ESTIMATION AND PREDICTION OF TWO RAYLEIGH LIFETIME DISTRIBUTIONS UNDER JOINT CENSORING SCHEME

by

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Comparative life tests of a life product have received considerable attention in the past few years for measuring the relative merits of a product in competing duration. In this paper, we are adopting the problem of estimation and prediction of Rayleigh lifetime distributions under joint Type-I censoring scheme. The point and interval estimation is formulated with maximum likelihood and the Bayes methods for the unknown parameters. Also, Bayesian approach is applied in viewpoint both point and interval prediction. The results of estimation and prediction are discussed through analysis the set of real data and bulding Monte-Carlo simulation studies. Finally, we built a numerical discussion of the most important results and future recommendations.

Key words: *joint censoring scheme, Rayleigh distribution, Bayesian estimation, maximum likelihood estimation, Bayesian prediction*

Introduction

The early common censoring schemes in life tests experiments are the time censoring scheme (Type-I censoring scheme) and the failure censoring scheme (Type-II censoring scheme). In the Type-I censoring scheme (Type-I CS), we terminate the life testing experiment at a prescribed time, τ. But, in the Type-II censoring scheme (Type-II CS), we terminate the life testing experiment at fixed number of failure, r. In the two types of censoring, Type-I CS and Type-II CS, don't allow to remove any unit of the experiment at points other than the final point. Generally, the property of removed units through the experiment is allowed in progressive censoring scheme, for more details, Balakrishnan and Aggarwala [1], Balakrishnan [2], Kundu [3], and Soliman *et al*. [4]. In life testing experiments, specially products which come from different lines of production under the same facility in conducting comparative life-tests is presented by joint censoring scheme. To be more precise suppose, manufactured products produce the same product under the same conditions from two different lines of production. Suppose, we are selected two independent samples of sizes κ_1 and κ_2 from these lines to place simultaneously under a life-testing experiment. The mechanism of joint Type-II CS is described by, all the devices put on life testing experiment simultaneously, to record the successive failure times. The test is running until specified number of failures is occurred. The experimenter decide under considerations of cost and time to terminate the test experiment as soon as a pre-specified number of failures is observed. Also, the mechanism of joint Type-I CS can be described by, the ideal test time, τ is prior proposed

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and all the devices are put under life testing experiment simultaneously. When the test is running the successive failure times are recorded until the specified time, τ is observed. In this paper, we are adopted the joint Type-I CS when the units of product come from two different lines of production and its life have the Rayleigh lifetime distribution. In the literature, different work are presented in the case of joint censoring scheme and its inferential methods, for early worke see Rao *et al*. [5], Basu [6], Johnson and Mehrotra [7], Mehrotra and Johnson [8], Bhattacharyya and Mehrotra [9], and Mehrotra and Bhattacharyya [10]. The inference of two exponential populations Balakrishnan and Rasouli [11]. The estimation and prediction of two exponential populations Shafay *et al*. [12] and Al-Matrafi and Abd-Elmougod [13]. Also, recently the statistical inference under joint censoring scheme, see Algarni *et al*. [14], Mondal-Kundu [15], and Mondal-Kundu [16]. The problem of statistical inference of competing risks model under joint censoring schemes is handeled by Almarashi *et al*. [17]

The Rayleigh distribution (RD) is defined as a special case of the Weibull distribution, it has wide applications, in engineering and communication, Dyer and Whisenand [18] and Dyer and Whisenand [19]. In life testing of electrovacum devices, Polovko [20] and recently Algarni *et al*. [21]. The random variable T is called Rayleigh random variable if has cumulative distribution function (CDF) with parameter φ_i, $i = 1, 2$ defined:

$$F_i(t) = 1 - \exp(-\varphi_i t^2),\ \ \varphi_i > 0,\ \ i = 1,2 \ \text{ and } \ t > 0 \tag{1}$$

The coresponding probability density functions (PDF), reliability function $R(t)$ and hazard rate function $h(t)$ of the RD given, respectively:

$$f_i(t) = 2\varphi_i t \exp\left(-\varphi_i t^2\right) \tag{2}$$

$$R_j(t) = \exp\left(-\varphi_i t^2\right) \tag{3}$$

and

$$H_j(t) = 2\varphi_i t \tag{4}$$

where RD has the property that increasing linear failure rate function and decreasing reliability function with higher rate than exponential distribution. Several authors were discussed the statistical inferences of the RD, Harter and Moore [22] obtained the explicit form of ML estimators under Type-II censored scheme, Dyer and Whisenand [18, 19] presented under complete sample and censored samples the best linear unbiased estimator. Lalitha and Mishra [23], and Kong and Hosain [24] considered the doubly censored samples. Also, Bayesian approach in the problem of estimation and prediction discussed by Howlader *et al*. [25], Fernandez [26], AL-Hussaini and Ahmad [27], and AL-Hussaini and Ahmad [28].

Our objective of this paper is building the statistical inference of estimation and prediction techniques for Rayleigh lifetime populations under joint Type-I censoring scheme. Therefore, the problem of measuring the relative merits of two competing duration of Rayleigh lifetime products are studied. Then, the comparative life-tests of products under joint Type-I censoring cheme are discussed under consideration that, a sample is selected from two different lines of production. For this puorpose, we suppose the product is manufactured taken from two different lines Λ_1 and Λ_2 of production under the same conditions and two independent samples of sizes κ_1 and κ_2 are put under the test, the obtained data are used in the problem of estimation and prediction under maximum likelihood and Bayes methods.

Model description and the likelihood function

Let Λ_1 and Λ_2 are the two distinct lines of production with the same facility. From a line Λ_1, κ_1 unites are randomly selected and the identical independent distributed (i.i.d.) lifetimes $X_1, X_2, \ldots X_{\kappa_1}$ have the population with CDF $F_1(x)$ and PDF $f_1(x)$. Also, from a line Λ_2, κ_2 unties are randomly selected and the i.i.d. lifetimes $Y_1, Y_2 \ldots Y_{\kappa_2}$, have the population with CDF $F_2(y)$ and PDF $f_2(y)$. The ordered lifetimes $W_1 \le W_2 \le \ldots \le W_r$ of $\{X_1, X_2, \ldots X_{\kappa_{1r}}, Y_1, Y_2, \ldots Y_{\kappa_{2r}}\}$ satisfies that $W_r < \tau$ with the prior test time τ and $r = \kappa_{1r} + \kappa_{2r} \le \kappa_1 + \kappa_2$ are recorded and called the jointly Type-I censored sample. Under the joint Type-I censored sample (W, Z) consist of $W = (W_1, W_2, \ldots W_r)$ with $1 < r \le \kappa_1 + \kappa_2$ and $Z = (Z_1, Z_2, \ldots Z_r)$ with ($Z_i = 1$ if $W_i = X_i$ and $Z_i = 0$ if $W_i = Y_i$). The joint likelihood function of (W, Z) under consideration $\kappa_{1r} = \Sigma_{i=1}^{r} Z_i$ are X's failures among $(W_1, W_2, \ldots W_r)$ and $\kappa_{2r} = \Sigma_{i=1}^{r} (1 - ZI)$ are Y's failures, is constructed:

$$L(w,\mathbf{z}) = \frac{\kappa_1!\kappa_2!}{(\kappa_1-\kappa_{1r})!(\kappa_2-\kappa_{2r})!}\left[\prod_{i=1}^{r}[f_1(w_i)]^{z_i}[f_2(w_i)]^{1-z_i}\right][S_1(\tau)]^{\kappa_1-\kappa_{1r}}[S_2(\tau)]^{\kappa_2-\kappa_{2r}} \tag{5}$$

Maximum likelihood estimation

For given τ and the first r order statistics $W = (W_1, W_2, \ldots W_r)$ of $\{X_1, X_2, \ldots X_{\kappa_{1r}}, Y_1, Y_2, \ldots Y_{\kappa_{2r}}\}$ are distributed with Rayleigh lifetime distributions. Then, the likelihood function (5) with distribution functions (1) and (2) under the jointly Type-I censoring sample W, presented:

$$L(\varphi_1,\varphi_2 \mid \underline{w}) \propto \varphi_1^{\kappa_{1r}}\varphi_2^{\kappa_{2r}} \exp\left\{-\varphi_1\sum_{i=1}^{r} z_i w_i^2 - \varphi_2\sum_{i=1}^{r}(1-z_i)w_i^2 - (\kappa_1-\kappa_{1r})\varphi_1\tau^2 - (\kappa_2-\kappa_{2r})\varphi_2\tau^2\right\} \tag{6}$$

After taken the logarithm of eq. (6), the log-likelihood function is given:

$$\ell(\varphi_1,\varphi_2 \mid \underline{w}) \propto \kappa_{1r}\log\varphi_1 + \kappa_{2r}\log\varphi_2 - \varphi_1\sum_{i=1}^{r} z_i w_i^2 - \varphi_2\sum_{i=1}^{r}(1-z_i)w_i^2 - (\kappa_1-\kappa_{1r})\varphi_1\tau^2 - (\kappa_2-\kappa_{2r})\varphi_2\tau^2 \tag{7}$$

The point and interval estimates of parameters φ_1 and φ_2 from log-likelihood function (7) is presented:

The MLE (point estimators)

The first partial derivatives of eq. (7) with respect to φ_1 and φ_2 presented:

$$\frac{\partial\ell(\varphi_1,\varphi_2 \mid \underline{w})}{\partial\varphi_1} = \frac{\kappa_{1r}}{\varphi_1} - \sum_{i=1}^{r} z_i w_i^2 - (\kappa_1-\kappa_{1r})\tau^2 = 0 \tag{8}$$

and

$$\frac{\partial\ell(\varphi_1,\varphi_2 \mid \underline{w})}{\partial\varphi_2} = \frac{\kappa_{2r}}{\varphi_2} - \sum_{i=1}^{r}(1-z_i)w_i^2 - (\kappa_2-\kappa_{2r})\tau^2 = 0 \tag{9}$$

hence

$$\hat{\varphi}_1 = \frac{\kappa_{1r}}{\sum_{i=1}^{r} z_i w_i^2 + (\kappa_1-\kappa_{1r})\tau^2} \tag{10}$$

and

$$\hat{\varphi}_2 = \frac{\kappa_{2r}}{\sum_{i=1}^{r}(1-z_i)w_i^2 + (\kappa_2-\kappa_{2r})\tau^2} \tag{11}$$

Remark: The eqs. (10) and (11) showed that, the conditional estimators of the model-parameters depend on the discrete random variable κ_{ir}.

Hence, the estimate $\hat{\varphi}_1$ and $\hat{\varphi}_2$ does not exist for $\kappa_{1r} = 0$ or r and κ_{2r} or r, respectively. The problem of exact distributions for estimators $\hat{\varphi}_1$ and $\hat{\varphi}_2$ are defined as mixture of discrete and continuous distributions, hence as given in Kundu and Joarder [29] is difficult to obtain

The MLE (interval estimators)

The second partial derivatives of eq. (7) with respect to φ_1 and φ_2 presented:

$$\frac{\partial^2 \ell(\varphi_1, \varphi_2 \mid \underline{w})}{\partial \varphi_1^2} = \frac{-\kappa_{1r}}{\varphi_1^2} \tag{12}$$

$$\frac{\partial^2 \ell(\varphi_1, \varphi_2 \mid \underline{w})}{\partial \varphi_2^2} = \frac{-\kappa_{2r}}{\varphi_2^2} \tag{13}$$

and

$$\frac{\partial^2 \ell(\varphi_1, \varphi_2 \mid \underline{w})}{\partial \varphi_1 \partial \varphi_2} = \frac{\partial^2 \ell(\varphi_1, \varphi_2 \mid \underline{w})}{\partial \varphi_{21} \partial \varphi_1} = 0 \tag{14}$$

Hence the negative expected of the second derivative present Fisher information matrix $\rho(\varphi_1, \varphi_2)$ can be defined:

$$\rho(\varphi_1, \varphi_2) = \begin{bmatrix} \dfrac{\kappa_{1r}}{\varphi_1^2} & 0 \\ 0 & \dfrac{\kappa_{2r}}{\varphi_2^2} \end{bmatrix} \tag{15}$$

Then the variance covariance matrix $\eta(\varphi_1, \varphi_2) = \rho^{-1}(\varphi_1, \varphi_2)$ is written:

$$\eta(\varphi_1, \varphi_2) = \begin{bmatrix} \dfrac{\varphi_1^2}{\kappa_{1r}} & 0 \\ 0 & \dfrac{\varphi_2^2}{\kappa_{2r}} \end{bmatrix} \tag{16}$$

With eqs. (10) and (11) the MLE of variance covariance matrix $\eta(\varphi_1, \varphi_2)$ is written:

$$\eta(\hat{\varphi}_1, \hat{\varphi}_2) = \begin{bmatrix} \dfrac{1}{\kappa_{1r}} \left[\sum_{i=1}^{r} z_i w_i^2 + (\kappa_1 - \kappa_{1r})\tau^2 \right]^2 & 0 \\ 0 & \dfrac{1}{\kappa_{2r}} \left[\sum_{i=1}^{r} (1 - z_i) w_i^2 + (\kappa_2 - \kappa_{2r})\tau^2 \right]^2 \end{bmatrix} \tag{17}$$

Then the bivariate normal distribution of MLE estimators is presented:

$$(\hat{\varphi}_1, \hat{\varphi}_2) \twoheadrightarrow ND\left[(\varphi_1, \varphi_2), \eta(\hat{\varphi}_1, \hat{\varphi}_2) \right] \tag{18}$$

Under the normal approximation of MLE the the $100(1 - 2\gamma)$% approximate confidence intervals of model parameters φ_1 and φ_2 is given:

$$\left(1 \mp \frac{z_\gamma}{\sqrt{\kappa_{1r}}} \right) \left[\sum_{i=1}^{r} z_i w_i^2 + (\kappa_1 - \kappa_{1r})\tau^2 \right] \tag{19}$$

and

$$\left(1\mp\frac{z_\gamma}{\sqrt{\kappa_{2r}}}\right)\left[\sum_{i=1}^{r}(1-z_i)w_i^2+(\kappa_2-\kappa_{2r})\tau^2\right] \tag{20}$$

respectively, where z_γ is the tabulated standard normal value with significant level equal γ.

Bayesian estimation

In this section, we adopted Bayesian point and interval estimation of parameters φ_1 and φ_2 under consideration independent gamma priors with prior functions $\psi_1^*(\varphi_1)$ and $\psi_2^*(\varphi_2)$ of parameters φ_1 and φ_2, respectively:

$$\psi_1^*(\varphi_1)\propto\varphi_1^{a_1-1}\exp(-b_1\varphi_1),\varphi_1>0,\ (a_1,b_1>0) \tag{21}$$

and

$$\psi_2^*(\varphi_2)\propto\varphi_2^{a_2-1}\exp(-b_2\beta_2),\varphi_2>0,\ (a_2,d_2>0) \tag{22}$$

Bayesian point estimation

From the joint prior density $\psi_1^*(\varphi_1)\times\psi_2^*(\varphi_2)$ and the likelihood function (6), the joint posterior density is given:

$$\psi(\varphi_1,\varphi_2\,|\,\underline{w})\propto L(\varphi_1,\varphi_2\,|\,\underline{w})\times\psi_1^*(\varphi_1)\times\psi_2^*(\varphi_2)$$

$$\propto\varphi_1^{a+\kappa_{1r}-1}\exp\left[-\varphi_1\left(b_1+\sum_{i=1}^{r}z_iw_i^2,+(\kappa_1-\kappa_{1r})\tau^2\right)\right]\times$$

$$\times\varphi_2^{a_2+\kappa_{2r}-1}\exp\left[-\varphi_2\left(b_2+\sum_{i=1}^{r}(1-z_i)w_i^2,+(\kappa_2-\kappa_{2r})\tau^2\right)\right]=\text{Gamma}(Q_1,Q_2)\times\text{Gamma}(Q_3,Q_4) \tag{23}$$

where

$$Q_1=a_1+\kappa_{1r},\ Q_2=b_1+\sum_{i=1}^{r}z_iw_i^2,+(\kappa_1-\kappa_{1r})\tau^2 \tag{24}$$

and

$$Q_3=a_2+\kappa_{2r},\ Q_4=b_2+\sum_{i=1}^{r}(1-z_i)w_i^2,+(\kappa_2-\kappa_{2r})\tau^2 \tag{25}$$

Different loss function, can be employed in Bayesian approach, we are adopted symmetric square error (SE) loss function and asymmetric linear exponential (LINEX) loss function.

Under SE loss function

From the posterior distribution (22), and the jointly Type-I censoring data, the Bayes estimators for the parameter $\varphi_{1,\text{SE}}$ and $\varphi_{2,\text{SE}}$, is presented:

$$\hat{\varphi}_{1,\text{SE}}=E_\psi(\varphi_1)=\frac{a_1+\kappa_{1r}}{b_1+\sum_{i=1}^{r}z_iw_i^2,+(\kappa_1-\kappa_{1r})\tau^2} \tag{26}$$

and

$$\varphi_{2,\text{SE}}=E_\psi(\varphi_1)=\frac{a_2+\kappa_{2r}}{b_2+\sum_{i=1}^{r}(1-z_i)w_i^2,+(\kappa_2-\kappa_{2r})\tau^2} \tag{27}$$

Under a LINEX loss function

From the posterior distribution eq. (22), and the jointly Type-I censoring data, the Bayes estimators for the parameter φ_1 and φ_2 under LINEX loss function, is presented:

$$\hat{\varphi}_{1,\text{LINEX}} = -\frac{1}{c}\log\left(E\psi[\exp(-c\varphi_1)]\right) = \frac{-\left(a_1+\kappa_{1r}\right)}{c}\log\left[\frac{b_1+\sum_{i=1}^{r} z_i w_i^2,+(\kappa_1-\kappa_{1r})\tau^2}{c+b_1+\sum_{i=1}^{r} z_i w_i^2,+(\kappa_1-\kappa_{1r})\tau^2}\right] \tag{28}$$

and

$$\varphi_{2,\text{LINEX}} = -\frac{1}{c}\log(E\psi[\exp(-c\varphi_2)]) = \frac{-(a_2+\kappa_{2r})}{c}\log\left[\frac{b_2+\sum_{i=1}^{r}\left(1-z_i\right)w_i^2,+\left(\kappa_2-\kappa_{2r}\right)\tau^2}{c+b_2+\sum_{i=1}^{r}\left(1-z_i\right)w_i^2,+\left(\kappa_2-\kappa_{2r}\right)\tau^2}\right] \tag{29}$$

Bayesian credible intervals

We are shown that, in the concept of ML the parameter values are treated as a fixed values but, the bounds of confidence intervals are random. Although, the parameters in Bayesian approach are treated as a random variable but, the bound of credible intervals are considered as a fixed values. Interval, which the posterior density at its points is greater than the posterior density at any point outside it is called the HPD interval. Also, equal two side credibal interval satisfy the conditions:

$$\int_{L_1}^{\infty}\psi_1(\varphi_1\mid \underline{w})\mathrm{d}\varphi_1 = 1-\gamma \quad\text{and}\quad \int_{U_1}^{\infty}\psi_1(\varphi_1\mid \underline{w})d\varphi_1 = \gamma \tag{30}$$

and

$$\int_{L_2}^{\infty}\psi_2(\varphi_2\mid \underline{w})\mathrm{d}\varphi_1 = 1-\gamma \quad\text{and}\quad \int_{U_2}^{\infty}\psi_2(\varphi_2\mid \underline{w})\mathrm{d}\varphi_1 = \gamma \tag{31}$$

where $\psi_1(\varphi_1|\underline{w})$ and $\psi_2(\varphi_2|\underline{w})$ are the marginal posterior distributions. Then $(1 - 2\gamma)100\%$ credibal intervals for φ_1 and φ_2 is obtained by solve the two eqs. (29) and (30). The two eqs. (29) and (30) are reduced:

$$\frac{1}{\Gamma(Q_1)}\Gamma(Q_1, L_1Q_2) = (1-\gamma) \quad\text{and}\quad \frac{1}{\Gamma(Q_1)}\Gamma(Q_1, U_1Q_2) = \gamma \tag{32}$$

and

$$\frac{1}{\Gamma(Q_3)}\Gamma(Q_3, L_2Q_4) = (1-\gamma) \quad\text{and}\quad \frac{1}{\Gamma(Q_3)}\Gamma(Q_{31}, U_2Q_4) = \gamma \tag{33}$$

where $\Gamma(a, b)$ is the incomplete gamma function satisfies $\Gamma(a, b) = \int_b^\infty t^{a-1}\mathrm{e}^{-t}\mathrm{d}t$ and $\Gamma(a)$ is the gamma function satisfies $\Gamma(a) = \int_a^\infty t^{a-1}\mathrm{e}^{-t}\mathrm{d}t$. The analytically solution in this case is not possible then, we need numerical technique for solving these non-linear eqs. (31) and (32) such as Newton Rafson.

Bayesian prediction

Without loss of the generality, the Bayes point and interval predictions of future failure time W_{r+1} under jointly type-I censored failure times (**W**, ***Z***), where $\mathbf{W} = (W_1, W_1,...,W_r)$ with $1 < r \le \kappa_1 + \kappa_2$ are discussed. For more information about generalization of future failure times are discussed in [12]. The Bayesian prediction of future W_{r+1} can be described in three cases separately, as follows.

At $\kappa_{1r} < \kappa_1$ and $\kappa_1 = \kappa_{r2}$, then $W_{r+1} = X_{\kappa_{1r}+1}$ and the joint density function of (**W**, **Z**, W_{r+1}), is given:

$$f_1(\mathbf{w},\mathbf{z},w_{r+1}) = \frac{\kappa_1!\kappa_2!}{(\kappa_1-\kappa_{1r}-1)!}\prod_{i=1}^{r}\left[f_1(w_i)\right]^{z_i}\left[f_2(w_i)\right]^{1-z_i}[S_1(w_{r+1})]^{\kappa_1-\kappa_{1r}-1}f_1(w_{r+1}) \tag{34}$$

At $\kappa_{1r} = \kappa_1$ and $\kappa_{2r} < \kappa_2$, then $W_{r+1} = Y_{\kappa_{2r}+1}$ and the joint density function of (**W**, **Z**, W_{r+1}), is given:

$$f_2(\mathbf{w},\mathbf{z},w_{r+1}) = \frac{\kappa_1!\kappa_2!}{(\kappa_2-\kappa_{2r}-1)!}\prod_{i=1}^{r}\left[f_1(w_i)\right]^{z_i}\left[f_2(w_i)\right]^{1-z_i}[S_2(w_{r+1})]^{\kappa_2-\kappa_{2r}-1}f_2(w_{r+1}) \tag{35}$$

At $\kappa_{1r} = \kappa_1$ and $\kappa_{2r} < \kappa_2$, then $W_{r+1} = Y_{\kappa_{1r}+1}$ or $Y_{\kappa_{2r}+1}$ and the joint density function of (**W**, **Z**, W_{r+1}, Z_{r+1}), is given:

$$\begin{aligned}f_3(\mathbf{w},\mathbf{z},w_{r+1},z_{r+1}) &= \frac{\kappa_1!\kappa_2!}{(\kappa_1-\kappa_{1r}-1)!(\kappa_2-\kappa_{2r}-1)!}\prod_{i=1}^{r}\left[f_1(w_i)\right]^{z_i}\left[f_2(w_i)\right]^{1-z_i}\times\\ &\times[S_1(w_{r+1})]^{\kappa_1-\kappa_{1r}-1}[S_2(w_{r+1})]^{\kappa_2-\kappa_{2r}-1}\left[f_1(w_{r+1})\right]^{z_{r+1}}\left[f_2(w_{r+1})\right]^{1-z_{r+1}}\end{aligned} \tag{36}$$

The conditional probability density functions of eqs. (33)-(35) of W_{r+1} are reduced:

$$f_1(w_{r+1}\,|\,w,\mathbf{z}) = (\kappa_1-\kappa_{1r})f_1(w_{r+1})\frac{[S_1(w_{r+1})]^{\kappa_1-\kappa_{1r}-1}}{[S_1(\tau)]^{\kappa_1-\kappa_{1r}}} \tag{37}$$

$$f_2(w_{r+1}\,|\,w,\mathbf{z}) = (\kappa_2-\kappa_{2r})f_2(w_{r+1})\frac{[S_2(w_{r+1})]^{\kappa_2-\kappa_{2r}-1}}{[S_2(\tau)]^{\kappa_2-\kappa_{2r}}} \tag{38}$$

and

$$\begin{aligned}f_3(w_{r+1}\,|\,w,\mathbf{z}) &= (\kappa_1-\kappa_{1r})(\kappa_2-\kappa_{2r})\left[f_1(w_{r+1})\right]^{z_{r+1}}\left[f_2(w_{r+1})\right]^{1-z_{r+1}}\times\\ &\times\frac{[S_1(w_{r+1})]^{\kappa_1-\kappa_{1r}-1}[S_2(w_{r+1})]^{\kappa_2-\kappa_{2r}-1}}{[S_1(\tau)]^{\kappa_1-\kappa_{1r}}[S_2(\tau)]^{\kappa_2-\kappa_{2r}}}\end{aligned} \tag{39}$$

Under consideration the distribution functions given by eqs. (1) and (2) the conditional density functions of W_{r+1} eqs. (36)-(38) reduced to three cases is given:

$$f_1(w_{r+1}\,|\,w,\mathbf{z}) = 2(\kappa_1-\kappa_{1r})\varphi_1 w_{r+1}\exp\left[-\varphi_1(\kappa_1-\kappa_{1r})(w_{r+1}^2-\tau^2)\right] \tag{40}$$

$$f_2(w_{r+1}\,|\,w,\mathbf{z}) = 2(\kappa_2-\kappa_{2r})\varphi_2 w_{r+1}\exp\left[-\varphi_2(\kappa_2-\kappa_{2r})(w_{r+1}^2-\tau^2)\right] \tag{41}$$

and

$$\begin{aligned}f_3(w_{r+1}\,|\,w,\mathbf{z}) &= 4(\kappa_1-\kappa_{1r})(\kappa_2-\kappa_{2r})\varphi_1\varphi_2 w_{r+1}^2\cdot\\ &\cdot\exp\left\{\left[-\varphi_1(\kappa_1-\kappa_{1r})-\varphi_2(\kappa_2-\kappa_{2r})\right](w_{r+1}^2-\tau^2)\right\}\end{aligned} \tag{42}$$

The conditional Bayes predictive density functions for given the original data (**W**, **Z**) from eqs. (39)-(41), is presented:

$$f^*(w_{r+1} \mid w, \mathbf{z}) = \int_0^\infty \int_0^\infty f^*(w_{r+1}, z_{r+1} \mid \mathbf{w}, \mathbf{z}) \psi(\varphi_1, \varphi_2 \mid \underline{w}) \mathrm{d}\varphi_1 \mathrm{d}\varphi_2 \tag{43}$$

hence

$$f_1^*(w_{r+1} \mid w, \mathbf{z}) = \int_0^\infty \int_0^\infty (\kappa_1 - \kappa_{1r}) \varphi_1 w_{r+1} \exp\left[-\varphi_1 (\kappa_1 - \kappa_{1r})(w_{r+1} - \tau)\right] \times$$

$$\times \mathrm{Gamma}(Q_1, Q_2) \times \mathrm{Gamma}(Q_3, Q_4) \mathrm{d}\varphi_1 \mathrm{d}\varphi_2 = (\kappa_1 - \kappa_{1r}) Q_1 Q_2^{Q_1} \frac{w_{r+1}}{\left[Q_2 + (\kappa_1 - \kappa_{1r})(w_{r+1} - \tau)\right]^{Q_1 + 1}} \tag{44}$$

$$f_2^*(w_{r+1} \mid w, \mathbf{z}) = \int_0^\infty \int_0^\infty (\kappa_2 - \kappa_{2r}) \varphi_2 w_{r+1} \exp\left[-\varphi_2 (\kappa_2 - \kappa_{2r})(w_{r+1} - \tau)\right] \times$$

$$\times \mathrm{Gamma}(Q_1, Q_2) \times \mathrm{Gamma}(Q_3, Q_4) \mathrm{d}\varphi_1 \mathrm{d}\varphi_2 = (\kappa_2 - \kappa_{2r}) Q_3 Q_4^{Q_3} \frac{w_{r+1}}{\left[Q_4 + (\kappa_2 - \kappa_{2r})(w_{r+1} - \tau)\right]^{Q_3 + 1}}$$

$$f_3^*(w_{r+1} \mid w, \mathbf{z}) = 4(\kappa_1 - \kappa_{1r})(\kappa_2 - \kappa_{2r}) w_{r+1}^2 \int_0^\infty \int_0^\infty \varphi_1 \varphi_2 \exp\left\{\left[-\varphi_1 (\kappa_1 - \kappa_{1r}) - \varphi_2 \times\right.\right.$$

$$\left.\left. \times (\kappa_2 - \kappa_{2r})\right](w_{r+1}^2 - \tau^2)\right) \mathrm{Gamma}(Q_1, Q_2) \mathrm{Gamma}(Q_3, Q_4) \mathrm{d}\varphi_1 \mathrm{d}\varphi_2 =$$

$$= 4(\kappa_1 - \kappa_{1r})(\kappa_2 - \kappa_{2r}) w_{r+1}^2 Q_1 Q_2 \left[\frac{Q_2}{Q_2 + (\kappa_1 - \kappa_{1r})(w_{r+1}^2 - \tau^2)}\right]^{Q_1 + 1} \times$$

$$\times Q_3 Q_4 \left[\frac{Q_4}{Q_4 + (\kappa_2 - \kappa_{2r})(w_{r+1}^2 - \tau^2)}\right]^{Q_3 + 1} \tag{45}$$

where Q_1, Q_2, Q_3, and Q_4 are given by eqs. (23) and (24).

Simulation studies

The problem of assessment of the results and the quality of proposed model are studied in this section through Monte-Carlo simulation study. Therefore, we test the change of sample size $\kappa_1 + \kappa_2$ and affect time τ as well as parameters vector $\varphi = \{\varphi_1, \varphi_2\}$. Then, we adopted two sets of the parameters values $\varphi = \{0.6, 1.0\}$ and $\varphi = \{1.2, 2.0\}$. The value of τ is taken to near from $E(X)$ and different combination of (κ_1, κ_2) reported in the tabs. 1-4 of simulation study. The simulation study is done with respect to 1000 simulated data sets. The prior parameters are selected to satisfies the property that $E(\varphi_i \simeq a_i/b_i)$The tools that used to test the point estimate is the mean estimate (ME) and the corresponding mean squared error (MSE). But, the interval estimate test under the mean interval length (ML) and probability coverage (PC). For the prior information, we adopted non-informative informative priors, P^0: (ai, bi) = (0.0001, 0.0001), i = 1, 2,..., 6 for non-informative and P^1 for informative prior). The results of simulation study are reported in tabs. 1-4.

The Monte-Carlo simulation study and the corresponding numerical results which presented by tabs. 1-4 have shown that, the model is acceptable and the corresponding estimation methods serve well. Also, from the numerical results we observe some point reported:
– The large value of τ serve very well than the small values.
– The model quailty improving at increasing κ_1 and κ_2.
– The numerical reslults under ML or non-informative Bayes methods are both closed.
– Informative prior Bayes estimate present the best choose in estimation or prediction.
– Estimations results under two choose of parameters of Rayleigh distribution are more acceptable.

Table 1. Mean and MSE of ML and Bayes (p^0, p^1) estimats at φ = {0.6, 1.0}

τ		τ = 0.7				τ = 1.4			
(κ_1, κ_2)		φ_1		φ_2		φ_1		φ_2	
		ME	MSE	ME	MSE	ME	MSE	ME	MSE
(30,30)	ML	0.654	0.154	1.425	1.254	0.645	0.115	1.224	0.887
	$Bayes^0$	0.646	0.118	1.342	1.182	0.636	0.101	1.204	0.784
	$Bayes^1$	0.632	0.099	1.245	0.998	0.622	0.097	1.144	0.654
(30, 40)	ML	0.635	0.110	1.366	1.211	0.632	0.102	1.199	0.874
	$Bayes^0$	0.632	0.101	1.325	1.101	0.624	0.100	1.214	0.869
	$Bayes^1$	0.614	0.094	1.211	1.000	0.617	0.087	1.171	0.614
(60, 30)	ML	0.628	0.103	1.354	1.199	0.619	0.097	1.200	0.837
	$Bayes^0$	0.615	0.099	1.323	1.187	0.617	0.089	1.198	0.811
	$Bayes^1$	0.610	0.095	1.207	1.003	0.608	0.082	1.166	0.588
(60, 60)	ML	0.594	0.088	1.215	0.876	0.611	0.082	1.113	0.711
	$Bayes^0$	0.573	0.082	1.211	0.862	0.615	0.081	1.100	0.654
	$Bayes^1$	0.562	0.059	1.189	0.756	0.603	0.049	1.002	0.499

Table 2. Mean length and CP of of ML and Bayes (p^0, p^1) estimats at φ = {0.6, 1.0}

τ		τ = 0.7				τ = 1.4			
(κ_1, κ_2)		φ_1		φ_2		φ_1		φ_2	
		ML	CP	ML	CP	ML	CP	ML	CP
(30,30)	ML	1.721	0.88	2.323	0.87	1.514	0.90	2.214	0.90
	$Bayes^0$	1.701	0.89	2.285	0.88	1.501	0.90	2.204	0.91
	$Bayes^1$	1.622	0.88	2.099	0.88	1.422	0.91	2.075	0.92
(30, 40)	ML	1.690	0.89	2.301	0.89	1.580	0.91	2.147	0.91
	$Bayes^0$	1.672	0.89	2.252	0.90	1.571	0.90	2.125	0.92
	$Bayes^1$	1.600	0.90	2.021	0.90	1.477	0.93	2.003	0.95
(60, 30)	ML	1.671	0.90	2.287	0.90	1.504	0.91	2.103	0.93
	$Bayes^0$	1.654	0.90	2.214	0.89	1.501	0.93	2.097	0.95
	$Bayes^1$	1.583	0.92	2.003	0.93	1.425	0.96	1.981	0.94
(60, 60)	ML	1.625	0.91	2.264	0.90	1.451	0.93	2.001	0.92
	$Bayes^0$	1.618	0.92	2.211	0.91	1.444	0.93	1.972	0.96
	$Bayes^1$	1.521	0.91	1.987	0.92	1.362	0.93	1.841	0.95

Table 3. Mean and MSE of ML and Bayes (p^0, p^1) estimats at φ = {1.2, 2.0}

τ		τ = 0.5				τ = 1.0			
(κ_1, κ_2)		φ_1		φ_2		φ_1		φ_2	
		ME	MSE	ME	MSE	ME	MSE	ME	MSE
(30,30)	ML	1.454	0.451	2.389	1.741	1.401	0.370	2.274	0.941
	Bayes^0	1.422	0.436	2.355	1.711	1.398	0.381	2.285	0.925
	Bayes^1	1.388	0.254	2.265	1.621	1.300	0.215	2.224	0.874
(30, 40)	ML	1.441	0.422	2.342	1.688	1.374	0.341	2.225	0.825
	Bayes^0	1.408	0.418	2.327	1.691	1.366	0.335	2.216	0.888
	Bayes^1	1.362	0.223	2.225	1.594	1.301	0.201	2.194	0.719
(60, 30)	ML	1.447	0.418	2.335	1.675	1.298	0.307	2.199	0.842
	Bayes^0	1.403	0.414	2.328	1.688	1.289	0.301	2.192	0.800
	Bayes^1	1.354	0.220	2.214	1.591	1.255	0.185	2.100	0.684
(60, 60)	ML	1.412	0.380	2.286	1.601	1.200	0.288	2.171	0.784
	Bayes^0	1.385	0.371	2.274	1.592	1.187	0.275	2.154	0.777
	Bayes^1	1.311	0.189	2.187	1.503	1.155	0.165	2.077	0.562

Table 4. Mean length and CP of of ML and Bayes (p^0, p^1) estimats at φ = {0.6, 1.0}

τ		τ = 0.7				τ = 1.4			
(κ_1, κ_2)		φ_1		φ_2		φ_1		φ_2	
		ML	CP	ML	CP	ML	CP	ML	CP
(30,30)	ML	2.854	0.89	3.582	0.88	2.665	0.90	3.354	0.91
	Bayes^0	2.822	0.89	3.549	0.90	2.645	0.91	3.344	0.90
	Bayes^1	2.645	0.89	3.424	0.89	2.425	0.93	3.275	0.92
(30, 40)	ML	2.782	0.90	3.502	0.90	2.571	0.91	3.315	0.91
	Bayes^0	2.744	0.90	3.487	0.91	2.563	0.90	3.300	0.93
	Bayes^1	2.600	0.91	3.390	0.92	2.392	0.94	3.225	0.94
(60, 30)	ML	2.777	0.91	3.498	0.92	2.515	0.93	3.274	0.93
	Bayes^0	2.735	0.90	3.469	0.91	2.502	0.94	3.251	0.91
	Bayes^1	2.597	0.92	3.379	0.93	2.344	0.95	3.207	0.92
(60, 60)	ML	2.725	0.91	3.415	0.93	2.459	0.93	3.215	0.94
	Bayes^0	2.704	0.94	3.403	0.91	2.436	0.93	3.205	0.94
	Bayes^1	2.521	0.94	3.307	0.94	2.302	0.93	3.166	0.96

Data analysis

In this section, we consider the real data given in Hoel [30] in two group of the lifetime of strain male mice obtained from a laboratory experiment under radiation dose of 300*r* at an age of 5-6 weeks. In two group of data consider the case which failure happen with other causes of failure to reported in tab. 5. For easy computation the data given in tab. 5 dividing by 1000. The observed joint Type-I censoring data taken from the lines of production Λ_1 and Λ_2 given $\tau = 0.77$ and $\kappa_1 = 39$, $\kappa_1 = 37$ given in tab. 6. From the real data reported in tab. 6 we have

{r = 59, κ_{1r} = 39, and κ_{2r} = 20. Bayesian approach applied under non-informative prior information (mean $a_i = b_i = 0.0001$, i = 1, 2). The ML and Bayes estimate is compute for point and interval estimation and reported in tab. 7. Also, from the data reported in tab. 6, we observe that $\kappa_1 = \kappa_{1r} = 39$. Therefore, the Bayesian prediction of future $W_{r+}1$ described in case two (mean from the line Λ_2). Therefor, the Bayes point and interval predictive can be computed from eq. (44).

Table 5. Two lines of the real data presented by [30]

Λ_1	Other cases	40	42	51	62	163	179	206	222	228	249	252
		282	324	333	341	366	385	407	420	431	441	461
		462	482	517	517	524	564	567	586	619	620	621
		622	647	651	686	761	763					
Λ_2	Other cases	136	246	255	376	421	565	616	617	652	655	658
		660	662	675	681	734	736	737	757	769	777	800
		807	825	855	857	864	868	870	873	882	895	910
		934	942	1015	1019							

Table 6. The joint Type-I censoring data for given τ = 0.5

	0.04	0.042	0.051	0.062	0.136	0.163	0.179	0.206	0.222	0.228
	0.246	0.249	0.252	0.255	0.282	0.324	0.333	0.341	0.366	0.376
Data	0.385	0.407	0.42	0.421	0.431	0.441	0.461	0.462	0.482	0.517
	0.517	0.524	0.564	0.565	0.567	0.586	0.616	0.617	0.619	0.62
	0.621	0.622	0.647	0.651	0.652	0.655	0.658	0.66	0.662	0.675
	0.681	0.686	0.734	0.736	0.737	0.757	0.761	0.763	0.769	
Type	1	1	1	1	0	1	1	1	1	1
	0	1	1	0	1	1	1	1	1	0
	1	1	1	0	1	1	1	1	1	1
	1	1	1	0	1	1	0	0	1	1
	1	1	1	1	0	0	0	0	0	0
	0	1	0	0	0	0	1	1	0	

Table 7. The Point and corresponding 95% ML and Bayes interval estimate

	(.)ML	(.)B-SE	(.)B-LE		ACI	CI
			c = 2.0	c = –20		
φ_1	4.74535	4.7453	4.24726	5.43763	(3.256, 6.2347)	(3.37438, 6.34629)
φ_1	1.14353	1.14353	1.08274	1.21437	(0.6424, 1.6447)	(0.698499, 1.69648)

Conclusion

The problem of application of joint Type-I censoring scheme to built statistical inference of Rayleigh lifetime populations is discussed in this paper. Therefore, we formulated point estimators of the unknown parameters with ML and Bayes methods. In Bayesian approach, we used non-informative and informative prior information under symmetric and asymmetric loss function. We have also, constructed asymptotic confidence intervals and Bayes credible intervals.

All proposed method assessed and compared through Monte-Carlo simulation study. Finally, we are used the real data set to illustrate the proposed methods. Therefore, we can acceptable the proposed method and recommended it to applied in different branches specially products in competing durations. Also, the results can be utilized in the field of comparative life testing.

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