SOME EXACT SOLUTIONS OF COUPLED NON-LINEAR HELMHOLTZ EQUATION

by

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In this study, we obtained some exact solutions of the coupled non-linear Helmholtz equation through the sub-equation method. The solutions were hyperbolic and trigonometric. We observed through MATHEMATICA 11.2 that these solutions provided the equations, and we presented graphs of some solutions in the last section.

Key words: *coupled non-linear Helmholtz equation, sub-equation method, exact solutions*

Introduction

The PDE equations represent the relationships between several partial derivatives of a multivariable function. They are often used in mathematics-based sciences such as physics and engineering. They establish the foundation of modern scientific logic of several concepts such as sound, heat, diffusion, electrostatics, electrodynamics, hydrodynamics, elasticity, and quantum mechanics. We focused on non-linear PDE in this study.

Several methods have been developed in recent years and is still being developed to provide solutions to such equations. Some of these methods are presented in [1-17]. In most of these methods, non-linear PDE are solved through transforming them into ODE using a transformation. In this study, we obtained some exact solutions of the coupled non-linear Helmholtz equation [13] through the sub-equation method. This method was developed by Zayed *et al.*, [1].

Application

Let us consider the coupled non-linear Helmholtz equation [13]:

$$\begin{gathered} iu_t + \alpha u_{tt} + \frac{1}{2}u_{xx} + s_1 |u|^2 u + s_2 |v|^2 u = 0 \\ iv_t + \alpha v_{tt} + \frac{1}{2}v_{xx} + s_1 |u|^2 v + s_2 |v|^2 v = 0, \quad \left(i = \sqrt{-1}\right) \end{gathered} \tag{1}$$

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If the following transformation is applied to eq. (1):

$$u(x,t)=e^{i(\theta)}U(\xi), \quad v(x,t)=e^{i(\theta)}V(\xi) \tag{2}$$

where $\xi=(m_3x-m_3m_4t), \quad \theta=(-m_1x+m_2t+\kappa)$.

It transforms into the following ODE. The real and imaginary parts of the equation are separated as:

$$\begin{gathered}
\left(-m_3^2-2\alpha m_3^2m_4^2\right)U''+\left(m_1^2+2m_2+2\alpha m_2^2\right)U+\left(-2s_1U^3-2s_2UV^2\right)=0\\
\left(2m_1m_3+2m_3m_4+4\alpha m_2m_3m_4\right)U'=0\\
\left(-m_3^2-2\alpha m_3^2m_4^2\right)V''+\left(m_1^2+2m_2+2\alpha m_2^2\right)V+\left(-2s_1U^2V-2s_2V^3\right)=0\\
\left(2m_1m_3+2m_3m_4+4\alpha m_2m_3m_4\right)V'=0
\end{gathered} \tag{3}$$

Here, if we apply the transformation of $V = \beta U$ to eq. (3), we obtain the following form of the equation:

$$\Psi_1U+\Psi_2U''+\Psi_3U^3=0 \tag{4}$$

$$i\left(2m_1m_3+2m_3m_4+4\alpha m_2m_3m_4\right)U'=0 \tag{5}$$

We obtain,

$$m_4=\frac{-m_1}{1+2\alpha m_2}$$

in eq. (5). In addition, $\boldsymbol{\Psi}_1=(m_1^2+2m_2+2\alpha m_2^2)$, $\boldsymbol{\Psi}_2=(-m_3^2-2\alpha m_3^2m_4^2)$, $\boldsymbol{\Psi}_3=(-2s_1-2s_2\beta^2)$. In eq. (4), if U'' and U^3 are balanced, we get n = 2 and m = 1. According to the sub-equation method [1], the solution function is as follows:

$$U(\zeta)=\frac{\alpha_0+\alpha_1\Omega(\xi)+\alpha_2\Omega^2(\xi)}{B_0+B_1\Omega(\xi)} \tag{6}$$

α_0, α_1, α_2, B_0 ve B_1 obtained in the previous solution function are constants. If solution (6) are substituted in eq. (4), the system of algebraic equations below is obtained:

$$\begin{gathered}
B_0^2\alpha_0\boldsymbol{\Psi}_1-prB_0B_1\alpha_0\boldsymbol{\Psi}_2+2p^2B_1^2\alpha_0\boldsymbol{\Psi}_2+prB_0^2\alpha_1\boldsymbol{\Psi}_2-\\
-2p^2B_0B_1\alpha_1\boldsymbol{\Psi}_2+2p^2B_0^2\alpha_2\boldsymbol{\Psi}_2+\alpha_0^3\boldsymbol{\Psi}_3=0\\
2B_0B_1\alpha_0\boldsymbol{\Psi}_1+B_0^2\alpha_1\boldsymbol{\Psi}_1-2pqB_0B_1\alpha_0\boldsymbol{\Psi}_2-r^2B_0B_1\alpha_0\boldsymbol{\Psi}_2+3prB_1^2\alpha_0\boldsymbol{\Psi}_2+\\
+2pqB_0^2\alpha_1\boldsymbol{\Psi}_2+r^2B_0^2\alpha_1\boldsymbol{\Psi}_2-3prB_0B_1\alpha_1\boldsymbol{\Psi}_2+6prB_0^2\alpha_2\boldsymbol{\Psi}_2+3\alpha_0^2\alpha_1\boldsymbol{\Psi}_3=0\\
B_1^2\alpha_0\boldsymbol{\Psi}_1+2B_0B_1\alpha_1\boldsymbol{\Psi}_1+B_0^2\alpha_2\boldsymbol{\Psi}_1-3qrB_0B_1\alpha_0\boldsymbol{\Psi}_2+2pqB_1^2\alpha_0\boldsymbol{\Psi}_2+r^2B_1^2\alpha_0\boldsymbol{\Psi}_2+\\
+3qrB_0^2\alpha_1\boldsymbol{\Psi}_2-2pqB_0B_1\alpha_1\boldsymbol{\Psi}_2-r^2B_0B_1\alpha_1\boldsymbol{\Psi}_2+8pqB_0^2\alpha_2\boldsymbol{\Psi}_2+4r^2B_0^2\alpha_2\boldsymbol{\Psi}_2+\\
+3prB_0B_1\alpha_2\boldsymbol{\Psi}_2+3\alpha_0\alpha_1^2\boldsymbol{\Psi}_3+3\alpha_0^2\alpha_2\boldsymbol{\Psi}_3=0\\
B_1^2\alpha_1\boldsymbol{\Psi}_1+2B_0B_1\alpha_2\boldsymbol{\Psi}_1-2q^2B_0B_1\alpha_0\boldsymbol{\Psi}_2+qrB_1^2\alpha_0\boldsymbol{\Psi}_2+2q^2B_0^2\alpha_1\boldsymbol{\Psi}_2-qrB_0B_1\alpha_1\boldsymbol{\Psi}_2+\\
+10qrB_0^2\alpha_2\boldsymbol{\Psi}_2+6pqB_0B_1\alpha_2\boldsymbol{\Psi}_2+3r^2B_0B_1\alpha_2\boldsymbol{\Psi}_2+prB_1^2\alpha_2\boldsymbol{\Psi}_2+\alpha_1^3\boldsymbol{\Psi}_3+6\alpha_0\alpha_1\alpha_2\boldsymbol{\Psi}_3=0\\
B_1^2\alpha_2\boldsymbol{\Psi}_1+6q^2B_0^2\alpha_2\boldsymbol{\Psi}_2+9qrB_0B_1\alpha_2\boldsymbol{\Psi}_2+2pqB_1^2\alpha_2\boldsymbol{\Psi}_2+r^2B_1^2\alpha_2\boldsymbol{\Psi}_2+3\alpha_1^2\alpha_2\boldsymbol{\Psi}_3+3\alpha_0\alpha_2^2\boldsymbol{\Psi}_3=0\\
6q^2B_0B_1\alpha_2\boldsymbol{\Psi}_2+3qrB_1^2\alpha_2\boldsymbol{\Psi}_2+3\alpha_1\alpha_2^2\boldsymbol{\Psi}_3=0,\ 2q^2B_1^2\alpha_2\boldsymbol{\Psi}_2+\alpha_2^3\boldsymbol{\Psi}_3=0
\end{gathered}$$

Through the solution of this system, the following coefficients are found:

$$q\boldsymbol{\Psi}_2\boldsymbol{\Psi}_3 \neq 0,\ p = \frac{-2\boldsymbol{\Psi}_1 + r^2\boldsymbol{\Psi}_2}{4q\boldsymbol{\Psi}_2},\ r\alpha_1 \neq 0,\ \alpha_2 = \frac{2q\alpha_1}{r},\ B_1 = \pm\frac{i\alpha_2\sqrt{\boldsymbol{\Psi}_3}}{\sqrt{2}q\sqrt{\boldsymbol{\Psi}_2}} \tag{7}$$

If the coefficients found in (7) are substituted in (6), and then the obtained solutions are replaced in solution (2), the following solutions of eq. (1) is obtained:

$$u_1(x,t) = -\frac{i\boldsymbol{\Psi}_1\sqrt{\boldsymbol{\Psi}_2}\left(\begin{array}{c}-2\cosh\left(\sqrt{2}\xi\sqrt{\dfrac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)A_1\sqrt{\dfrac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}} - 2\sinh\left(\sqrt{2}\xi\sqrt{\dfrac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)A_2\sqrt{\dfrac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}} + \\ +\sqrt{\dfrac{4A_1^2\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2} - \dfrac{4A_2^2\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2} + \dfrac{\mu^2\boldsymbol{\Psi}_2}{\boldsymbol{\Psi}_1}}\end{array}\right)e^{i(\theta)}}{\left(2\sinh\left(\sqrt{2}\xi\sqrt{\dfrac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)A_1\boldsymbol{\Psi}_1 + 2\cosh\left(\sqrt{2}\xi\sqrt{\dfrac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)A_2\boldsymbol{\Psi}_1 - \mu\boldsymbol{\Psi}_2\right)\sqrt{\boldsymbol{\Psi}_3}} \tag{8}$$

$$v_1(x,t) = -\frac{\beta i\boldsymbol{\Psi}_1\sqrt{\boldsymbol{\Psi}_2}\left(\begin{array}{c}-2\cosh\left(\sqrt{2}\xi\sqrt{\dfrac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)A_1\sqrt{\dfrac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}} - 2\sinh\left(\sqrt{2}\xi\sqrt{\dfrac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)A_2\sqrt{\dfrac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}} + \\ +\sqrt{\dfrac{4A_1^2\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2} - \dfrac{4A_2^2\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2} + \dfrac{\mu^2\boldsymbol{\Psi}_2}{\boldsymbol{\Psi}_1}}\end{array}\right)e^{i(\theta)}}{\left(2\sinh\left(\sqrt{2}\xi\sqrt{\dfrac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)A_1\boldsymbol{\Psi}_1 + 2\cosh\left(\sqrt{2}\xi\sqrt{\dfrac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)A_2\boldsymbol{\Psi}_1 - \mu\boldsymbol{\Psi}_2\right)\sqrt{\boldsymbol{\Psi}_3}} \tag{9}$$

In solutions (8) and (9), if we assume that $A_1 \neq 0$, $A_2 = 0$, $\mu = 0$, we obtain the following solutions of the equation:

$$u_2(x,t) = \frac{i\,\mathrm{csch}\left(\sqrt{2}\xi\sqrt{\dfrac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)\left(\cosh\left(\sqrt{2}\xi\sqrt{\dfrac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)A_1\sqrt{\dfrac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}} - \sqrt{\dfrac{A_1^2\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)\sqrt{\boldsymbol{\Psi}_2}\,e^{i(\theta)}}{A_1\sqrt{\boldsymbol{\Psi}_3}} \tag{10}$$

$$v_2(x,t) = \frac{\beta i\,\mathrm{csch}\left(\sqrt{2}\xi\sqrt{\dfrac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)\left(\cosh\left(\sqrt{2}\xi\sqrt{\dfrac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)A_1\sqrt{\dfrac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}} - \sqrt{\dfrac{A_1^2\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)\sqrt{\boldsymbol{\Psi}_2}\,e^{i(\theta)}}{A_1\sqrt{\boldsymbol{\Psi}_3}} \tag{11}$$

Again, if we assume in solutions (8) and (9) that $A_1 = 0$, $A_2 \neq 0$, $\mu = 0$, we obtain different solutions of the equation as follows:

$$u_3(x,t) = \frac{i\,\mathrm{sech}\left(\sqrt{2}\xi\sqrt{\dfrac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)\left(\sinh\left(\sqrt{2}\xi\sqrt{\dfrac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)A_2\sqrt{\dfrac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}} - \sqrt{-\dfrac{A_2^2\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)\sqrt{\boldsymbol{\Psi}_2}\,e^{i(\theta)}}{A_2\sqrt{\boldsymbol{\Psi}_3}} \tag{12}$$

$$v_3(x,t)=\frac{\beta i\,\mathrm{sech}\left(\sqrt{2}\xi\sqrt{\frac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)\left(\sinh\left(\sqrt{2}\xi\sqrt{\frac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)A_2\sqrt{\frac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}-\sqrt{-\frac{A_2^2\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)\sqrt{\boldsymbol{\Psi}_2}\,e^{i(\theta)}}{A_2\sqrt{\boldsymbol{\Psi}_3}} \tag{13}$$

in the solutions above, $\lambda=(4pq-r^2)<0,\ \sigma_1=(A_1^2-A_2^2),\ (\lambda^2\sigma_1+\mu^2)>0$:

$$u_4(x,t)=-\frac{i\boldsymbol{\Psi}_1\sqrt{\boldsymbol{\Psi}_2}\left(\begin{array}{c}-2\cos\left(\sqrt{2}\xi\sqrt{-\frac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)A_1\sqrt{-\frac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}+2\sin\left(\sqrt{2}\xi\sqrt{-\frac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)A_2\sqrt{-\frac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}+\\+\sqrt{-\frac{4A_1^2\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}-\frac{4A_2^2\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}+\frac{\mu^2\boldsymbol{\Psi}_2}{\boldsymbol{\Psi}_1}}\end{array}\right)e^{i(\theta)}}{\left(2\sin\left(\sqrt{2}\xi\sqrt{-\frac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)A_1\boldsymbol{\Psi}_1+2\cos\left(\sqrt{2}\xi\sqrt{-\frac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)A_2\boldsymbol{\Psi}_1-\mu\boldsymbol{\Psi}_2\right)\sqrt{\boldsymbol{\Psi}_3}} \tag{14}$$

$$v_4(x,t)=-\frac{\beta i\boldsymbol{\Psi}_1\sqrt{\boldsymbol{\Psi}_2}\left(\begin{array}{c}-2\cos\left(\sqrt{2}\xi\sqrt{-\frac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)A_1\sqrt{-\frac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}+2\sin\left(\sqrt{2}\xi\sqrt{-\frac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)A_2\sqrt{-\frac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}+\\+\sqrt{-\frac{4A_1^2\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}-\frac{4A_2^2\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}+\frac{\mu^2\boldsymbol{\Psi}_2}{\boldsymbol{\Psi}_1}}\end{array}\right)e^{i(\theta)}}{\left(2\sin\left(\sqrt{2}\xi\sqrt{-\frac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)A_1\boldsymbol{\Psi}_1+2\cos\left(\sqrt{2}\xi\sqrt{-\frac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)A_2\boldsymbol{\Psi}_1-\mu\boldsymbol{\Psi}_2\right)\sqrt{\boldsymbol{\Psi}_3}} \tag{15}$$

In solutions (14) and (15), if we consider that $A_1 \neq 0$, $A_2 \neq 0$, $\mu = 0$, the following solutions of the equation are obtained:

$$u_5(x,t)=\frac{i\csc\left(\sqrt{2}\xi\sqrt{-\frac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)\left(\cos\left(\sqrt{2}\xi\sqrt{-\frac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)A_1\sqrt{-\frac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}-\sqrt{-\frac{A_1^2\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)\sqrt{\boldsymbol{\Psi}_2}\,e^{i(\theta)}}{A_1\sqrt{\boldsymbol{\Psi}_3}} \tag{16}$$

$$v_5(x,t)=\frac{\beta i\csc\left(\sqrt{2}\xi\sqrt{-\frac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)\left(\cos\left(\sqrt{2}\xi\sqrt{-\frac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)A_1\sqrt{-\frac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}-\sqrt{-\frac{A_1^2\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)\sqrt{\boldsymbol{\Psi}_2}\,e^{i(\theta)}}{A_1\sqrt{\boldsymbol{\Psi}_3}} \tag{17}$$

Again in solutions (14) and (15), if $A_1 = 0$, $A_2 \neq 0$, $\mu = 0$, we obtain the following different solutions of the equation:

$$u_6(x,t)=-\frac{i\sec\left(\sqrt{2}\xi\sqrt{-\frac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)\left(\sin\left(\sqrt{2}\xi\sqrt{-\frac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)A_2\sqrt{-\frac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}+\sqrt{-\frac{A_2^2\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)\sqrt{\boldsymbol{\Psi}_2}\,e^{i(\theta)}}{A_2\sqrt{\boldsymbol{\Psi}_3}} \tag{18}$$

$$v_6\left(x,t\right)=-\frac{\beta i\sec\left(\sqrt{2}\xi\sqrt{-\frac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)\left(\sin\left(\sqrt{2}\xi\sqrt{-\frac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)A_2\sqrt{-\frac{\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}+\sqrt{-\frac{A_2^2\boldsymbol{\Psi}_1}{\boldsymbol{\Psi}_2}}\right)\sqrt{\boldsymbol{\Psi}_2}\,e^{i(\theta)}}{A_2\sqrt{\boldsymbol{\Psi}_3}} \tag{19}$$

in the previous solutions:

$$\xi=\left(m_3x-m_3m_4t\right),\ \theta=\left(-m_1x+m_2t+\kappa\right),\ \lambda=\left(4pq-r^2\right)>0$$

$$\sigma_2=\left(A_1^2+A_2^2\right),\ \left(\lambda^2\sigma_2-\mu^2\right)>0,\ \boldsymbol{\Psi}_1=\left(m_1^2+2m_2+2\alpha m_2^2\right)$$

$$\boldsymbol{\Psi}_2=\left(-m_3^2-2\alpha m_3^2m_4^2\right),\ \boldsymbol{\Psi}_3=\left(-2s_1-2s_2\beta^2\right)$$

Graphics

In this section, graphs of some solutions found are presented. In drawing 3-D graphic, 2-D graphic, and contour graphic, the range of $-3\le x\le 3$, $-3\le t\le 3$ for solution (8) and $-5\le x\le 5$, $-5\le t\le 5$ for Solution (10) and (14) were used.

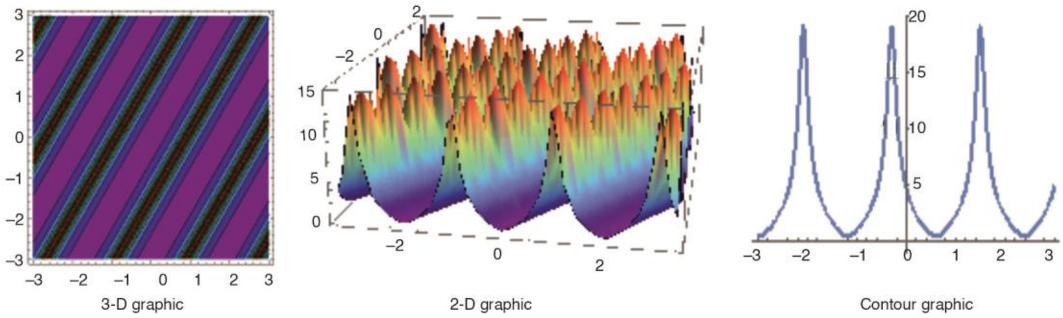

Figure 1. The 3-D surfaces of solution (8) for:
$m_1=-3$, $m_2=1$, $m_3=1$, $s_1=-2$, $s_2=1$, $A_1=3$, $A_2=1$, $\alpha=2$, $\beta=1$
The 2-D surfaces of solution (8) for:
$m_1=-3$, $m_2=1$, $m_3=1$, $s_1=-2$, $s_2=1$, $A_1=3$, $A_2=1$, $\alpha=2$, $\mu=2$, $\beta=1$, $t=1$
The Contour surfaces of solution (8) for:
$m_1=-3$, $m_2=1$, $m_3=1$, $s_1=-2$, $s_2=1$, $A_1=3$, $A_2=1$, $\alpha=2$, $\mu=2$, $\beta=1$

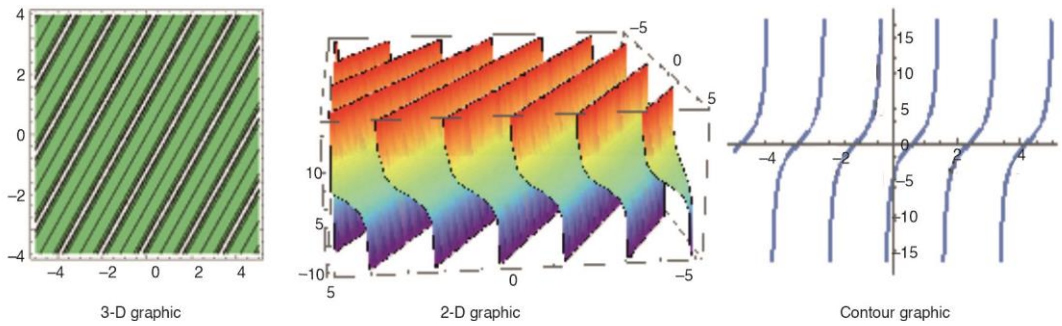

Figure 2. The 3-D surfaces of solution (10) for:
$m_1=-3$, $m_2=1$, $m_3=1$, $s_1=-2$, $s_2=1$, $A_1=1$, $\alpha=2$, $\beta=1$
The 2-D surfaces of solution (10) for:
$m_1=-3$, $m_2=1$, $m_3=1$, $s_1=-2$, $s_2=1$, $\alpha=2$, $\mu=2$, $\beta=1$, $t=1$
The Contour surfaces of solution (10) for:
$m_1=-3$, $m_2=1$, $m_3=1$, $s_1=-2$, $s_2=1$, $\alpha=2$, $\beta=1$

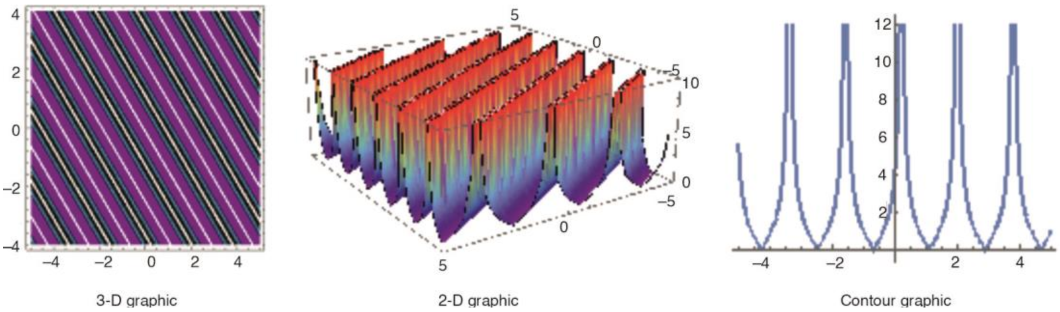

Figure 3. The 3-D surfaces of solution (14) for:
$\boldsymbol{m_1 = 3, m_2 = 1, m_3 = 1, s_1 = 2, s_2 = 1, A_1 = 3, A_2 = 1, \alpha = 2, \beta = 1, \mu = 2}$
The 2-D surfaces of solution (14) for:
$\boldsymbol{m_1 = 3, m_2 = 1, m_3 = 1, s_1 = 2, s_2 = 1, A_1 = 3, A_2 = 1, \alpha = 2, \mu = 2, \beta = 1, t = 1}$
The Contour surfaces of solution (14) for:
$\boldsymbol{m_1 = 3, m_2 = 1, m_3 = 1, s_1 = 2, s_2 = 1, A_1 = 3, A_2 = 1, \alpha = 2, \beta = 1, \mu = 2}$

Conclusion

Consequently, we obtained some exact solutions of the coupled nonlinear Helmholtz equation through the sub-equation method. The obtained solutions were hyperbolic and trigonometric. We observed using MATHEMATICA 11.2 that the solutions provided the equations. In addition, we presented the graphics performance of some of the obtained solutions. This method has recently been used to obtain exact solutions of nonlinear partial differential equations.

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