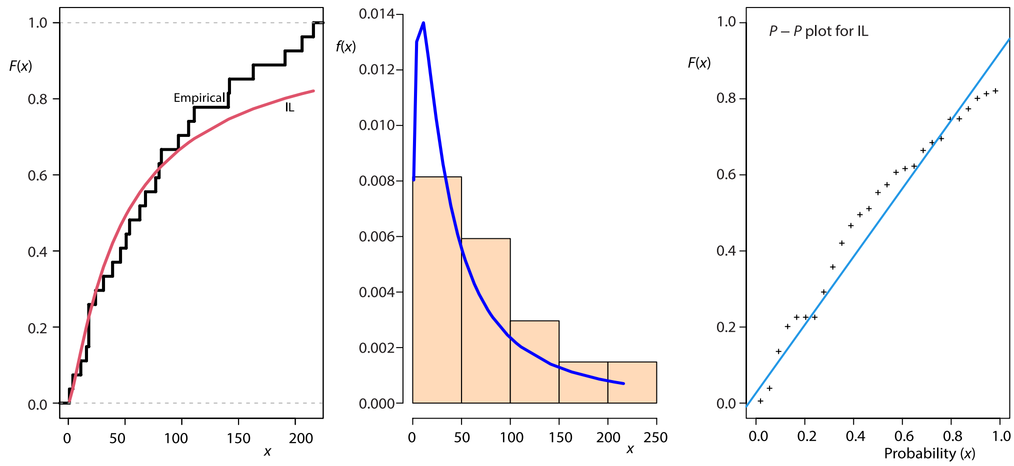

Figure 5. Estimated cdf, pdf and pp-plot for strength variable for second data

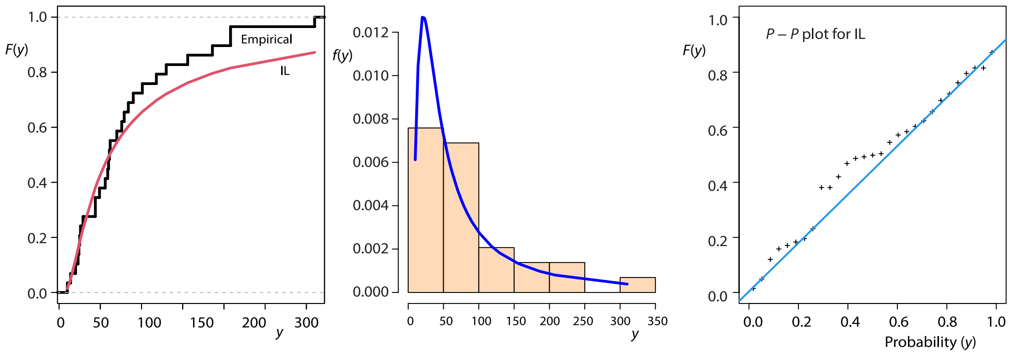

Figure 6. Estimated cdf, pdf and pp-plot for stress variable for second data

Table 6. The KS test for the strength and stress variable for second data

	x	y
DKS	0.1792	0.1503
PVKS	0.3509	0.5292

Table 7. Classical and Bayesian estimation based on stress-strength model for second real data

	MLE		MPS		Bayeisna	
	Estimates	SE	Estimates	SE	Estimates	SE
α_1	3.3111	1.5052	2.3896	3.7527	3.5419	1.1656
α_2	2.2060	0.8479	1.6119	2.2741	2.3442	0.6526
β	16.5056	8.4668	22.4831	6.8787	16.7764	5.9869
R	0.6001		0.5972		0.6017	

Figures 3 and 7 show trace and normal curve of posterior distribution for MCMC estimation for the first and the second data, respectively. Also theses figures confirmed convergence of MCMC results. Figures 8 and 4 show the MCMC samples as a pairs plot, with the scatter plot matrix in the top plot, correlation coefficients in the bottom plot, and marginal frequency of proposed distribution as normal distribution for each parameter on the diagonal. The parameters (P1 with P3) and (P2 with P3) have a negative correlation in this diagram, where P3 is the joint parameter β of the strength and stress variables.

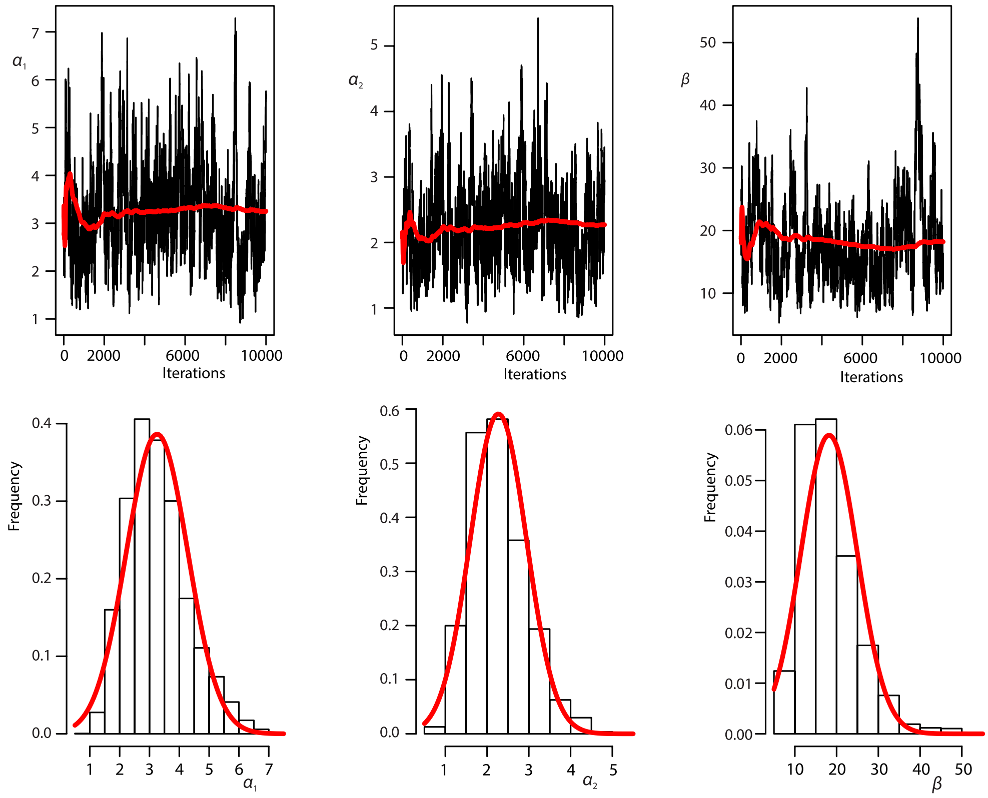

Figure 7. Iterations and convergence of MCMC results for second data of stress-strength model

We note the Bayesian estimation method have the smallest standard error (SE), then this the best estimation methods. Also noted as k increase then increases reliability stress-strength of IL model. The value of reliability stress-strength is larger in MPS than anther methods.

Conclusion

With X and Y being independent inverse Lomax random variables, the new method of estimating fuzzy stress-strength reliability $R_F = P(X > Y)$, is receiving a lot of attention due to the characteristics of R_F that make the analysis more sensitive and trustworthy. To present the point and interval estimations of all parameters as well as the fuzzy stress-strength reliability function R_F, the Bayesian techniques as well as the maximum likelihood and maximum product spacing are introduced. Additionally, the highest posterior density (HPD) intervals for the fuzzy reliability function and the unknown parameters of the inverse Lomax distribution are

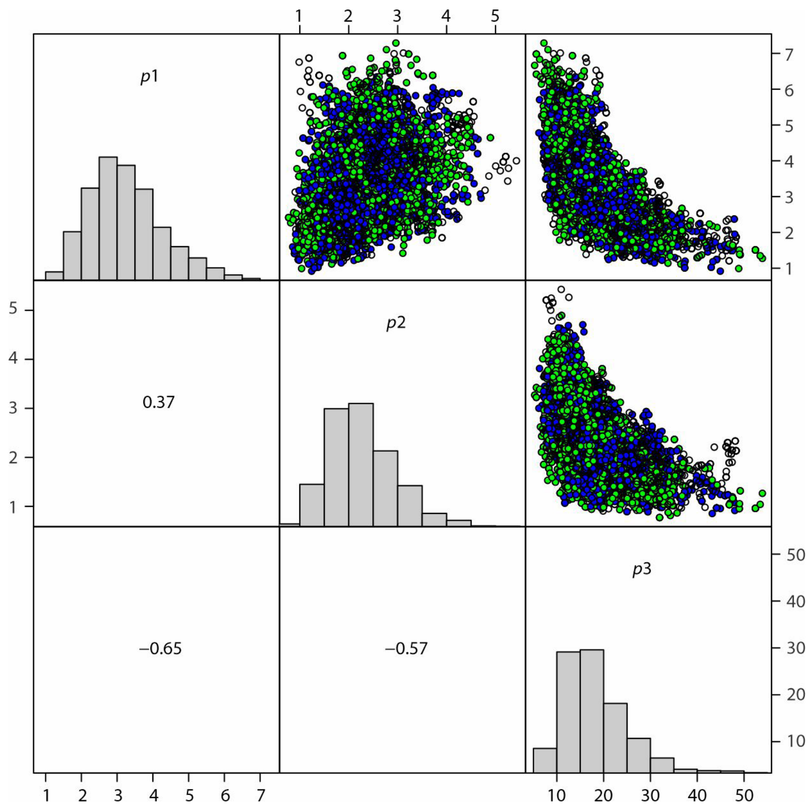

Figure 8. Scatter-plot matrices of MCMC results for second data of stress-strength model

examined, as well as the bootstraps *p* and *t* CI. Therefore, MCMC samples are produced from the posterior density function using the Metropolis-Hasting algorithm. To evaluate the effectiveness of the method used here, a simulated data set is used. The ML and Bayes estimates are supplied together with the appropriate CI lengths. Additionally, the the reliability stress-strength function was developed using two real data sets as well as a simulated data set.

References

[1] Bamber, D., The Area above the Ordinal Dominance Graph and the Area below the Receiver Operating Characteristic Graph, *Journal of Mathematical Psychology*, *12* (1975), 4, pp. 387-415

[2] Coolen, F. P. A., Newby, M. J., *A Note on the Use of the Product of Spacings in Bayesian Inference*, Department of Mathematics and Computing Science, University of Technology, Eindhoven, The Netherlands, 1990

[3] Raqab, M. Z., *et al.*, Estimation of $p\ (y < x)$ for the Three-Parameter Generalized Exponential Distribution, *Communications in Statistics – Theory and Methods*, *37* (2008), 18, pp. 2854-2864

[4] Wong, A., Interval Estimation of $p\ (y < x)$ for Generalized Pareto Distribution, *Journal of Statistical Planning and Inference*, *142* (2012), 2, pp. 601-607

[5] El-Sagheer, R. M., *et al.*, Inferences for Stress-Strength Reliability Model in the Presence of Partially Accelerated Life Test to Its Strength Variable, *Computational Intelligence and Neuroscience*, *2022* (2022), 4710536

[6] Asgharzadeh, A., *et al.*, Estimation of the Stress-Strength Reliability for the Generalized Logistic Distribution, *Statistical Methodology*, *15* (2013), Nov., pp. 73-94

[7] Akgül, F. G., Senoglu, B., Estimation of p ($x < y$) Using Ranked Set Sampling for the Weibull Distribution, *Quality Technology and Quantitative Management*, *14* (2017), 3, pp. 296-309
[8] Al-Omari, A. I, *et al.*, Estimation of the Stress-Strength Reliability for Exponentiated Pareto Distribution Using Median and Ranked set Sampling Methods, *Comput Mater Contin*, *64* (2020), 2, pp. 835-857
[9] Liu, Y., *et al.*, Reliability and Mean Time to Failure of Unrepairable Systems with Fuzzy Random Lifetimes, *IEEE Transactions on Fuzzy Systems*, *15* (2007), 5, pp. 1009-1026
[10] Mohamed, R. A. H., *et al*., Inference of Reliability Analysis for Type II Half Logistic Weibull Distribution with Application of Bladder Cancer, *Axioms*, *11* (2022), 386
[11] Huang, H.-Z., Reliability Analysis Method in the Presence of Fuzziness Attached to Operating Time, *Microelectronics Reliability*, *35* (1995), 12, pp. 1483-1487
[12] Huang, H.-Z., *et al.*, Bayesian Reliability Analysis for Fuzzy Lifetime Data, *Fuzzy Sets and Systems*, *157* (2006), 12, pp. 1674-1686
[13] Wu, H.-C., Fuzzy Reliability Estimation Using Bayesian Approach, Computers and Industrial Engineering, *46* (2004), 3, pp. 467-493
[14] Wu, H.-C., Fuzzy Bayesian System Reliability Assessment Based on Exponential Distribution, *Applied Mathematical Modelling*, *30* (2006), 6, pp. 509-530
[15] Buckley, J. J., *Fuzzy Probability and Statistic*s, Springer, Heibelberg, Germany, 2006
[16] Chen, G., Pham, T. T., *Introduction Fuzzy Sets, Fuzzy Logic, and Fuzzy Control Systems*, CRC Press, Boca Raton, Fla., USA, 2000
[17] Lee, I.-S., *et al*., Comparison of Interval Estimations for p ($x < y$) in Marshall-Olkin's Model, *Journal of the Korean Data and Information Science Society*, *7* (1996), 1, pp. 93-104
[18] Zardasht, V., *et al*., On Non-Parametric Estimation of a Reliability Function, *Munications in Statistics-Theory and Methods*, *41* (2012), 6, pp. 983-999
[19] Neamah, M. W., Ali, B. K., Fuzzy Reliability Estimation for Frechet Distribution by Using Simulation, *Periodicals of Engineering and Natural Sciences (PEN)*, *8* (2020), Jan., pp. 632-646
[20] Sabry, M. A. H., *et al*., Inference of Fuzzy Reliability Model for Inverse Rayleigh Distribution, *AIMS Mathematics*, *6* (2021), 9, pp. 9770-9785
[21] Kleiber, C., Kotz, S., *Statistical Size Distributions in Economics and Actuarial Sciences*, John Wiley and Sons, New York, USA, 2003, Vol. 70
[22] Ziegel, E. R., Statistical Size Distributions in Economics and Actuarial Sciences, *Technometrics*, *46* (2004), 4, pp. 499-500
[23] McKenzie, D., *et al*., *The Landscape Ecology of Fire*, Springer Science and Business Media, Berlin, Germany, 2011
[24] Greene, W., *et al*., *Estimating Econometric Models with Fixed Effects*, Department of Economics, Stern School of Business, New York University, New York, USA, 2001
[25] Washington, S., *et al*., *Statistical and Econometric Methods for Transportation Data Analysis*, Chapman and Hall/CRC, London, UK, 2020
[26] Almongy, H. M., *et al*., The Weibull Generalized Exponential Distribution with Censored Sample: Estimation and Application on Real Data, *Complexity*, *2021* (2021), ID6653534
[27] Muhammed, H. Z., Almetwally, E. M., Bayesian and Non-Bayesian Estimation for the Bivariate Inverse Weibull Distribution under Progressive Type-II Censoring, *Annals of Data Science*, *3* (2020), Nov., pp. 1-32
[28] Abd El-Raheem, A. M., *et al*., Accelerated Life Tests for Modified Kies Exponential Lifetime Distribution: Binomial Removal, Transformers Turn Insulation Application and Numerical Results, *AIMS Mathematics*, *6* (2021), 5, pp. 5222-5255
[29] Efron, B., Tibshirani, R. J., *An Introduction the Bootstrap*, CRC Press, Boca Raton, Fla., USA, 1994
[30] Hall, P., Theoretical Comparison of Bootstrap Confidence Intervals, *The Annals of Statistics*, *16* (1988), 3, pp. 927-953
[31] Tolba, A. H., *et al*., Bayesian Estimation of a One Parameter Akshaya Distribution with Progressively Type II Censord Data, *Journal of Statistics Applications and Probability An International Journa*, *11* (2022), 2, pp. 565-579
[32] Nassr, S. G., *et al*., Statistical Inference for the Extended Weibull Distribution Based on Adaptive Type-II Progressive Hybrid Censored Competing Risks Data, *Thailand Statistician*, *19* (2021), Jan., pp. 547-564
[33] Albert, J., *Bayesian Computation with R*., Springer Science and Business Media, Berlin Germany, 2009
[34] Abushal, T. A., *et al*., Estimation for Akshaya Failure Model with Competing Risks under Progressive Censoring Scheme with Analyzing of Thymic Lymphoma of Mice Application, *Complexity*, *2022* (2022), ID5151274

[35] Agiwal, V., Bayesian Estimation of Stress Strength Reliability from Inverse Chen Distribution with Application on Failure Time Data, *Annals of Data Science*, *3* (2021), Jan., pp. 1-31
[36] Chen, M.-H., Shao, Q.-M., Monte Carlo Estimation of Bayesian Credible and HPD Intervals, *Journal of Computational and Graphical Statistics*, *8* (1999), 1, pp. 69-92
[37] Caramanis, M., *et al*., Probabilistic Production Costing: An Investigation of Alternative Algorithms, *International Journal of Electrical Power and Energy Systems*, *5* (1983), 2, pp. 75-86
[38] Mazumdar, M., Gaver, D. P., On the Computation of Power-Generating System Reliability Indexes, *Technometrics*, *26* (1984), 2, pp. 173-185
[39] Khan, M. J. S., Arshad, M., Umvu Estimation of Reliability Function and Stress-Strength Reliability from Proportional Reversed Hazard Family Based on Lower Records, *American Journal of Mathematical and Management Sciences*, *35* (2016), 2, pp. 171-181
[40] Proschan, F., Theoretical Explanation of Observed Decreasing Failure Rate, *Technometrics*, *5* (1963), 3, pp. 375-383

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