PHOTON STATISTICS AND NON-LOCAL PROPERTIES OF A TWO-QUBIT-FIELD SYSTEM IN THE EXCITED NEGATIVE BINOMIAL DISTRIBUTION

by

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In this paper, a quantum scheme for a two-qubit system (2QS) and field initially prepared in the excited negative binomial distribution is presented. The field photon statistics is detected from the evolution of the Mandel parameter, while the evolution of von Neumann entropy detects the nonlocal correlation between the 2QS and radiation field. The concurrence is used to detect the qubit-qubit entanglement during the time evolution. The dynamical properties of single-qubit and two-qubit quantum Fisher information are investigated. We visualize the number of photon excitations on the field in negative binomial states with influence of photon success probability. A connection is provided between the dynamical behaviors of these statistical quantities. We have found that the proposed quantities are strongly influenced by the number of excited photons of the field in negative binomial states and photon success probability.

Keywords: *single (two)-qubit quantum Fisher information, Mandel parameter, excited negative binomial distribution, 2QS, concurrence*

Introduction

Non-classical radiation field states such as number states, coherent states, and phase states play an important role in quantum optics and have been extensively studied [1, 2]. Barnett *et al*. [3] have used the negative binomial distribution (NBD) as an initial field state and investigated statistical properties of photons in the radiation field. The obtained results revealed similar properties when the operators of the creation and annihilation roles for the two field states were swapped. These states have also been found to have a strong squeezing effect and statistics considered as super-poissonian [4]. The NBD are also considered to have odd and even, superposed, deformed states and entangled as well as coherent states in non-linear case [5]. The estimation of odd and even binomial distributions (BD) has been investigated [6], and a quantum superposition of these states such as $(|\alpha\rangle \pm |\beta\rangle)/2$ has been introduced [7]. Non-linear coherent states, meanwhile, have piqued the interest of many researchers [8, 9].

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In recent years, significant progress has been made in applying non-classical and non-local properties of an initially excited field in quantum statistics and optics. Exciting the field state by adding photons to the optical photon distribution is crucial in quantum information processing and quantum statistics. Probability theory has been used to investigate non-classical states induced excitations of specific quantum states [10-12]. These states exhibit remarkable non-classical properties such as radiation field, squeezing, and sub-poissonian photon statistics. Several other excited quantum states have been studied as initial field distributions, for example odd and even, excited coherent states [13], non-linear coherent states added by deformed photon [14] and photon-subtracted squeezed thermal states [15].

A new non-local and non-classical property is obtained in a quantum system if an appropriate probability distribution of field photons is chosen. As a result, the quality and novelty of the obtained results relate to the field state setting. A study has been done on the relationship between the dynamical properties of the quantum Fisher information (QFI) and the tomographic entropy of a single qubit in an excited binomial state interacting with the initial field in the NBD [16]. Another study has been done on the effect of the field deformation parameter in an even binomial distribution of the field photon statistics of the, and on the statistical properties and non-local correlation between two qubits and an optical field [17]. Furthermore, the QFI has been applied in two modes of a Gaussian distribution to determine the statistical properties of the applied field. In addition, its measure of the amount of information that an observable extract contains about a parameter [18]. The Husimi Q-function and atomic version have been used to develop a new quantum formula for extropy [19]. Quantum extropy was successfully detected the non-local correlations of a system consisting of a qubit and inters field in the BD and NBD. Photon statistics and tomographic entropy have also been applied to a quantum scheme of a single qubit interacting with a radiation field in the even BD [20].

More information is obtained from quantum information technology, this advancement has raised awareness of non-local correlation. This phenomenon plays a crucial role in quantum estimation and quantum metrology [21-23]. The non-local correlation or entanglement is at the core of various quantum technologies [24-26]. This article investigates the statistical and non-classical properties of the interaction between two qubits and field initially in the excited NBD.

Model of a two-qubit system and field in the ENBD

Here by using the experimental set-up presented in [27], we assume that the photons of the field are prepared separately with an NBD and ENBD at $t = 0$ and then via a very small aperture is delivered into the cavity. Then, from the photonic field the initial state is:

$$\rho_{RF}\left(0\right)=\left|p;M\right\rangle\left\langle p;M\right| \tag{1}$$

where

$$\left|\, p;M\right\rangle_{NBS}=\sum\nolimits_{n=0}^{\infty}\sqrt{B_n\left(p,M\right)}\,\left|\, n\right\rangle \tag{2}$$

with

$$B_n\left(p;M\right)=\left(M+n-1\; n\right)p^n\left(1-p\right)^M\,,\quad n=0,1,\ldots\ldots \tag{3}$$

where M is the fixed positive integer and p – the success probability satisfying $0 < p < 1$. The photon distribution of the states in eq. (2) is $\langle p;M\rangle = B_n(p;M)$, which is the NBD. For a field described by the annihilation and creation operators $\hat{c}$ and $\hat{c}^\dagger$, respectively, the ENBD for a number of excited photons k is $|p;M,k\rangle_{\text{ENBS}} = \hat{c}^\dagger|p;M\rangle_{\text{NBS}}$

The Hamiltonian describing the interaction of each qubit of the 2QS has an upper (lower) state $|u_j\rangle(|l_j\rangle)$ coupled to a one-mode field as:

$$\hat{H}_I = \sum_{j=1}^{2} \varepsilon_j \left(\hat{c}_j |u_j\rangle\langle l_j| + \hat{c}_j^\dagger |l_j\rangle\langle u_j|\right) \tag{4}$$

where ε_j is the 2QS-F coupling, which is taken for the two indicial qubits as $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$. The wave function can be obtained as a function of the scaled time $T = \varepsilon t$ as:

$$|\Upsilon(T)\rangle = \exp\left[-i\hat{H}_I t\right]|\Upsilon(0)\rangle \tag{5}$$

The 2Qs-F system wave function at $t = 0$ is assumed to be:

$$|\Upsilon_{2Qs-F}(0)\rangle = |\Upsilon_{2Qs}(0)\rangle \otimes |\Upsilon_F(0)\rangle = \frac{1}{\sqrt{2}}\left(|u_1u_2\rangle + |l_1l_2\rangle\right) \otimes |p;M,k\rangle \tag{6}$$

At any time $T > 0$, the final 2QS-F state is formulated as:

$$|\Upsilon_{2Qs-F}(T)\rangle = \sum_{n=0}^{\infty} \begin{pmatrix} \Gamma_1(n,T)|n,u_1u_2\rangle + \Gamma_2(n,T)|n+1,u_1l_2\rangle + \\ +\Gamma_3(n,T)|n+1,u_2l_1\rangle + \Gamma_4(n,T)|n+2,l_1l_2\rangle \end{pmatrix} \tag{7}$$

From the total density matrix

$$\varrho_{2Qs-F}(T) = |\Upsilon_{2Qs-F}(T)\rangle\langle\Upsilon_{2Qs-F}(T)|$$

the reduced density matrix of the two-qubit field contributed by

$$\varrho_F(T)\left(\varrho_{2Qs-F}(T)\right)$$

is obtained as:

$$\varrho_{2Qs}(T) = Tr_F\left\{\varrho_{2Qs-F}(T)\right\} = \sum_{j=1}^{4}\sum_{m=1}^{4} \varrho_{jm|j\rangle\langle m|} \tag{8}$$

$$\varrho_F(T) = Tr_{2Qs}\left\{\varrho_{2Qs-F}(T)\right\} = \sum_i \varrho_i |i\rangle\langle i| \tag{9}$$

The density matrix elements in eq. (8) are used to evaluate the other dynamical quantities of the qubit-qubit entanglement for single-qubit QFI (SQQFI) and two-qubit QFI (TQQFI). Via the evolution of the Mandel parameter in terms of the field density matrix in eq. (9) the statistics of the field photons is investigated.

Time evolution and properties of statistical quantities

The photon statistics of a field initially in NBD and ENBD will be studied on the basis of the dynamics of the Mandel parameter, Q_M. The concurrence, C_{QQ}, determines how the two qubits-field entanglement when the two qubits in maximally entangled states at $T = 0$ and the field follow the NBD and ENBD. We also examine the influence of excited photons in negative binomial states and the NBD photon success probability on the non-classical properties of the field and the non-local correlation between the two qubits and the field, and the effect of time evolution of the SQQFI and TQQFI contributed by F_{SQ} and F_{2Qs}, respectively.

Mandel parameter dynamics and field photon statistics

The Mandel parameter applied to measure the non-classicality of the field or its photon statistics within the interaction time. Mandel [28, 29] first attempted to highlight the non-classicality of a quantum state. The Mandel parameter can be formulated in terms of the mean photon number

$$Tr\left(\hat{c}^{\dagger}\hat{c}\right)=\left\langle \Upsilon\left(t\right)\right\rangle =\left\langle \hat{n}\left(T\right)\right\rangle$$

and the corresponding variance

$$\left\langle \hat{n}^{2}\left(T\right)\right\rangle -\left(\left\langle \hat{n}\left(T\right)\right\rangle\right)^{2}$$

as:

$$Q_{M}=\frac{Tr\left(\hat{c}^{\dagger}\hat{c}\right)^{2}}{Tr\left(\hat{c}^{\dagger}\hat{c}\right)}-Tr\left(\hat{c}^{\dagger}\hat{c}\right)-1 \tag{10}$$

The field photon statistics are indicated by the Mandel parameter as: The field is governed by super-Poissonian statistics for $Q_M > 0$, Poissonian statistics for $Q_M = 0$, and sub-Poissonian statistics for $Q_M < 0$.

Qubit–qubit entanglement based on concurrence

Quantum correlation, especially entanglement, has potential applications in information sciences and quantum algorithms [30-34]. In this context, the concurrence is used to measure nonlocal correlation between the two qubits in the 2QS [35, 36]. The concurrence is defined as [37]:

$$C_{QQ}\coloneqq\max\left\{0,\mu_{1}-\mu_{2}-\mu_{3}-\mu_{4}\right\} \tag{11}$$

where μ_j defines the eigenvalues given in decreasing order of the matrix $\varrho_{QQ}\tilde{\varrho}_{QQ}$ where $\tilde{\varrho}_{Q_1Q_2}$ is the density matrix related to the Pauli matrix σ_Y and ϱ_{QQ}^{*} (the conjugate of ϱ_{QQ}) is defined as:

$$\tilde{\varrho}_{Q_1Q_2}\coloneqq\left(\sigma_{Y}\otimes\sigma_{Y}\right)\varrho_{QQ}^{*}\left(\sigma_{Y}\otimes\sigma_{Y}\right) \tag{12}$$

The two qubits are in a pure state for $C_{QQ} = 0$, while the maximally entangled state is obtained for $C_{QQ} = 1$.

Single- and two-qubit QFI based on parameter estimation

The TQQFI relies on the estimator parameter $\hat{\vartheta}=\beta$ which is motivated by the phase shift parameter according to

$$U_{\hat{\vartheta}}=\frac{1}{\sqrt{2}}\left[\exp\left(i\beta\right)\left|uu\right\rangle\left\langle uu\right|+\left|ll\right\rangle\left\langle ll\right|\right]$$

Thus, the optimal target state is

$$U_{\hat{\vartheta}}\left|\Upsilon\left(0\right)\right.$$

according to:

$$\left|\Upsilon\left(0\right)\right\rangle_{\text{opt}}=\frac{1}{\sqrt{2}}\left[\exp\exp\left(i\beta\right)\left|uu\right\rangle\left\langle uu\right|+\left|ll\right\rangle\left\langle ll\right|\right]\otimes\left|\Upsilon_{F}\left(0\right)\right\rangle \tag{13}$$

The QFI of a given structure with an unknown parameter, β, is formulated as [38-40]:

$$F_{2SQ}(T) = tr\left\{ \varrho_{QQ}(\beta,T) R_{2Qs}(\beta,T)^2 \right\} \tag{14}$$

where the two-qubit density operator ϱ_{QQ} is related to the symmetric logarithmic derivative operator $R(\beta,T)$ [39, 40] via:

$$2\frac{\partial R_{2Qs}(\beta,T)}{\partial T} = \varrho_{QQ}(\beta,T) R_{2Qs}(\beta,T) + R_{2Qs}(\beta,T)\ \varrho_{QQ}(\beta,T) \tag{15}$$

In terms of the single-qubit density matrix $\varrho_{SQ} = \varrho_{QA} = tr_B\{\varrho_{QQ}\}$, the SQQFI is expressed as:

$$F_{SQ}(T) = tr\left\{ \varrho_{SQ}(\beta,T) R_{SQ}(\beta,T)^2 \right\} \tag{16}$$

with

$$2\frac{\partial R_{SQ}(\beta,T)}{\partial T} = \varrho_{QQ}(\beta,T) R_{SQ}(\beta,T) + R_{SQ}(\beta,T)\ \varrho_{QQ}(\beta,T) \tag{17}$$

where Q_A and Q_B indicate the first and second qubit respectively.

On the basis of eqs. (10), (11), (14), and (16) we plot the Mandel parameter, concurrence, TQQFI, and SQQFI respectively in fig. 1. For the field photon statistics, the dynamical behavior is analyzed when the 2QS interacts with the field in the NBD and ENBD with doubly success probability, p. As shown in fig. 1(a), $Q_M > 0$ for an NBD field with a success probability p =1/4, which means that the field has a sub-poissonian distribution during the time evolution. This completely changes when we increase the success probability to p = 3/4 where the field is super-poissonian, fig. 1(e). Comparing figs. 1(a) and 1(e) shows that the photon statistics of the field is very sensitive to the success probability, p. In figs. 1(b) and 1(f), the entanglement between the qubits in the 2QS is studied via the evolution of the concurrence C_{QQ} for a field initially in the NBD for a single-photon transition. As seen from fig. 1(b), the concurrence starts from its maximum value, which means that the two qubits are in a maximally entangled state. The concurrence decreases with time, and its behavior becomes chaotic and irregular when a significant amount of time passes. These results show that small values of the success probability lead to weak entanglement between the two qubits and the field. Furthermore, the maximum value does not exceed 0.5 with heavy oscillations. In fig. 1(f), the qubit-qubit entanglement is enhanced by increasing the success probability to p = 3/4 and reaches its maximum value periodically. These findings show that the concurrence or the entanglement between the two qubits is strongly connected with the field distribution parameters.

Figures 1(c) and 1(g) depict the evolution of the TQQFI for p = 1/4 and p = 3/4, respectively. The initial optimal state corresponds to F_{2QS} = 1. We investigate the dynamics of the TQQFI when the 2QS starts in the maximally entangled state for each fixed value of the NBD success probability, p. The TQQFI starts with the maximum value and gradually decreases with time. After a short time, the TQQFI drops to zero and is revived as time goes on with some monotonic behavior and with qubit–qubit entanglement. The function F_{2QS} also exhibits interesting behavior as the success probability increases, figs. 1(d) and 1(h).

To examine the influence of the excited photons on the field distribution, we plot the dynamical behavior of the proposed quantities for five-photon excitation in fig. 2. Comparing figs. 1 and 2 shows that the statistical properties of the field are affected by the number of added photons as seen in figs. 2(a) and 2(e). The field exhibits sub-poissonian statistics for small and large values of the success probability. The figures also show that the number of excited photons has a small effect on the behavior of other quantities except for the Mandel parameter.

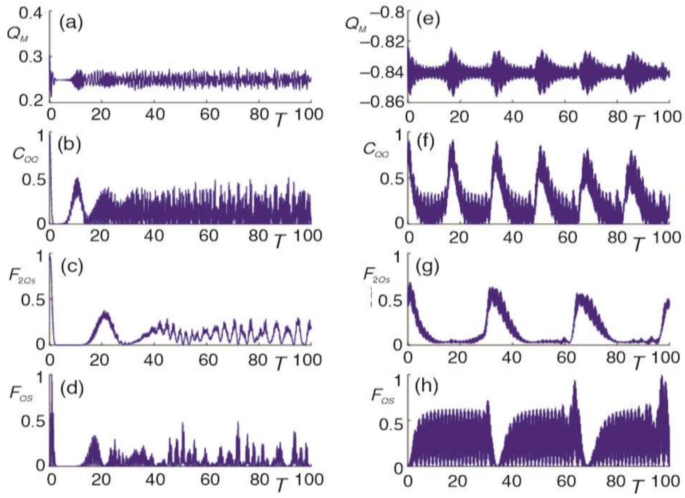

Figure 1. Time evolution of; (a) Mandel parameter Q_M, (b) concurrence, C_{QQ}, (c) TQQFI F_{2Qs}, and (d) SQQFI F_{SQ} for the 2QS interaction with the radiation field following the NBD with success probability M = 30 and p = 1/4; panels (e)-(h) are the same as (a)-(d) but for success probability p = 3/4

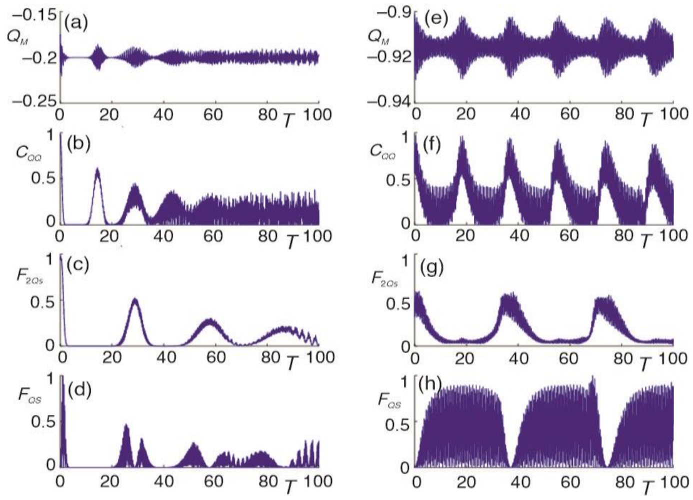

Figure 2. Effects of five-photon excitation of the field state on the time evolution of; (a) Mandel parameter Q_M, (b) concurrence Q_{QQ}, (c) TQQFI F_{2Qs}, and (d) SQQFI F_{SQ} for 2QS interaction with the radiation field following the NBD with M = 30 and p = 1/4; panels (e)-(h) are the same as (a)-(d) but for success probability p = 3/4

Conclusion

In this paper, the NBD and ENBD were used as initial states for a radiation field. We used the Mandel parameter to quantify the statistical properties of the field during the interaction time. The concurrence was considered as a tool for providing the how the entanglement or nonlocal correlations between the two qubits. The time evolution of the SQQFI and TQQFI of the phase shift estimator is based on the optimal initial state (maximally entangled state). We also examined the effect of low and high values of the success probability on the dynamical behavior of the concurrence for the SQQFI, TQQFI, and photon statistics. The results show that the statistical and nonclassical properties of the field is very sensitive to the success probability and number of excited field photons. Additionally, the concurrence, SQQFI, and TQQFI were more strongly affected by the success probability than the number of excited photons.

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