BAYESIAN AND NON-BAYESIAN ESTIMATION METHODS TO INDEPENDENT COMPETING RISKS MODELS WITH TYPE II HALF LOGISTIC WEIBULL SUB-DISTRIBUTIONS WITH APPLICATION TO AN AUTOMATIC LIFE TEST

by

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In the survival data analysis, competing risks are commonly overlooked, and conventional statistical methods are used to analyze the event of interest. There may be more than one cause of death or failure in many experimental investigations of survival analysis. A competing risks model will be derived statistically applying Type-II half logistic weibull sub-distributions. Type-II half logistic weibull lifetimes failure model with independent causes. It is possible to estimate parameters and parametric functions using Bayesian and classical methods. A Bayes estimation is obtained by the Markov chain Monte-Carlo method. The posterior density function and the Metropolis-Hasting algorithm are used to calculate the Markov chain Monte-Carlo samples. Simulation data is used to evaluate the performance of the two methods according to the Type-II censored system. As a test of the discussed model, a real data set is provided.

Key words: *maximum likelihood estimator, Markov chain Monte-Carlo, competing risks models, Bayesian method, mean squared error*

Introduction

In time-to-event data, competing risks are often recognized, and regression analysis of such data has just received appropriate analytical advancements. Models for estimating the lifetimes of a certain risk have been produced in recent years, however, taking into account competing risk variables. The time of failure and the indicator variable that indicates the specific reason for the failure of an individual or an item are the data for competing risk models. Analysis of competing risk data in most cases assumes that failure has independent causes. Despite the fact that the basic concepts of model dependence may be more achievement, there is some attention on its feasibility [1, 2] and several other authors, such as [3-12], verified that

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the data could not be used to examine the hypotheses of independent failure times without information on covariance. See [13-21] of different applications of lifetime distributions in survival analysis. Competing risks and censoring are two important characteristics in competing risks analysis. The time elapsed between occurrence of an event and when it is over is denoted by T. In order to fully see this T, we must first censor it. When we know that an event has not occurred for some time C, but we can no longer follow the individual to measure T accurately, we observe $Z = \min(C, T)$ as a result of the right-censoring phenomenon.

In order to use time-to-event analytical techniques, we must make the key assumption that the censoring process is not informative. Therefore, knowing an individual's *censoring time* provides additional information about their future chance of experiencing a particular traumatic incident. In other words, the instant probability of being censored does not depend on the future event time.

Using biomedical engineering modelling of age data was proposed by [22]. Since the intensity function of the TIIHLW distribution has many-forms (right-skewed, symmetric, unimodel inverse, and *J*-shaped), so a variety of real datasets are analyzed correctly [22]. The method of maximum likelihood is used when the cause of the failure either known or unknown.

In this paper, we study and analyze the first type of data when there are $k \geq 2$ failure causes. It is necessary to analyze lifetime data for the purpose of the three parameters of TIIHLW distribution [22] since the hazard is likely to be increased or have the shape of a bathtub. Only the TIIHLW distribution allows a bathtub shaped hazard. Different parameters of these distributions are estimated using maximum likelihood and Bayesian methods under Type-II censoring data. On the other hand, we obtain asymptotically confident intervals and estimate failure risks.

Description model

The model describes as follows, it is assumed identical N and independent elements in a systems. Each element is allocated corresponding one of k, $k \geq 2$ different failure modes. Every subject is examined until it fails or time runs out. The item fails due to a single cause in the failure situation. The test will end when all objects fail, the censored times are reached, or a combination of the two. There will be two observable values when an item fails: T, denotes the object's lifetime, and $\delta \in \{1, 2,..., k\}$ denotes the cause of failure. We simply keep track of the censoring time in a censored condition. We use $\delta = 0$ for the censoring example to keep the notations simple. Furthermore, the following assumptions must be followed throughout the paper:

- The survival function is $S(.)$, $F(.)$ represents the cumulative distribution function, $f(.)$ denotes to the probability density function, and the object's lifetime is T_i, $i = 1, 2,..., N$.
- The T_{ji} represents the lifetime of failure that reason j ($j = 1, 2,..., k$), will object to i, $i = 1, 2,..., N$ at a random time. The T_{ji}, for $i = 1, 2,..., N$ are identically distributed random variables since the N objects on the life test are identical. We also assume that T_{ji}, for $i = 1, 2,..., N$, has the cumulative distribution relation $F_j(.)$ (also known as the sub-distribution form of cause j), the survival function $S_j(.)$, the probability density $f_j(.)$, and the hazard rate function $h_j(.)$, for $j = 1, 2,..., k$.

Using $f_j(t)$, $h_j(t)$, and $S_j(t)$, the survival function $S(t)$ is given:

$$S(t) = \prod_{j=1}^{k} S_j(t) \tag{1}$$

the density function $f(t)$ is constructed:

$$f(t)=\sum_{j=1}^{k} f_j(t)\prod_{\substack{l=1\\ l\neq j}}^{k} S_l(t)=\sum_{j=1}^{k} h_j(t)\prod_{l=1}^{k} S_l(t) \tag{2}$$

and the formulation of $h(t)$ is constructed:

$$h(t)=\sum_{j=1}^{k} h_j(t) \tag{3}$$

The sum of the probabilities of the various reasons is the instantaneous chance of failure.

– We let T_{ji} follows TIIHLW distributions with unknown parameters α_j, β_j, and λ_j, denoted by TIIHLW(α_j, β_j, λ_j), for i = 1, 2,..., N and j = 1, 2,..., k. That is, the cumulative distribution function is T_{ji}.

$$F(t;\alpha_j,\beta_j,\lambda_j)=\frac{2\left[1-\mathrm{e}^{-\alpha_j t^{\beta_j}}\right]^{\lambda_j}}{1+\left[1-\mathrm{e}^{-\alpha_j t^{\beta_j}}\right]^{\lambda_j}},\quad \alpha_j,\beta_j,\lambda_j>0 \tag{4}$$

the probability density is calculated:

$$f(t;\alpha_j,\beta_j,\lambda_j)=\frac{2\alpha_j\,\beta_j\,\lambda_j\,t^{\beta_j-1}\mathrm{e}^{-\alpha_j t^{\beta_j}}\left[1-\mathrm{e}^{-\alpha_j t^{\beta_j}}\right]^{\lambda_j-1}}{\left[1+\left[1-\mathrm{e}^{-\alpha_j t^{\beta_j}}\right]^{\lambda_j}\right]^2},\quad \alpha_j,\beta_j,\lambda_j>0 \tag{5}$$

the survival rate function:

$$S_j(t;\alpha_j,\beta_j,\lambda_j)=\frac{1-\left[1-\mathrm{e}^{-\alpha_j t^{\beta_j}}\right]^{\lambda_j}}{1+\left[1-\mathrm{e}^{-\alpha_j t^{\beta_j}}\right]^{\lambda_j}},\quad \alpha_j,\beta_j,\lambda_j>0 \tag{6}$$

and the $h_j(t)$ function can be written:

$$h_j(t)=\frac{2\lambda_j\alpha_j\beta_j t^{\beta_j-1}\mathrm{e}^{-\alpha_j t^{\beta_j}}\left(1-\mathrm{e}^{-\alpha_j t^{\beta_j}}\right)^{\lambda_j-1}}{\left[1-\left(1-\mathrm{e}^{-\alpha_j t^{\beta_j}}\right)\right]^{2\lambda_j}} \tag{7}$$

where λ_j and β_j are shape parameters and α_j is the scale parameter.

Maximum-likelihood estimation

More elements T_{ij} is considered independent and identically distributed, (iid) on the causes of failure $j = 1, 2,..., k$. The first n items are assumed to be failures, and the subsequent $(N - n)$ observations are censored, without losing generality. Whereas the first n observations include both failure observations and failure causes, the other $(N - n)$ items include only the censored times and no failure. It is possible to describe the available data (T_1, δ_1), (T_2, δ_2),..., (T_N, δ_N), where:

$$T_i = \begin{cases} T_{ij} & \text{if } \delta_i = j, \quad j \in \{1,2,...k\} \\ T_i & \text{if } \delta_i = 0\,(T_i \text{ is a censoring time}) \end{cases} \tag{8}$$

Then the likelihood function is written:

$$L(\omega\,|\,.) = \prod_{i=1}^{N}\left[\prod_{j=1}^{k}\left[f_j(t_i)\prod_{\substack{l=1\\l\neq j}}^{k}S_l(t_i)\right]^{I(\delta_i=j)}\quad[S(t_i)]^{I(\delta_i=0)}\right] \tag{9}$$

where ω is the vector of k: unknown parameters

$$\omega = (\alpha_1,\alpha_2,...,\alpha_k,\beta_1,\beta_2,...,\beta_k,\lambda_1,\lambda_2,...,\lambda_k)$$

The model assumptions and the well-known correlations between the reliability metrics are used to calculate the survival function, hazard rate function, and probability density function. The likelihood equation is given:

$$L(\omega\,|\,.) = \left[\prod_{i=1}^{N}\prod_{j=1}^{k}\left[h_j(t_i)\right]^{I(\delta_i=j)}\right]\left[\prod_{i=1}^{N}\prod_{j=1}^{k}S_j(t_i)\right] \tag{10}$$

As a result of calculating the log-likelihood function for both sides of eq. (10), we obtain:

$$\ell = \sum_{i=1}^{N}\sum_{j=1}^{K}\left[I(\delta_i = j)\left(\log h_j(t_i)\right) + \log S_j(t_i)\right] \tag{11}$$

where $I(A)$ represents an indicator function, where $I(A) = 1$ for A is true and 0, otherwise. Equations (6) and (7) can be substituted for eqs. (10) and (11) to obtain for competing risks model with unknown cause parameters for a TIIHLW distribution, construct the likelihood and log-likelihood functions are written:

$$L(\omega\,|\,.) = \left[\prod_{i=1}^{N}\prod_{j=1}^{k}\left(\frac{2\alpha_j\,\beta_j\,\lambda_j\,t_i^{\beta_j-1}\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\left[1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right]^{\lambda_j-1}}{1-\left[1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right]^{2\lambda_j}}\right)^{I(\delta_i=j)}\right]\left[\prod_{i=1}^{N}\prod_{j=1}^{k}\frac{1-\left[1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right]^{\lambda_j}}{1+\left[1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right]^{\lambda_j}}\right] \tag{12}$$

and

$$\ell=\sum_{i=1}^{N}\sum_{j=1}^{K}\left[\begin{array}{c}2I(\delta_i=j)\left(\begin{array}{c}\log\alpha_j+\log\beta_j+\log\lambda_j+(\beta_j-1)\log t_i-\alpha_j t_i^{\beta_j}+\\+(\lambda_j-1)\log\left(1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right)-\log\left(1-\left[1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right]^{2\lambda_j}\right)\end{array}\right)+\\+\log\left(1-\left[1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right]^{\lambda_j}\right)-\log\left(1+\left[1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right]^{\lambda_j}\right)\end{array}\right] \tag{13}$$

The set of values for components $\hat{\omega}$ maximizes the log-likelihood function is known as the maximum likelihood point estimate.

The derivatives for ℓ in order to both of α_ℓ, β_ℓ, and λ_ℓ, $\ell = 1, 2,..., k$ are:

$$\frac{\partial\ell}{\partial\alpha_l}=\sum_{i=1}^{N}\sum_{j=1}^{K}\delta_{jl}\left[\begin{array}{c}2I(\delta_i=j)\left(\frac{1}{\alpha_j}-t_i^{\beta_j}+\frac{(\lambda_j-1)t_i^{\beta_j}\mathrm{e}^{-\alpha_j t_i^{\beta_j}}}{1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}}+\frac{2\lambda_j t_i^{\beta_j}\mathrm{e}^{-\alpha_j t_i^{\beta_j}}}{1-\left[1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right]^{2\lambda_j}}\right)-\\-\frac{\lambda_j t_i^{\beta_j}\mathrm{e}^{-\alpha_j t_i^{\beta_j}}}{1-\left[1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right]^{\lambda_j}}-\frac{\lambda_j t_i^{\beta_j}\mathrm{e}^{-\alpha_j t_i^{\beta_j}}}{1+\left[1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right]^{\lambda_j}}\end{array}\right] \tag{14}$$

$$\frac{\partial\ell}{\partial\beta_l}=\sum_{i=1}^{N}\sum_{j=1}^{K}\delta_{jl}\left[\begin{array}{c}2I(\delta_i=j)\left(\begin{array}{c}\frac{1}{\beta_j}+\log t_i-\alpha_j t_i^{\beta_j}\log t_i+\frac{(\lambda_j-1)\alpha_j t_i^{\beta_j}\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\log t_i}{\left(1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right)}+\\+\frac{2\lambda_j\alpha_j t_i^{\beta_j}\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\log t_i\left[1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right]^{2\lambda_j-1}}{1-\left[1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right]^{2\lambda_j}}+\end{array}\right)-\\-\frac{\lambda_j\alpha_j t_i^{\beta_j}\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\log t_i\left[1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right]^{\lambda_j-1}}{1-\left[1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right]^{\lambda_j}}-\frac{\lambda_j\alpha_j t_i^{\beta_j}\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\log t_i\left[1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right]^{\lambda_j-1}}{1+\left[1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right]^{\lambda_j}}\end{array}\right] \tag{15}$$

and

$$\frac{\partial \ell}{\partial \lambda_l}=\sum_{i=1}^{N}\sum_{j=1}^{K}\delta_{jl}\left[\begin{array}{c} 2I(\delta_i=j)\left(\dfrac{1}{\lambda_j}+\log\left(1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right)-\dfrac{2\lambda_j\left[1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right]^{2\lambda_j-1}\log\left[1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right]}{1-\left[1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right]^{2\lambda_j}}\right)- \\ -\dfrac{\lambda_j\left[1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right]^{\lambda_j-1}\log\left[1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right]}{1-\left[1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right]^{\lambda_j}}-\dfrac{\lambda_j\left[1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right]^{\lambda_j-1}\log\left[1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right]}{1+\left[1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right]^{\lambda_j}} \end{array}\right] \quad (16)$$

The Koronecker delta is expressed by the constant δ_{ij}, $l, j = 1, 2, ..., k$. Set

$$\frac{\partial \ell}{\partial \alpha_l}=0,\ \frac{\partial \ell}{\partial \beta_\ell},\ \text{and}\ \frac{\partial \ell}{\partial \lambda_l}=0,\ l=1,2,...,k$$

The system of k equations defined for the parameters α_j, β_j, and λ_j, $j = 1, 2,..., k$ are solved in order to obtain maximum-likelihood point estimates for the parameters. To estimate the parameters, the Newton-Raphson method is used, while the obtained system is a non-closed form, and therefore, does not have an analytic solution. To get the information matrix of ℓ, we require the second partial derivative with respect to α_j, β_j, and λ, which are

$$\frac{\partial^2 \ell}{\partial \alpha_l \partial \alpha_m},\ \frac{\partial^2 \ell}{\partial \beta_l \partial \beta m},\ \text{and}\ \frac{\partial^2 \ell}{\partial \lambda_l \partial \lambda m},\ \text{with}\ m,\ l=1,2,3,...,k.$$

Confidence intervals

As a result, it is impossible to obtain the parameter maximum likelihood estimators in analytic form and determine their real distributions. The asymptotic distribution for the parameters $\omega = (\alpha_1,..., \alpha_k, \beta_1,..., \beta_k, \lambda_1,..., \lambda_k)$ is constructed by using the maximum likelihood estimators. the CI for $\omega = (\alpha_1,..., \alpha_k, \beta_1,..., \beta_k, \lambda_1,..., \lambda_k)$.

It is well known that:

$$\hat{\omega}\to N_{3k}\left[\omega, I^{-1}(\hat{\omega})\right] \quad (17)$$

where N_{3k} represents $3k$-multidimensional the normal distribution and the covariance matrix, $I^{-1}(\hat{\omega})$,which consists of the inverse of the information matrix for $(\hat{\omega})$. Using the second partial derivative the information matrix for ℓ is obtained by estimating the maximum likelihood point with respect to the unknown parameters:

$$I_{ij}(\omega)=-E\left[\frac{\partial^2 \ell(\omega)}{\partial \omega_i \partial \omega_j}\right],\ i,j-1,2,...,k \quad (18)$$

and $\omega = (\alpha_1,..., \alpha_k, \beta_1,..., \beta_k, \lambda_1,..., \lambda_k)$. Thus, $100(1 - \gamma)\%$ two sided CI for ω is constructed:

$$\hat{\omega}\pm Z_{\gamma/2}\sqrt{I^{-1}(\hat{\omega})} \quad (19)$$

with $Z_{\gamma/2}$ represents the upper $\gamma^{\text{th}}/2$ percentile of a standard normal distribution.

Bootstrap CI

The [23, 24] are used to create the percentile bootstrap (Boot-p) and bootstrap -t confidence intervals for the TIIHLWD unknown parameters α_j, β_j, and λ_j. For more information of bootstrap CI for competing risks model see [25-28].

Boot-p algorithm

The steps for the Boot-p algorithm are written as:

Step 1. Generate a sample with replacement the TIIHLW(α_j, β_j, and λ_j) distribution and compute the estimate $\omega_i = (\alpha_{1i},..., \alpha_{ki}, \beta_{1i},..., \beta_{ki}, \lambda_{1i},..., \lambda_{ki})$. Next, obtain the sample X using ω_i and then compute ω_i.

Step 2. Repeat Step 1, B times.

Step 3. Let $\hat{\omega}_{\text{Boot}} - p(x) = \hat{F}_1^{-1}(x)$, with $\hat{F}_1(x) = \text{P}(\hat{\omega}_i^* \leq \text{x})$ indicates to the CDF of $\hat{\omega}_i^*$. Then, the 100(1 – p)% CI for ω_i is obtained:

$$\left[\hat{\omega}_{i_{Boot}-p}\left(\frac{P}{2}\right), \hat{\omega}_{i_{Boot}-p}\left(1-\frac{P}{2}\right)\right] \tag{20}$$

Boot-t algorithm

Step 1. Same as the Boot-*p* algorithm.

Step 2. Compute the following statistic.

$$T^* = \frac{\hat{\omega}_i^* - \hat{\omega}_i}{\sqrt{\hat{V}ar(\hat{\omega}_i^*)}}$$

Step 3. Repeat Step 2, B times.

Step 4. Now, let

$$\hat{\omega}_{i_{Boot}-t}(x) = \hat{\omega}_i + \sqrt{\hat{V}ar(\hat{\omega}_i^*)}\hat{F}_2^{-1}(x), \text{ where } \hat{F}_2(x) = P(\hat{T}^* \leq x)$$

denotes the CDF of T^*. The approximate 100(1 – p)% CI for ω_i, i = 1, 2,..., N_{Boot} is then given:

$$\left[\hat{\omega}_{i_{\text{Boot}}-t}\left(\frac{P}{2}\right), \hat{\omega}_{i_{\text{Boot}}-t}\left(1-\frac{P}{2}\right)\right] \tag{21}$$

Competing risks under Type II censored data

It is assumed that the N independent elements undergo a lifetime test, with k taken into account prior the experiment, [29]. The initial failure, $T_{1:k}$, as well as the cause of the failure, δ_1, are both recorded. The second failure $T_{2:k}$ was also recorded, as well as the explanation for failure δ_2. The experiment is repeated until the k^{th} failure, $T_{k:k}$ and its cause δ_1 are observed, along with its cause δ_1. Because it contains the following variables, it is referred to Type-II competing risk data: $(T_{1:k}, \delta_1) < (T_{2:k}, \delta_2) < ...< (T_{k:k}, \delta_k)$. Under the Type-II competing risk sample, the joint likelihood function: $t = \{(T_{1:k}, \delta_1), T_{2:k}, \delta_2),...,(T_{k:k}, \delta_k)\}$is given:

$$f(t) = \frac{n!}{(n-m)!}\left[S_1(t_k)S_2(t_k)\right]^{n-m}\prod_{i=1}^{m}\left[f_1(t_i)S_2(t_i)\right]^{I(\delta_i=1)}\left[f_2(t_i)S_1(t_i)\right]^{I(\delta_i=2)} \tag{22}$$

where S(.) is the survival function and

$$I(\delta_i = j) = \begin{cases} 1 & ,\delta_i = j \\ 0 & ,\delta_i \neq j \end{cases}, j = 1,2 \text{ for } 0 < t_1 < \ldots < t_m < \infty$$

Let the causes follow the TIIHLW distribution with unknown parameters, we obtain the likelihood function for the Type II censored scheme under competing risk:

$$L(\alpha_1,\alpha_2,\beta,\lambda)=\frac{n!}{(n-m)!}\left(\frac{1-\left[1-\mathrm{e}^{-\alpha_1 t_k^{\beta}}\right]^{\lambda}}{1+\left[1-\mathrm{e}^{-\alpha_1 t_k^{\beta}}\right]^{\lambda}}\frac{1-\left[1-\mathrm{e}^{-\alpha_2 t_k^{\beta}}\right]^{\lambda}}{1+\left[1-\mathrm{e}^{-\alpha_2 t_k^{\beta}}\right]^{\lambda}}\right)^{n-m}\cdot$$

$$\cdot\prod_{k=1}^{m}\left[\frac{2\alpha_1\,\beta\,\lambda\,t_k^{\beta-1}\mathrm{e}^{-\alpha_1 t_k^{\beta}}\left[1-\mathrm{e}^{-\alpha_1 t_k^{\beta}}\right]^{\lambda-1}\left(1-\left[1-\mathrm{e}^{-\alpha_2 t_k^{\beta}}\right]^{\lambda}\right)}{\left[1+\left[1-\mathrm{e}^{-\alpha_1 t_k^{\beta}}\right]^{\lambda}\right]^2\left(1+\left[1-\mathrm{e}^{-\alpha_2 t_k^{\beta}}\right]^{\lambda}\right)}\right]^{I(\delta_i=1)}\cdot$$

$$\cdot\left[\frac{2\alpha_2\,\beta\,\lambda\,t_k^{\beta-1}\mathrm{e}^{-\alpha_2 t_k^{\beta}}\left[1-\mathrm{e}^{-\alpha_2 t_k^{\beta}}\right]^{\lambda-1}\left(1-\left[1-\mathrm{e}^{-\alpha_1 t_k^{\beta}}\right]^{\lambda}\right)}{\left[1+\left[1-\mathrm{e}^{-\alpha_2 t_k^{\beta}}\right]^{\lambda}\right]^2\left(1+\left[1-\mathrm{e}^{-\alpha_1 t_k^{\beta}}\right]^{\lambda}\right)}\right]^{I(\delta_i=2)} \tag{23}$$

Bayesian estimation

Due to the computational progress in the past few years, Bayesian estimation is one of the most important statistical methods for estimating parameters of parametric survival models. In time-event analysis, the setting of competing risks can be used whenever the presence of censored observations has to be taken into account. To evaluate the unknown parameters of Gompertz distribution in a competing risk model with a very general censoring scheme, Bakoban and Abd-Elmougod [30] employed Bayesian estimation based on the MCMC method. Almarashi *et al.* [29] applies the MCMC algorithm to the Nadarajaha and Haghighi distribution based on type-II competing risk data. Sarhan *et al*. [12] discussed the maximum likelihood method and Bayesian method to estimate the parameters of the lifetime Weibull sub-distribution. Bantan *et al*. [31] discussed the Bayesian analysis of partially accelerated life tests for weighted Lomax distributions. The Bayesian estimation for modified Kies exponential lifetime distribution under accelerated life tests was presented in [32].

In this section, we consider Bayesian inference of the unknown parameters α_j, β_j, and λ_j of the TIIHLWD. It is assumed that α_j, β_j, and λ_j have the independent gamma prior distributions with shape parameter a_{j1}, b_{j1}, and c_{j1} and scale parameter a_{j1}, b_{j1}, and c_{j1}, respectively.

Thus, the joint prior density of $\omega = (\alpha_1,..., \alpha_k, \beta_1,..., \beta_k, \lambda_1,..., \lambda_k)$, up to *a* constant is:

$$g(\omega)\propto\prod_{j=1}^{k}\alpha_j^{a_{j1}-1}\mathrm{e}^{-a_{j2}\alpha_j}\,\beta_j^{b_{j1}-1}\mathrm{e}^{-b_{j2}\beta_j}\,\lambda_j^{c_{j1}-1}\mathrm{e}^{-c_{j2}\lambda_j},\quad \alpha_j>0,\ \beta_j>0,\ \lambda_j>0 \tag{24}$$

In order to generate the joint posterior density function ω, we combined the joint prior density eq. (19) and the likelihood function (12) and applied the Bayes' theorem:

$$g(\omega|.)\propto\left[\prod_{i=1}^{N}\prod_{j=1}^{k}\left(\frac{2\alpha_j\,\beta_j\,\lambda_j\,t_i^{\beta_j-1}\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\left[1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right]^{\lambda_j-1}}{1-\left[1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right]^{2\lambda_j}}\right)^{I(\delta_i=j)}\right]$$

$$g(\omega|.)\propto\left[\prod_{i=1}^{N}\prod_{j=1}^{k}\left(\frac{2\alpha_j\,\beta_j\,\lambda_j\,t_i^{\beta_j-1}\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\left[1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right]^{\lambda_j-1}}{1-\left[1-\mathrm{e}^{-\alpha_j t_i^{\beta_j}}\right]^{2\lambda_j}}\right)^{I(\delta_i=j)}\right] \tag{25}$$

where

$$n_j=\sum_{i=1}^{N}I(\delta_i=j)$$

The posterior mean of any function of the vector of unknown parameters ω, say $\nu(\omega)$, is the Bayesian estimate of that function under the quadratic loss function:

$$\hat{\nu}=E_{\omega|.}(\nu(\omega))=\int_0^\infty \nu(\omega)\,g(\omega|.)\,\mathrm{d}\omega \tag{26}$$

Analytical solutions do not exist for the integral in eq. (21) and the normalizing constant. As a result, Bayesian analysis of the underlying model should be performed using numerical methods. A Markov chain Monte-Carlo (MCMC) simulation technique will be used for the analysis among other techniques. To obtain random raws from a joint posterior distribution in eq. (20), the MCMC algorithm can be used, without having to calculate a normalised constant. A model parameter or a model characteristic study can be performed using random draws.

Markov chain Monte-Carlo method

Techniques such as MCMC have been proven successful in modern Bayesian statistical analysis. The MCMC is a method of summarising posterior distributions that does not require calculation of the normalised constants. According to [1], the Bayesian statistical inference process highly depends on MCMC techniques. This modified version of the MCMC algorithm is called the Metropolis Hastings sampler. There are two requirements for an effective MCMC proposal, it must be simple to replicate and must closely resemble the desired posterior distribution function. By using acceptance-rejection rule, we choose our target posterior distribution at random. Random draws from the posterior distribution $g(\omega)$ are simulated using the Metropolis-Hastings algorithm:

- Suppose the initial values $\omega^{(0)}$.
- Set a limit on the number of trails that obtained for random drawings, say M.
- Repeat the following steps for $i = 1,..., M$.
- Set $\omega = \omega(i-1)$.
- A proposal distribution $P(\omega^*|\omega)$ can generate a candidate ω^*.

– Calculate the acceptance probability

$$\eta_\omega = \min\left[1, \frac{g(\omega^* \mid .)\, P(\omega^* \mid \omega)}{g(\omega^* \mid .)\, P(\omega^* \mid \omega)}\right]$$

– Generate a u_1 from a Uniform (0, 1) distribution. If $u_1 < \eta_\omega$, accept the proposal and set $\omega^{(i)} = \omega^*$, otherwise $\omega^{(i)} = \omega^{(i-1)}$.

The relative risks

The failure probability relation of each cause of failure in the presence of all others will be explored, as well as the risk associated with each cause of failure relation. In the existence of all causes, the failure relation of cause j at time x is defined [7, 8]:

$$F_{C_j}(x) = \int_0^x h_j(e) \prod_{l=1}^{k} S_l(e)\,\mathrm{d}e \tag{27}$$

The risk cause is defined:

$$\pi_j = \lim_{x\to\infty} F_{C_j}(x) = \int_0^x h_j(x) \prod_{l=1}^{k} S_l(x)\,\mathrm{d}x \tag{28}$$

The reasons for failure follow the TIIHLW distribution, π_j can be derived by calculating the integral by substituting eqs. (5) and (6) into eq. (28):

$$\pi_j = \int_0^\infty \frac{2\alpha_j\,\beta_j\,\lambda_j\, x^{\beta_j - 1} \mathrm{e}^{-\alpha_j x^{\beta_j}} \left[1 - \mathrm{e}^{-\alpha_j x^{\beta_j}}\right]^{\lambda_j - 1}}{1 - \left[1 - \mathrm{e}^{-\alpha_j x^{\beta_j}}\right]^{2\lambda_j}} \prod_{l=1}^{k} \frac{1 - \left[1 - \mathrm{e}^{-\alpha_l x^{\beta_l}}\right]^{\lambda_l}}{1 + \left[1 - \mathrm{e}^{-\alpha_l x^{\beta_l}}\right]^{\lambda_l}}\,\mathrm{d}x \tag{29}$$

When the integrated function in eq. (29) is analyzed at the maximum likelihood estimates of the parameters, the maximum likelihood estimate of eq. (29) can be determined using numerical integration using the invariant property. We will combine the random drawings from the joint probability relation with the integral in eq. (29) Bayesian analysis to obtain random draws from the posterior distribution of π_j, which we can then use in Bayesian analysis for π_j, $j = 1, 2,..., k$.

Simulation study and comparisons

The MSE and CI for the bias estimators of MLE and Bayesian estimation methods which considered in this paper are not obtain in closed form. For this reason, we will conduct a simulation study in which assess all of the aforementioned estimators for α_j, β_j, and λ_j; $j = 1, 2$ of the TIIHLW distribution under competing risks with Type-II censoring. The simulation study has the following inputs.

The number of iterations is 10000 iteration, and the number of sample obtained by bootstrap is 1000 sample.

The generated samples from TIIHLW distribution under competing risks with Type-II censored are calculated by using different sample sizes, n, as 50, 100, and 200. The number of observed failure times from Type-II censored are calculated by using different ratio of sample size, r, as 60% and 90%.

Different exact values of parameter of the TIIHLW distribution are determined as:

$$\text{Case } 1: \alpha_1 = 0.25,\ \beta_1 = 0.25,\ \lambda_1 = 0.4,\ \alpha_2 = 0.35,\ \beta_2 = 0.3,\ \lambda_2 = 0.45$$

$$\text{Case } 2: \alpha_1 = 1.7, \beta_1 = 0.6, \lambda_1 = 1.6, \alpha_2 = 1.9, \beta_2 = 0.45, \lambda_2 = 1.45$$

Equation (29) is used to produce MLE estimators of TIIHLW distribution under competing risks with Type-II censored. The Newton-Raphson algorithm can be used to discover the best MLE solution. Moreover, Bayesian estimates of α_j, β_j, and λ_j; $j = 1, 2$ will be derived by using the gamma distribution with scale and shape parameters as the prior of the hyper-parameter. As detailed in [33], the hyper-parameters of the informative priors are obtained using the same technique. The 95% confidence interval (CI) length for each parameter is calculated using the asymptotic confidence interval (ACI), the credible confidence interval (CCI), and the bootstrap confidence intervals (*Bt* and *Bp*).

Table 1. Point and interval estimation method for parameters of the TIIHLW distribution under competing risks based on Type II censored sample: Case 1, *r* = 0.6

Case 1		Point estimation				CI		Bootstrap			
$r = 0.6$		MLE		Bayes		MLE	Bayes	MLE		Bayes	
n		Bias	MES	Bias	MES	ACI	PCI	Bt	Bp	Bt	Bp
50	λ_1	0.4535	0.5189	0.4639	0.2488	2.1949	2.0485	0.0679	0.0723	0.0619	0.0705
	α_1	–0.2003	0.0534	–0.1916	0.0381	0.4513	0.0875	0.0143	0.0139	0.0028	0.0029
	β_1	0.0371	0.0195	0.0486	0.0191	0.5283	0.1952	0.0167	0.0159	0.0069	0.0067
	λ_2	0.2645	0.0958	0.2218	0.0823	0.6306	0.6081	0.0199	0.0199	0.0207	0.0194
	α_2	–0.2272	0.0544	–0.1396	0.0263	0.2064	0.0669	0.0066	0.0066	0.0021	0.0021
	β_2	0.1319	0.0725	0.0790	0.0324	0.9209	0.8236	0.0307	0.0307	0.0260	0.0263
100	λ_1	0.3153	0.2656	0.4128	0.1996	1.5990	1.5802	0.0540	0.0502	0.1222	0.1198
	α_1	–0.2233	0.0561	–0.1810	0.0348	0.3093	0.0551	0.0095	0.0097	0.0017	0.0018
	β_1	0.0546	0.0155	0.0753	0.0122	0.4391	0.0945	0.0134	0.0140	0.0031	0.0030
	λ_2	0.2594	0.0900	0.3903	0.0877	0.5912	0.5905	0.0181	0.0181	0.0192	0.0181
	α_2	-0.2307	0.0558	-0.1264	0.0232	0.2008	0.0623	0.0066	0.0066	0.0020	0.0020
	β_2	0.0579	0.0324	0.0439	0.0170	0.6689	0.4558	0.0215	0.0215	0.0147	0.0145
200	λ_1	0.1756	0.0828	0.3540	0.0749	0.8940	0.8152	0.0285	0.0274	0.0808	0.0672
	α_1	-0.2425	0.0604	-0.2208	0.0325	0.1568	0.0259	0.0055	0.0048	0.0008	0.0008
	β_1	0.0812	0.0144	0.0685	0.0116	0.3473	0.0683	0.0110	0.0114	0.0022	0.0020
	λ_2	0.2352	0.0642	0.2355	0.0637	0.3691	0.2578	0.0120	0.0120	0.0085	0.0082
	α_2	-0.2412	0.0590	-0.1833	0.0377	0.1087	0.0464	0.0036	0.0036	0.0015	0.0015
	β_2	0.0052	0.0113	0.0258	0.0129	0.4170	0.2604	0.0132	0.0132	0.0098	0.0081

From tabs. 1-4 we can conclude the following:

- Based on Type II censored samples, the TIIHLW probability has a decreasing bias, MSE, and CI length as the sample size increases.
- Bias, MSE, and L.CI values for the parameters of the TIIHLW distribution under competing risks decrease for the number of observed failure times increases as r increases based on Type II censored sample.
- Based on Type II censored samples, Bayesian estimates have a significantly increased efficiency than MLE for most parameters of TIIHLW distribution.
- The length of CCI is smaller than the length of ACI.
- The length of bootstrap-t CI is smaller than the length of bootstrap-*p*.

Table 2. Point and interval estimation method for parameters of the TIIHLW distribution under competing risks based on Type II censored sample: Case 1, r = 0.9

Case 1		Point estimation				CI		Bootstrap			
r = 0.9		MLE		Bayes		MLE	Bayes	MLE		Bayes	
n		Bias	MES	Bias	MES	ACI	PCI	Bt	Bp	Bt	Bp
50	λ_1	1.0576	0.4668	0.4854	0.2690	2.9071	1.5285	0.0875	0.0962	0.0846	0.0843
	α_1	0.0042	0.0524	–0.0136	0.0229	0.8978	0.5729	0.0289	0.0287	0.0196	0.0182
	β_1	0.0244	0.0173	0.0243	0.0159	0.6501	0.2820	0.0202	0.0220	0.0090	0.0088
	λ_2	0.2158	0.0922	0.1398	0.0921	0.9818	0.8499	0.0324	0.0324	0.0609	0.0602
	α_2	–0.1210	0.0356	0.0264	0.0228	0.5673	0.2846	0.0190	0.0190	0.0108	0.0091
	β_2	0.4281	0.0702	0.2412	0.0694	0.9056	0.9121	0.0280	0.0280	0.0286	0.0278
100	λ_1	0.9775	0.2588	0.4883	0.2469	2.2617	1.4345	0.0684	0.0730	0.0623	0.0623
	α_1	–0.0238	0.0351	–0.0152	0.0261	0.7285	0.3001	0.0224	0.0220	0.0100	0.0094
	β_1	0.0108	0.0148	0.0214	0.0137	0.5242	0.1521	0.0165	0.0165	0.0048	0.0050
	λ_2	0.1804	0.0672	0.1409	0.0621	0.7303	0.9859	0.0238	0.0238	0.0393	0.0362
	α_2	–0.1516	0.0342	0.0116	0.0175	0.4153	0.1648	0.0135	0.0135	0.0060	0.0054
	β_2	0.4141	0.0281	0.1942	0.0623	0.7220	0.6827	0.0236	0.0236	0.0209	0.0219
200	λ_1	0.8811	0.0942	0.4531	0.0823	1.5975	1.0990	0.0501	0.0539	0.0413	0.0430
	α_1	–0.0573	0.0226	–0.0517	0.0231	0.5449	0.1441	0.0169	0.0180	0.0047	0.0043
	β_1	0.0114	0.0136	0.0192	0.0126	0.4545	0.1163	0.0146	0.0146	0.0040	0.0036
	λ_2	0.1528	0.0400	0.1142	0.0322	0.5065	0.6279	0.0155	0.0155	0.0218	0.0197
	α_2	–0.1760	0.0367	0.0072	0.0147	0.2966	0.0983	0.0096	0.0096	0.0033	0.0030
	β_2	0.3987	0.0175	0.1548	0.0141	0.5003	0.4308	0.0159	0.0159	0.0138	0.0137

Table 3. Point and interval estimation method for parameters of the TIIHLW distribution under competing risks based on Type II censored sample: Case 2, r = 0.6

Case 2		point estimation				CI		Bootstrap			
r = 0.6		MLE		Bayes		MLE	Bayes	MLE		Bayes	
n		Bias	MES	Bias	MES	ACI	PCI	Bt	Bp	Bt	Bp
50	λ_1	2.9554	8.2977	1.5128	3.4096	4.9035	2.5846	0.1550	0.1484	0.1098	0.1083
	α_1	0.3695	0.1988	0.0958	0.0626	0.9786	0.8006	0.0298	0.0314	0.0256	0.0253
	β_1	0.4288	0.3704	0.8494	1.0190	1.6940	3.0633	0.0561	0.0529	0.0496	0.0495
	λ_2	1.9778	4.8114	1.1040	2.4102	3.7199	2.4599	0.1134	0.1134	0.1066	0.1065
	α_2	0.2117	0.0744	0.0480	0.0436	0.6749	0.4135	0.0204	0.0204	0.0132	0.0138
	β_2	0.5123	0.4569	0.9228	1.2882	1.7291	2.5758	0.0554	0.0554	0.0819	0.0780
100	λ_1	2.0997	6.0752	1.6733	3.8736	4.1051	2.0354	0.1274	0.1317	0.1028	0.1020
	α_1	0.3836	0.1859	0.1221	0.0619	0.7719	0.6770	0.0233	0.0251	0.0213	0.0209
	β_1	0.3052	0.1794	0.7030	0.7000	1.1521	1.0484	0.0375	0.0369	0.0338	0.0335
	λ_2	1.0971	2.3616	1.2381	2.5918	2.7067	2.3942	0.0838	0.0838	0.0744	0.0743
	α_2	0.2267	0.0670	0.0790	0.0441	0.4905	0.3030	0.0159	0.0159	0.0099	0.0096
	β_2	0.3717	0.2300	0.7142	0.7752	1.1888	1.1084	0.0365	0.0365	0.0352	0.0354
200	λ_1	0.9623	1.1667	1.8852	4.3521	3.9992	1.2967	0.1556	0.1624	0.1283	0.1358
	α_1	0.4901	0.2667	0.1702	0.0618	0.6384	0.6829	0.0238	0.0238	0.0269	0.0263
	β_1	0.0995	0.0486	0.5024	0.3620	0.7713	0.3019	0.0286	0.0313	0.0124	0.0116
	λ_2	0.8439	1.0528	1.3115	2.6003	2.9867	2.1487	0.1133	0.1133	0.0972	0.0970
	α_2	0.3123	0.1138	0.0957	0.0393	0.5004	0.3675	0.0196	0.0196	0.0144	0.0139
	β_2	0.1820	0.0803	0.5646	0.4807	0.8520	0.4434	0.0319	0.0319	0.0169	0.0165

Table 4. Point and interval estimation method for parameters of the TIIHLW distribution under competing risks based on Type II censored sample: Case 2, r = 0.9

Case 2		point estimation				CI		Bootstrap			
r = 0.9		MLE		Bayes		MLE	Bayes	MLE		Bayes	
n		Bias	MES	Bias	MES	ACI	PCI	Bt	Bp	Bt	Bp
50	λ_1	1.7677	3.5536	1.2829	2.6252	2.5683	2.0895	0.0770	0.0827	0.0743	0.0742
	α_1	0.0081	0.0203	–0.0871	0.0194	0.5579	0.1359	0.0177	0.0185	0.0043	0.0043
	β_1	–0.0087	0.1114	0.2240	0.1029	1.3087	0.8739	0.0427	0.0421	0.0293	0.0278
	λ_2	1.0865	1.4353	1.0092	1.1012	1.9796	1.0598	0.0636	0.0636	0.0521	0.0521
	α_2	–0.0491	0.0137	–0.0635	0.0135	0.4162	0.0807	0.0129	0.0129	0.0028	0.0026
	β_2	0.1626	0.1764	0.3572	0.1521	1.5190	1.3645	0.0480	0.0480	0.0445	0.0445
100	λ_1	1.2926	4.0898	1.3190	2.7160	2.0424	1.5037	0.0784	0.0763	0.0648	0.0646
	α_1	0.0433	0.0168	–0.0828	0.0143	0.4797	0.1577	0.0150	0.0153	0.0051	0.0046
	β_1	–0.1983	0.0946	0.0621	0.0982	0.9220	0.4224	0.0302	0.0286	0.0143	0.0128
	λ_2	1.1049	1.3891	1.0010	1.3597	1.6086	0.9148	0.0512	0.0512	0.0421	0.0420
	α_2	–0.0358	0.0081	–0.0594	0.0078	0.3230	0.0746	0.0103	0.0103	0.0027	0.0024
	β_2	–0.0357	0.0631	0.1424	0.0623	0.9748	0.3328	0.0315	0.0315	0.0107	0.0104
200	λ_1	1.0215	4.5930	1.3608	2.6291	1.7915	0.9716	0.0914	0.0845	0.0748	0.0752
	α_1	0.0632	0.0194	–0.0707	0.0135	0.4861	0.1619	0.0154	0.0151	0.0051	0.0052
	β_1	–0.3270	0.1414	–0.0973	0.1205	0.7278	0.4839	0.0237	0.0228	0.0147	0.0151
	λ_2	0.8151	1.5861	0.9968	1.7814	1.0059	0.8220	0.0616	0.0616	0.0527	0.0525
	α_2	–0.0245	0.0099	–0.0581	0.0083	0.3776	0.0960	0.0113	0.0113	0.0031	0.0029
	β_2	–0.1622	0.0662	–0.0039	0.0514	0.7829	0.3112	0.0264	0.0264	0.0098	0.0096

Application of electrical appliances

A real-life data set is examined in [34, p. 441]. The 36 small electronic components were put through an automatic life test and failures are divided to 18 different categories. However, we found that only seven modes were represented among the 33 identified failures, and that only modes 6-9 occurred more than twice. Failure mode 9 is strongly valued. As a result, the data set consists of two causes of failure δ = 1 (failure mode 9), δ = 2 (all other failure modes), and δ = 0 (failure time is censored). As a result, the following data illustrates the failure times in order and, if available, the cause of each failure. The data set was shown in tab. 5.

Table 5. Data set gives the times from 36 small electrical units under an automatic life test

(11, 2)	(35, 2)	(49, 2)	(170, 2)	(329, 2)	(381, 2)
(708, 2)	(958, 2)	(1062, 2)	(1167, 1)	(1594, 2)	(1925, 1)
(1990, 1)	(2223, 1)	(2327, 2)	(2400, 1)	(2451, 2)	(2471, 1)
(2551, 1)	(2565, 0)	(2568, 1)	(2694, 1)	(2702, 2)	(2761, 2)
(2831, 2)	(3034, 1)	(3059, 2)	(3112, 1)	(3214, 1)	(3478, 1)
(3504, 1)	(4329, 1)	(6367, 0)	(6976, 1)	(7846, 1)	(13403, 0)

Table 6 provides the maximum likelihood and Bayes point estimates of the four model parameters, based on the model mentioned in this paper, in order to assess the data. We used the *R* programming language in all calculations.

Table 6. Estimates the parameters of the TIIHLW distribution under competing risks by using MLE and Bayesian estimation method

	MLE				Bayes			
	Coef	St.E	LCI	U.CI	Coef	St.E	LCI	U.CI
λ_1	212.9505	4.6276	203.8803	222.0207	213.0388	4.4807	204.2565	221.8211
α_1	0.3678	0.1291	0.1148	0.6207	0.3764	0.0667	0.2457	0.5070
β_1	0.3246	0.0446	0.2370	0.4121	0.3243	0.0226	0.2799	0.3685
λ_2	1.0962	0.9292	-0.7250	2.9175	0.9273	0.2479	1.4131	0.4414
α_2	0.0031	0.0151	-0.0265	0.0327	0.0033	0.0021	0.0073	–0.0008
β_2	0.5857	0.4606	–0.3170	1.4884	0.5733	0.1293	0.8267	0.3199

The log-likelihood (ll) values are computed for the TIIHLW distribution under the competing risks model in tab. 7. As a result, consistent AIC (CAIC), the Akaike information criterion (AIC), Bayesian information criterion (BIC) goodness-of-fit measures and Hannan-Quinn information criterion (HQIC) are examined.

Table 7. The values of log-likelihood, AIC, BIC, CAIC, and HQIC for the TIIHLW distribution under competing risks

MLE	ll	AIC	BIC	HQIC	CAIC
Measure	–308.89	629.781	639.282	633.097	632.677

The trace plots and the marginal posterior pdf of parameters of the TIIHLW probability under competing risks by using Bayesian estimation are obtained in figs. 1 and 2. The auto-correlation plots demonstrate that the lag decreases over time, and the trace plots show a clear mix of the sampled drawings, indicating that they become practically independent with time and are drawn from the real posterior distribution, fig. 3. Without obtaining these marginal posterior relations. Random drawings from the joint posterior distribution θ were used to obtain random plots of risks π_j, j = 1, 2. Further the Bayes estimates π_j were calculated and posterior density functions were generated using those graphics. Figure 4 shows the posterior density for the MCMC results for all parameters, indicating a symmetric normal distribution identical to the proposed distribution.

Table 8 represents the estimation of relative risks and survival at different times for each method. Figure 5 shows the convergence diagrams for random draws of πj. In the fig. 6, different θ-values are used to generate random graphs for the sub-survivors and overall survivor. Confidence intervals for Bayesian estimations at 95% were constructed for these functions at different times. In fig. 7, the results of estimating MCMC for the posterior distribution of each of π_1 and π_2 were presented. The results showed that they have a symmetric normal distribution.

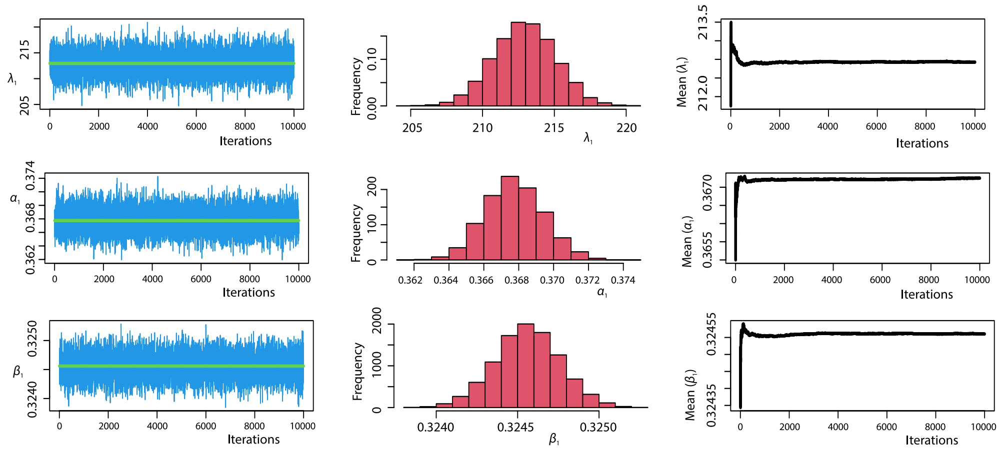

Figure 1. The trace plots, the marginal posterior pdf and convergence of the parameters λ_1, α_2, and β_1

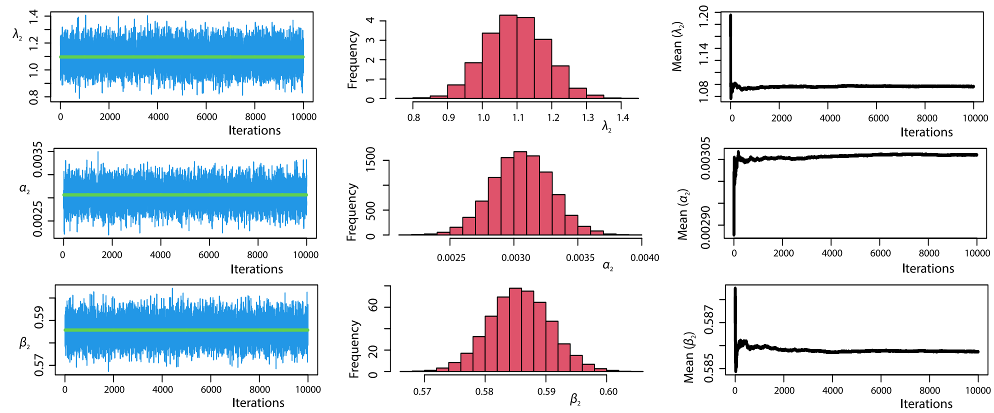

Figure 2. The trace plots and the marginal posterior pdf of the parameters λ_2, ω_2, and β_2

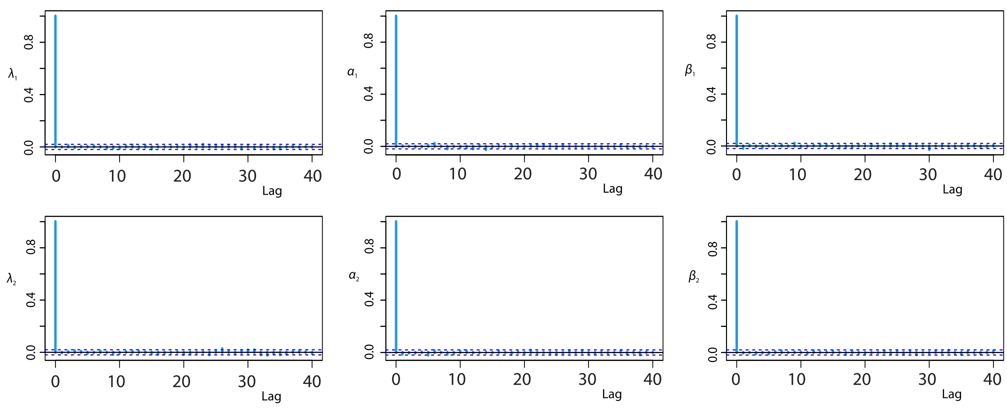

Figure 3. The autocorrelation of MCMC results

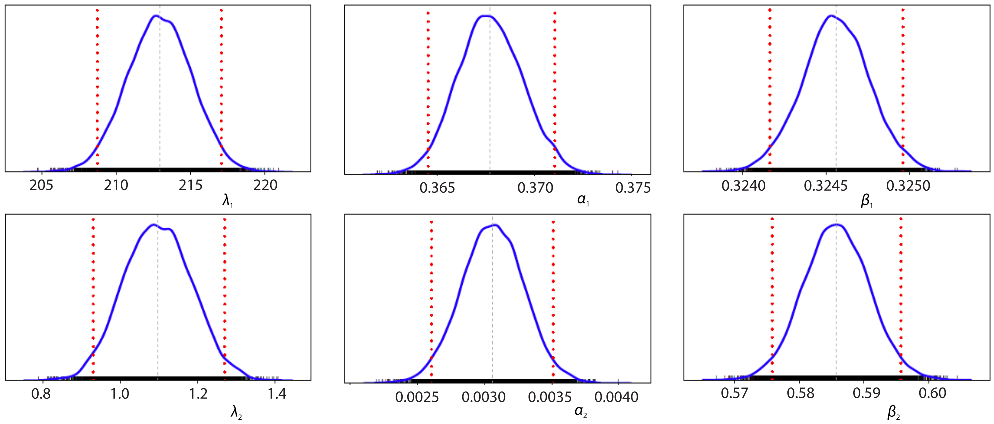

Figure 4. The posterior density of MCMC results

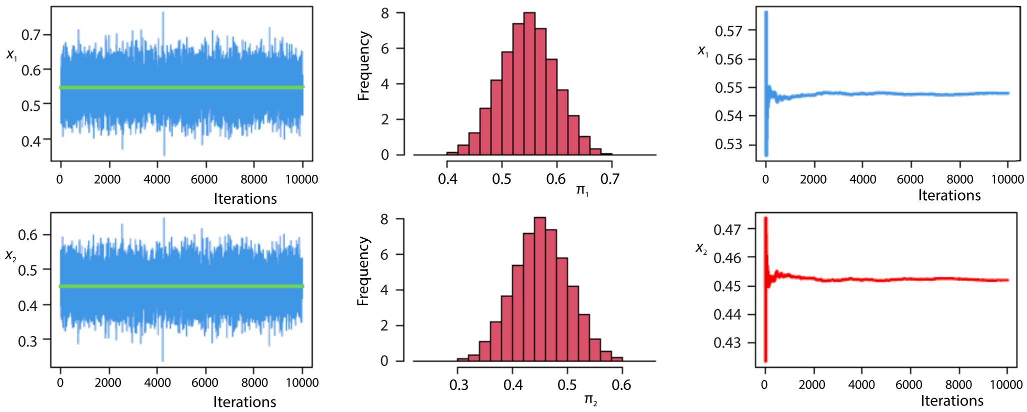

Figure 5. The trace plots, marginal posterior and convergence of the relative risks of MCMC results

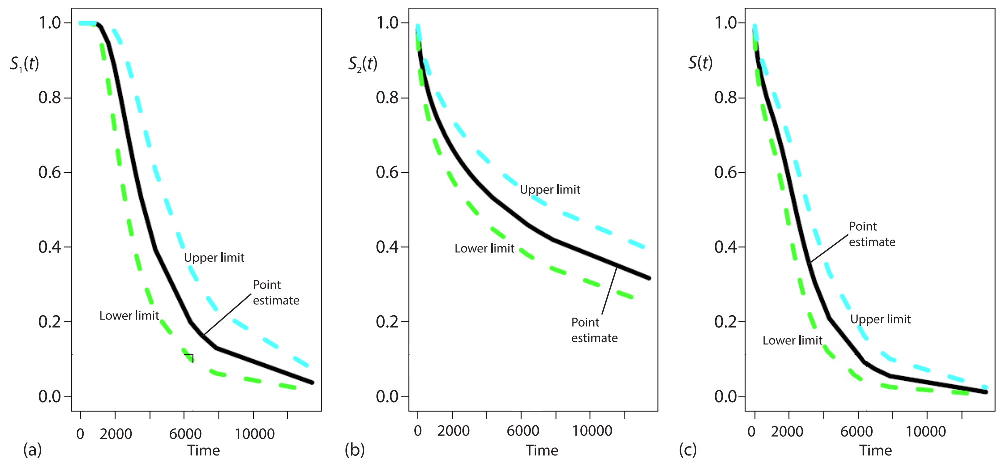

Figure 6. Bayes point and interval estimates of the sub-survivor functions and the overall survivor function; Bayes estimate of (a) $S_1(t)$, (b) $S_2(t)$, and (c) $S(t)$

Table 8. Estimated relative risks and survival

	π_1	π_2	$S_1(11)$	$S_2(11)$	$S_1(11)$	$S_1(2511)$	$S_2(2511)$	$S(2511)$
MLE	0.5476	0.4524	1	0.9839	0.9839	0.7641	0.6289	0.4805
Bayesian	0.5480	0.4520	1	0.9839	0.9839	0.7639	0.6293	0.4807

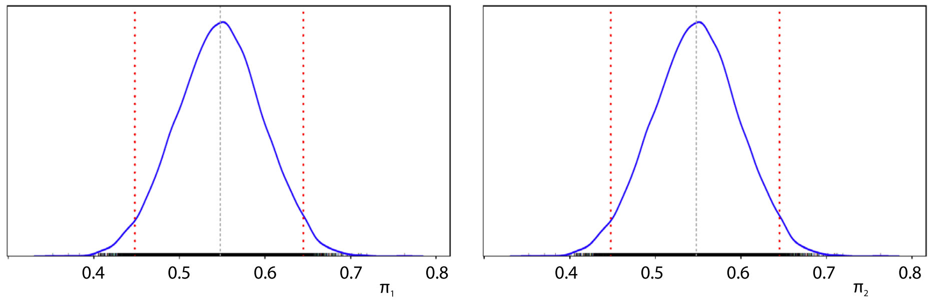

Figure 7. The posterior density for relative risks of MCMC results

Conclusion

Using k, $k \geq 2$, independent censoring data, we discussed competing risk models in this paper. Using the general likelihood equation of the model, the likelihood function was calculated for cases where the risks follow TIIHLW sub-distributions with scale parameters and unknown form. The Bayes estimates and the maximum likelihood of the model parameter estimates are described. The gamma prior distribution in Bayes analysis to get random drawing information is used to calculate the joint posterior-distribution function of all unknown parameters with known hyperparameters and the MCMC. For all unknown parameters, credible intervals, asymptotic confidence intervals, and bootstrap confidence intervals are provided. Sub-survivor, overall survivor function estimations and individual risk, were also included in the reliability analysis.

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