EXP-FUCTION METHOD FOR EXACT SOLUTIONS OF SOME NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

by

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In this study, we have obtained the exact solutions of (2+1) and (3+1)-D constant coefficient KdV equations by applying the exponential function method. These exact solutions we find are in the form of an exponential function. In addition, we have seen that these solutions provide the equations by using MATHEMATICA 11.3 program. Apart from that, we have shown the graphics performance of some of the solutions found.

Key words*: (2+1)-D constant coefficient KdV equation, exact solution, (3+1)-D constant coefficient KdV equation, exponential function method*

Introduction

Non-linear PDE have an important place in applied mathematics and physics. These equations are mathematical models of the physical phenomenon that occurs in engineering, chemistry, biology, mechanics, and physics. It is very important to have knowledge about the solutions of mathematical models. Solving these equations is necessary to better understand the mechanisms of mathematical models. Therefore, it has an important role in obtaining analytical solutions of non-linear PDE in applied sciences. Recently, solving these equations has become attractive. For this reason, some methods have been developed by scientists. Some of these are: Hirota method [1], Backlund transform [2], Cole-hopf transformation method [3], Generalized Miura Transform [4], inverse scattering method [5], Darboux transform [6], Painleve method [7], homogeneous balance method [8], similarity reduction method [9], and sine cosine method [10].

Besides these methods, there are many methods based on the use of an auxiliary equation. First, non-linear PDE are converted into non-linear ODE using these methods. Secondly, the obtained non-linear ODE are solved with the help of the auxiliary equation.

These methods can be listed tanh function method [11], extended tanh function method [12], modified extended tanh method [13], improved tanh function method [14], Jacobi elliptic function method [15], and the others [16-20].

In this study, we have obtained the exact solutions of these equations by applying the exponential function method to (2+1) and (3+1)-D constant coefficient KdV equations [21].

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Analysis of expansional function method

A two-variable PDE is given:

$$Q\left(u, u_t, u_x, u_{xx}, \ldots\right) = 0 \tag{1}$$

Let us apply the transformation $(x, t) = u(\xi)$, $\xi = kx + wt$ to this equation, where k and w are constants. As a result of the application of this transformation, eq. (1) transforms into an ordinary differential equation dependent on $u(\xi)$:

$$Q'\left(u', u'', u''', \ldots\right) = 0 \tag{2}$$

We are looking for the solution of this eq. (2):

$$u\left(\xi\right) = \frac{\sum_{n=-c}^{d} a_n \exp\left(n\xi\right)}{\sum_{m=-p}^{q} b_m \exp\left(m\xi\right)} \tag{3}$$

where c, d, p, and q are positive integers are a_n and b_m are the unknown constants. We assume that the solution of eq. (2) is:

$$u\left(\xi\right) = \frac{a_c \exp\left(c\xi\right) + \cdots + a_{-d} \exp\left(-d\xi\right)}{a_p \exp\left(p\xi\right) + \cdots + a_{-q} \exp\left(-q\xi\right)} \tag{4}$$

where c, d, p, and q are positive integers, and these numbers are found by balancing the highest order derivative term in eq. (2) with the highest order non-linear term. If the solution of (4) is replaced in eq. (2), an algebraic equation system for $\exp(\xi)$ is obtained. In this equation system obtained, all the coefficients of $\exp(\xi)$ are equal to zero and the constants a_n and b_m are found.

Examples

Example

Let us first consider the (2+1)-D constant coefficient KdV equation [21]:

$$u_{ty} + u_{xxxy} + \alpha u_{yx} u_x + \alpha u_y u_{xx} + \beta u_{xx} + \gamma u_{yy} = 0 \tag{5}$$

Applying the transformation of $(x, y, t) = u(\xi)$, $\xi = (x - \sigma y - kt)$ the equation becomes a common differential equation:

$$\sigma k u'' - \sigma u^{(4)} - 2\alpha\sigma u' u'' + \beta u'' + \gamma\sigma^2 u'' = 0 \tag{6}$$

Once equation (6) is integrated once, we get common differential equation:

$$\left(\sigma k + \beta + \gamma\sigma^2\right) u' - \alpha\sigma\left(u'\right)^2 - \sigma u''' = 0 \tag{7}$$

When u''' is balanced by $(u')^2$ in eq. (7):

$$\frac{c_1 \exp\left[\left(c + 7p\right)\xi\right] + \cdots}{c_2 \exp\left[8p\xi\right] + \cdots} = \frac{c_3 \exp\left[\left(2c + 6p\right)\xi\right] + \cdots}{c_4 \exp\left[8p\xi\right] + \cdots}$$

where $p = c$ is obtained. Similarly, when u''' and $(u')^2$ are balanced to determine the values of q and d:

$$\frac{\cdots + d_1 \exp\left[-\left(d + 7q\right)\xi\right]}{\cdots + d_2 \exp\left[-8q\xi\right]} = \frac{\cdots + d_3 \exp\left[-\left(2d + 6q\right)\xi\right]}{\cdots + d_4 \exp\left[-8q\xi\right]}$$

where $q = d$ is obtained. Here, if we take $p = c = 1$ and $q = d = 1$, the solution (4) is obtained:

$$u(\xi)=\frac{a_1 exp(\xi)+a_0+a_{-1}exp(-\xi)}{exp(\xi)+b_0+b_{-1}exp(-\xi)} \tag{8}$$

If this solution is written in eq. (7), an algebraic equation system is formed for the coefficients a_0, a_1, a_{-1}, b_0, and b_{-1} as follows:

$$\begin{gathered}\frac{1}{T}(-4\alpha\sigma a_{-1}^2-4\beta a_{-1}b_{-1}-32\sigma a_{-1}b_{-1}-4k\sigma a_{-1}b_{-1}-4\gamma\sigma^2 a_{-1}b_{-1}+2\alpha\sigma a_0^2 b_{-1}+\\ +8\alpha\sigma a_{-1}a_1 b_{-1}+4\beta a_1 b_{-1}^2+32\sigma a_1 b_{-1}^2+4k\sigma a_1 b_{-1}^2+4\gamma\sigma^2 a_1 b_{-1}^2-4\alpha\sigma a_1^2 b_{-1}^2-\\ -2\alpha\sigma a_{-1}a_0 b_0-2\alpha\sigma a_0 a_1 b_{-1}b_0-4\beta a_{-1}b_0^2+4\sigma a_{-1}b_0^2-4k\sigma a_{-1}b_0^2-4\gamma\sigma^2 a_{-1}b_0^2+\\ +2\alpha\sigma a_{-1}a_1 b_0^2+4\beta a_1 b_{-1}b_0^2-4\sigma a_1 b_{-1}b_0^2+4k\sigma a_1 b_{-1}b_0^2+4\gamma\sigma^2 a_1 b_{-1}b_0^2)=0\end{gathered}$$

$$\frac{1}{T}\left(-\beta a_0+\sigma a_0-k\sigma a_0-\gamma\sigma^2 a_0+\beta a_1 b_0-\sigma a_1 b_0+k\sigma a_1 b_0+\gamma\sigma^2 a_1 b_0\right)=0$$

$$\frac{1}{T}\left(\begin{gathered}\beta a_0 b_{-1}^3-\sigma a_0 b_{-1}^3+k\sigma a_0 b_{-1}^3+\gamma\sigma^2 a_0 b_{-1}^3-\beta a_{-1}b_{-1}^2 b_0+\\ +\sigma a_{-1}b_{-1}^2 b_0-k\sigma a_{-1}b_{-1}^2 b_0-\gamma\sigma^2 a_{-1}b_{-1}^2 b_0=0\end{gathered}\right)$$

$$\begin{gathered}\frac{1}{T}(-2\beta a_{-1}+8\sigma a_{-1}-2k\sigma a_{-1}-2\gamma\sigma^2 a_{-1}-\alpha\sigma a_0^2+2\beta a_1 b_{-1}-8\sigma a_1 b_{-1}+2k\sigma a_1 b_{-1}+\\ +2\gamma\sigma^2 a_1 b_{-1}-2\beta a_0 b_0-4\sigma a_0 b_0-2k\sigma a_0 b_0-2\gamma\sigma^2 a_0 b_0+2\alpha\sigma a_0 a_1 b_0+2\beta a_1 b_0^2+4\sigma a_1 b_0^2+\\ +2k\sigma a_1 b_0^2+2\gamma\sigma^2 a_1 b_0^2-\alpha\sigma a_1^2 b_0^2)=0\end{gathered}$$

$$\begin{gathered}\frac{1}{T}(-2\beta a_{-1}b_{-1}^2+8\sigma a_{-1}b_{-1}^2-2k\sigma a_{-1}b_{-1}^2-2\gamma\sigma^2 a_{-1}b_{-1}^2-\alpha\sigma a_0^2 b_{-1}^2+2\beta a_1 b_{-1}^3-8\sigma a_1 b_{-1}^3+\\ +2k\sigma a_1 b_{-1}^3+2\gamma\sigma^2 a_1 b_{-1}^3+2\alpha\sigma a_{-1}a_0 b_{-1}b_0+2\beta a_0 b_{-1}^2 b_0+4\sigma a_0 b_{-1}^2 b_0+2k\sigma a_0 b_{-1}^2 b_0+\\ +2\gamma\sigma^2 a_0 b_{-1}^2 b_0-\alpha\sigma a_{-1}^2 b_0^2-2\beta a_{-1}b_{-1}b_0^2-4\sigma a_{-1}b_{-1}b_0^2-2k\sigma a_{-1}b_{-1}b_0^2-2\gamma\sigma^2 a_{-1}b_{-1}b_0^2)=0\end{gathered}$$

$$\begin{gathered}\frac{1}{T}(4\alpha\sigma a_{-1}a_0 b_{-1}+\beta a_0 b_{-1}^2+23\sigma a_0 b_{-1}^2+k\sigma a_0 b_{-1}^2+\gamma\sigma^2 a_0 b_{-1}^2-4\alpha\sigma a_0 a_1 b_{-1}^2-4\alpha\sigma a_{-1}^2 b_0-\\ -6\beta a_{-1}b_{-1}b_0-18\sigma a_{-1}b_{-1}b_0-6k\sigma a_{-1}b_{-1}b_0-6\gamma\sigma^2 a_{-1}b_{-1}b_0+4\alpha\sigma a_{-1}a_1 b_{-1}b_0+5\beta a_1 b_{-1}^2 b_0-\\ -5\sigma a_1 b_{-1}^2 b_0+5k\sigma a_1 b_{-1}^2 b_0+5\gamma\sigma^2 a_1 b_{-1}^2 b_0+\beta a_0 b_{-1}b_0^2-\sigma a_0 b_{-1}b_0^2+k\sigma a_0 b_{-1}b_0^2+\gamma\sigma^2 a_0 b_{-1}b_0^2-\\ -\beta a_{-1}b_0^3+\sigma a_{-1}b_0^3-k\sigma a_{-1}b_0^3-\gamma\sigma^2 a_{-1}b_0^3)=0\end{gathered}$$

$$\begin{gathered}\frac{1}{T}(-4\alpha\sigma a_{-1}a_0-\beta a_0 b_{-1}-23\sigma a_0 b_{-1}-k\sigma a_0 b_{-1}-\gamma\sigma^2 a_0 b_{-1}+4\alpha\sigma a_0 a_1 b_{-1}-5\beta a_{-1}b_0+5\sigma a_{-1}b_0-\\ -5k\sigma a_{-1}b_0-5\gamma\sigma^2 a_{-1}b_0+4\alpha\sigma a_{-1}a_1 b_0+6\beta a_1 b_{-1}b_0+18\sigma a_1 b_{-1}b_0+6k\sigma a_1 b_{-1}b_0+6\gamma\sigma^2 a_1 b_{-1}b_0-\\ -4\alpha\sigma a_1^2 b_{-1}b_0-\beta a_0 b_0^2+\sigma a_0 b_0^2-k\sigma a_0 b_0^2-\gamma\sigma^2 a_0 b_0^2+\beta a_1 b_0^3-\sigma a_1 b_0^3+k\sigma a_1 b_0^3+\gamma\sigma^2 a_1 b_0^3)=0\end{gathered}$$

Here $T = [\exp(\xi) + b_0 + b_{-1}\exp(-\xi)]$. When this system is solved, the coefficients a_0, a_1, a_{-1}, b_0, b_{-1}, and k are found:

(i) $$\alpha \neq 0,\ a_1 = \frac{6}{\alpha}, a_{-1} = 0, b_{-1} = \frac{1}{36}\left(-\alpha^2 a_0^2 + 6\alpha a_0 b_0\right),\ \sigma \neq 0,\ k = \frac{-\beta + \sigma - \gamma\sigma^2}{\sigma},\ \beta \neq 0$$

(ii) $$\alpha \neq 0,\ a_1 = \frac{12}{\alpha},\ a_0 = 0, a_{-1} = 0, b_0 = 0,\ \sigma b_{-1} \neq 0,\ k = \frac{-\beta + 4\sigma - \gamma\sigma^2}{\sigma},\ \beta \neq 0$$

(iii) $$a_0 = 0,\ b_0 = 0,\ a_1\left(-12 + \alpha a_1\right) \neq 0, b_{-1} = \frac{\alpha a_{-1}}{-12 + \alpha a_1},\ \sigma\left(a_{-1} - a_1 b_{-1}\right) \neq 0,$$
$$k = \frac{-\beta + 4\sigma - \gamma\sigma^2}{\sigma},\ \beta \neq 0$$

(iv) $$a_1 = 0,\ a_0 = 0, b_0 = 0,\ b_{-1} = -\frac{1}{12}\alpha a_{-1}, \sigma \neq 0,\ k = \frac{-\beta + 4\sigma - \gamma\sigma^2}{\sigma},\ \beta a_{-1} \neq 0$$

(v) $$a_1 = 0,\ a_0 \neq 0,\ b_0 = \frac{6a_{-1} - \alpha a_0^2}{6a_0}, a_{-1} \neq 0,\ b_{-1} = -\frac{1}{6}\alpha a_{-1}, \sigma \neq 0,$$
$$k = \frac{-\beta + \sigma - \gamma\sigma^2}{\sigma},\ \alpha\beta \neq 0$$

If the coefficients (i)-(v) are written in their place in the solution (8), the solutions of eq. (5) are obtained:

$$u_1(x,y,t) = \frac{36\mathrm{e}^{x + \frac{t\beta}{\sigma} + t\gamma\sigma}}{6\mathrm{e}^{x + \frac{t\beta}{\sigma} + t\gamma\sigma}\alpha - \mathrm{e}^{t + y\sigma}\alpha^2 a_0 + 6\mathrm{e}^{t + y\sigma}\alpha b_0},\ \text{(see fig. 1)}$$

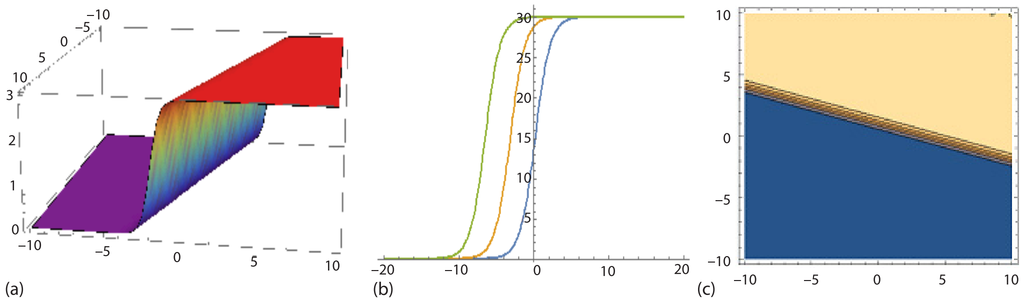

Figure 1. (a) The 3-D surfaces of the aforementioned solution for $\beta = 4$, $\gamma = 1$, $\sigma = 3$, $\alpha = 2$, $a_0 = 1$, $b_0 = 2$, $y = 1$, (b) the 2-D surfaces of the aforementioned solution for $\beta = 4$, $\gamma = 1$, $\sigma = 3$, $\alpha = 2$, $a_0 = 1$, $b_0 = 2$, $y = 1$, $t = 1$, $t = 2$, $t = 3$, and (c) the contour plot of the aforementioned solution for $\beta = 4$, $\gamma = 1$, $\sigma = 3$, $\alpha = 2$, $a_0 = 1$, $b_0 = 2$, $y = 1$

$$u_2(x,y,t) = \frac{12\mathrm{e}^{2x}}{\alpha\left(\mathrm{e}^{2x} + \mathrm{e}^{2y\sigma + t\left(8 - \frac{2\beta}{\sigma} - 2\gamma\sigma\right)}b_{-1}\right)},\ \text{(see fig. 2)}$$

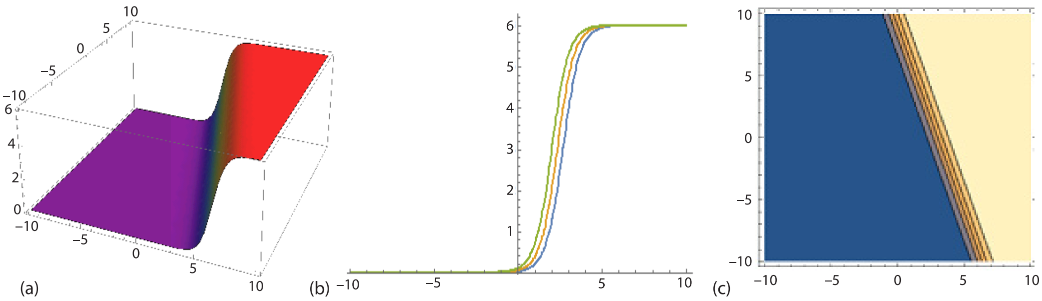

Figure 2. (a) The 3-D surfaces of the aforementioned solution for $\beta = 4$, $\gamma = 1$, $\sigma = 3$, $\alpha = 2$, $b_{-1} = 1$, (b) the 2-D surfaces of the aforementioned solution for $\beta = 4$, $\gamma = 1$, $\sigma = 3$, $\alpha = 2$, $b_{-1} = 1$, $y = 1$, $t = 1$, $t = 2$, $t = 3$, and (c) the contour plot of the aforementioned solution for $\beta = 4$, $\gamma = 1$, $\sigma = 3$, $\alpha = 2$, $a_0 = 1$, $b_0 = 2$, $y = 1$

$$u_3(x,y,t) = \frac{\left(\mathrm{e}^{2y\sigma + t\left(8 - \frac{2\beta}{\sigma} - 2\gamma\sigma\right)} a_{-1} + \mathrm{e}^{2x} a_1\right)(-12 + \alpha a_1)}{\mathrm{e}^{2y\sigma + t\left(8 - \frac{2\beta}{\sigma} - 2\gamma\sigma\right)} \alpha a_{-1} + \mathrm{e}^{2x}(-12 + \alpha a_1)}$$

$$u_4(x,y,t) = -\frac{12\mathrm{e}^{8t + 2y\sigma} a_{-1}}{-12\mathrm{e}^{2\left(x + \frac{t\beta}{\sigma} + t\gamma\sigma\right)} + \mathrm{e}^{8t + 2y\sigma}\alpha a_{-1}}, \quad u_5(x,y,t) = \frac{6\mathrm{e}^{t + y\sigma} a_0}{6\mathrm{e}^{x + \frac{t\beta}{\sigma} + t\gamma\sigma} - \mathrm{e}^{t + y\sigma}\alpha a_0}$$

Example

Now let's apply the exponential function method to the (3+1)-D constant coefficient KdV equation [21]:

$$u_{ty} + u_{xxxy} + \alpha u_{yx} u_x + \alpha u_y u_{xx} + \beta u_{xx} + \gamma u_{yy} + \delta u_{zy} = 0 \tag{9}$$

Applying the transformation $(x, y, z, t) = u(\xi)$, $\xi = (x - \sigma y + \rho z - kt)$ to eq. (9), a common differential equation is obtained:

$$\sigma k u'' - \sigma u^{(4)} - 2\alpha\sigma u' u'' + \beta u'' + \gamma\sigma^2 u'' - \delta\sigma\rho u'' = 0 \tag{10}$$

If the integral of this common differential equation is taken once:

$$\left(\sigma k + \beta + \gamma\sigma^2 - \delta\sigma\rho\right)u' - \alpha\sigma\left(u'\right)^2 - \sigma u''' = 0 \tag{11}$$

equation is obtained. When u''' is balanced by $(u')^2$ in eq. (11):

$$\frac{c_1 \exp\left[(c + 7p)\xi\right] + \cdots}{c_2 \exp\left[8p\xi\right] + \cdots} = \frac{c_3 \exp\left[(2c + 6p)\xi\right] + \cdots}{c_4 \exp\left[8p\xi\right] + \cdots}$$

where $p = c$ is obtained. Similarly, when u''' and $(u')^2$ are balanced to determine q and d values:

$$\frac{\cdots + d_1 \exp\left[-(d + 7q)\xi\right]}{\cdots + d_2 \exp\left[-8q\xi\right]} = \frac{\cdots + d_3 \exp\left[-(2d + 6q)\xi\right]}{\cdots + d_4 \exp\left[-8q\xi\right]}$$

where $q = d$ is obtained. Here, if we take $p = c = 1$ and $q = d = 1$, the solution (4) is obtained in the form of:

$$u(\xi)=\frac{a_1\exp(\xi)+a_0+a_{-1}\exp(-\xi)}{\exp(\xi)+b_0+b_{-1}\exp(-\xi)} \tag{12}$$

If the solution eq. (12) is substituted in the eq. (11), an algebraic equation system is formed for the coefficients a a_0, a_1, a_{-1}, b_0, and b_{-1}:

$$\frac{1}{T}\Big(-4\alpha\sigma a_{-1}^2-4\beta a_{-1}b_{-1}-32\sigma a_{-1}b_{-1}-4k\sigma a_{-1}b_{-1}+4\delta\rho\sigma a_{-1}b_{-1}-4\gamma\sigma^2 a_{-1}b_{-1}+2\alpha\sigma a_0^2 b_{-1}+$$
$$+8\alpha\sigma a_{-1}a_1 b_{-1}+4\beta a_1 b_{-1}^2+32\sigma a_1 b_{-1}^2+4k\sigma a_1 b_{-1}^2-4\delta\rho\sigma a_1 b_{-1}^2+4\gamma\sigma^2 a_1 b_{-1}^2-4\alpha\sigma a_1^2 b_{-1}^2-$$
$$-2\alpha\sigma a_{-1}a_0 b_0-2\alpha\sigma a_0 a_1 b_{-1}b_0-4\beta a_{-1}b_0^2+4\sigma a_{-1}b_0^2-4k\sigma a_{-1}b_0^2+4\delta\rho\sigma a_{-1}b_0^2-4\gamma\sigma^2 a_{-1}b_0^2+$$
$$+2\alpha\sigma a_{-1}a_1 b_0^2+4\beta a_1 b_{-1}b_0^2-4\sigma a_1 b_{-1}b_0^2+4k\sigma a_1 b_{-1}b_0^2-4\delta\rho\sigma a_1 b_{-1}b_0^2+4\gamma\sigma^2 a_1 b_{-1}b_0^2\Big)=0$$

$$\frac{1}{T}\Big(-\beta a_0+\sigma a_0-k\sigma a_0+\delta\rho\sigma a_0-\gamma\sigma^2 a_0+\beta a_1 b_0-\sigma a_1 b_0+k\sigma a_1 b_0-\delta\rho\sigma a_1 b_0+\gamma\sigma^2 a_1 b_0\Big)=0$$

$$\frac{1}{T}\Big(\beta a_0 b_{-1}^3-\sigma a_0 b_{-1}^3+k\sigma a_0 b_{-1}^3-\delta\rho\sigma a_0 b_{-1}^3+\gamma\sigma^2 a_0 b_{-1}^3-\beta a_{-1}b_{-1}^2 b_0+\sigma a_{-1}b_{-1}^2 b_0-$$
$$-k\sigma a_{-1}b_{-1}^2 b_0+\delta\rho\sigma a_{-1}b_{-1}^2 b_0-\gamma\sigma^2 a_{-1}b_{-1}^2 b_0\Big)=0$$

$$\frac{1}{T}\Big(-2\beta a_{-1}+8\sigma a_{-1}-2k\sigma a_{-1}+2\delta\rho\sigma a_{-1}-2\gamma\sigma^2 a_{-1}-\alpha\sigma a_0^2+2\beta a_1 b_{-1}-8\sigma a_1 b_{-1}+2k\sigma a_1 b_{-1}-$$
$$-2\delta\rho\sigma a_1 b_{-1}+2\gamma\sigma^2 a_1 b_{-1}-2\beta a_0 b_0-4\sigma a_0 b_0-2k\sigma a_0 b_0+2\delta\rho\sigma a_0 b_0-2\gamma\sigma^2 a_0 b_0+2\alpha\sigma a_0 a_1 b_0+$$
$$+2\beta a_1 b_0^2+4\sigma a_1 b_0^2+2k\sigma a_1 b_0^2-2\delta\rho\sigma a_1 b_0^2+2\gamma\sigma^2 a_1 b_0^2-\alpha\sigma a_1^2 b_0^2\Big)=0$$

$$\frac{1}{T}\Big(-2\beta a_{-1}b_{-1}^2+8\sigma a_{-1}b_{-1}^2-2k\sigma a_{-1}b_{-1}^2+2\delta\rho\sigma a_{-1}b_{-1}^2-2\gamma\sigma^2 a_{-1}b_{-1}^2-\alpha\sigma a_0^2 b_{-1}^2+2\beta a_1 b_{-1}^3-$$
$$-8\sigma a_1 b_{-1}^3+2k\sigma a_1 b_{-1}^3-2\delta\rho\sigma a_1 b_{-1}^3+2\gamma\sigma^2 a_1 b_{-1}^3+2\alpha\sigma a_{-1}a_0 b_{-1}b_0+2\beta a_0 b_{-1}^2 b_0+$$
$$+4\sigma a_0 b_{-1}^2 b_0+2k\sigma a_0 b_{-1}^2 b_0-2\delta\rho\sigma a_0 b_{-1}^2 b_0+2\gamma\sigma^2 a_0 b_{-1}^2 b_0-\alpha\sigma a_{-1}^2 b_0^2-2\beta a_{-1}b_{-1}b_0^2-$$
$$-4\sigma a_{-1}b_{-1}b_0^2-2k\sigma a_{-1}b_{-1}b_0^2+2\delta\rho\sigma a_{-1}b_{-1}b_0^2-2\gamma\sigma^2 a_{-1}b_{-1}b_0^2\Big)=0$$

$$\frac{1}{T}\Big(4\alpha\sigma a_{-1}a_0 b_{-1}+\beta a_0 b_{-1}^2+23\sigma a_0 b_{-1}^2+k\sigma a_0 b_{-1}^2-\delta\rho\sigma a_0 b_{-1}^2+\gamma\sigma^2 a_0 b_{-1}^2-4\alpha\sigma a_0 a_1 b_{-1}^2-$$
$$-4\alpha\sigma a_{-1}^2 b_0-6\beta a_{-1}b_{-1}b_0-18\sigma a_{-1}b_{-1}b_0-6k\sigma a_{-1}b_{-1}b_0+6\delta\rho\sigma a_{-1}b_{-1}b_0-6\gamma\sigma^2 a_{-1}b_{-1}b_0+$$
$$+4\alpha\sigma a_{-1}a_1 b_{-1}b_0+5\beta a_1 b_{-1}^2 b_0-5\sigma a_1 b_{-1}^2 b_0+5k\sigma a_1 b_{-1}^2 b_0-5\delta\rho\sigma a_1 b_{-1}^2 b_0+5\gamma\sigma^2 a_1 b_{-1}^2 b_0+$$
$$+\beta a_0 b_{-1}b_0^2-\sigma a_0 b_{-1}b_0^2+k\sigma a_0 b_{-1}b_0^2-\delta\rho\sigma a_0 b_{-1}b_0^2+\gamma\sigma^2 a_0 b_{-1}b_0^2-\beta a_{-1}b_0^3+\sigma a_{-1}b_0^3-$$
$$-k\sigma a_{-1}b_0^3+\delta\rho\sigma a_{-1}b_0^3-\gamma\sigma^2 a_{-1}b_0^3\Big)=0$$

$$\frac{1}{T}\Big(-4\alpha\sigma a_{-1}a_0-\beta a_0 b_{-1}-23\sigma a_0 b_{-1}-k\sigma a_0 b_{-1}+\delta\rho\sigma a_0 b_{-1}-\gamma\sigma^2 a_0 b_{-1}+4\alpha\sigma a_0 a_1 b_{-1}-$$
$$-5\beta a_{-1}b_0+5\sigma a_{-1}b_0-5k\sigma a_{-1}b_0+5\delta\rho\sigma a_{-1}b_0-5\gamma\sigma^2 a_{-1}b_0+4\alpha\sigma a_{-1}a_1 b_0+6\beta a_1 b_{-1}b_0+$$
$$+18\sigma a_1 b_{-1}b_0+6k\sigma a_1 b_{-1}b_0-6\delta\rho\sigma a_1 b_{-1}b_0+6\gamma\sigma^2 a_1 b_{-1}b_0-4\alpha\sigma a_1^2 b_{-1}b_0-\beta a_0 b_0^2+\sigma a_0 b_0^2-$$
$$-k\sigma a_0 b_0^2+\delta\rho\sigma a_0 b_0^2-\gamma\sigma^2 a_0 b_0^2+\beta a_1 b_0^3-\sigma a_1 b_0^3+k\sigma a_1 b_0^3-\delta\rho\sigma a_1 b_0^3+\gamma\sigma^2 a_1 b_0^3\Big)=0$$

Here $T = [\exp(\xi) + b_0 + b_{-1}\exp(-\xi)]$. When this algebraic system is solved, we obtain the values of coefficients a_0, a_1, a_{-1}, b_0, b_{-1}, and k as follows:

(i)
$$\alpha \neq 0, a_1 = \frac{6}{\alpha},\ a_{-1} = 0,\ b_{-1} = \frac{1}{36}\left(-\alpha^2 a_0^2 + 6\alpha a_0 b_0\right),\ \sigma \neq 0,$$
$$k = \frac{-\beta + \sigma + \delta\rho\sigma - \gamma\sigma^2}{\sigma},\ \beta \neq 0$$

(ii)
$$a_1 = 0,\ a_0 \neq 0,\ b_0 = \frac{6a_{-1} - \alpha a_0^2}{6a_0},\ a_{-1} \neq 0,\ b_{-1} = -\frac{1}{6}\alpha a_{-1},\ \sigma \neq 0,$$
$$k = \frac{-\beta + \sigma + \delta\rho\sigma - \gamma\sigma^2}{\sigma},\ \alpha\beta \neq 0$$

(iii)
$$\alpha \neq 0,\ a_1 = \frac{12}{\alpha},\ a_0 = 0,\ a_{-1} = 0,\ b_0 = 0,\ \sigma b_{-1} \neq 0,\ k = \frac{-\beta + 4\sigma + \delta\rho\sigma - \gamma\sigma^2}{\sigma},\ \beta \neq 0$$

(iv)
$$a_1 = 0,\ a_{-1} = 0,\ b_0 = -\frac{\alpha a_0}{6},\ b_{-1} = 0,\ \sigma \neq 0,\ k = \frac{-\beta + \sigma + \delta\rho\sigma - \gamma\sigma^2}{\sigma},\ \alpha\beta a_0 \neq 0$$

(v)
$$a_0 = 0,\ b_0 = 0,\ a_1\left(-12 + \alpha a_1\right) \neq 0,\ b_{-1} = \frac{\alpha a_{-1}}{-12 + \alpha a_1},\ \sigma\left(a_{-1} - a_1 b_{-1}\right) \neq 0,$$
$$k = \frac{-\beta + 4\sigma + \delta\rho\sigma - \gamma\sigma^2}{\sigma},\ \beta \neq 0$$

If the coefficients obtained in (i)-(v) are written in their place in the solution function expressed by (12), the solutions of eq. (9) are obtained:

$$u_1(x,y,z,t) = \frac{36\mathrm{e}^{x + z\rho + \frac{t\beta}{\sigma} + t\gamma\sigma}}{6\mathrm{e}^{x + z\rho + \frac{t\beta}{\sigma} + t\gamma\sigma}\alpha - \mathrm{e}^{t + t\delta\rho + y\sigma}\alpha^2 a_0 + 6\mathrm{e}^{t + t\delta\rho + y\sigma}\alpha b_0},\ \text{(see figs. 3 and 4)}$$

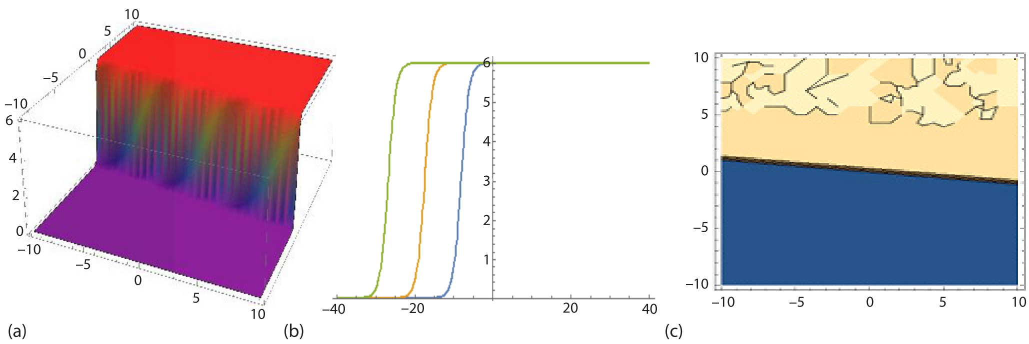

Figure 3. (a) The 3-D surfaces of the aforementioned solution for $\beta = 4$, $\gamma = 1$, $\sigma = 3$, $\alpha = 2$, $a_0 = 1$, $y = 1$, $z = 1$, $b_0 = 2$, $\delta = -2$, $\rho = 3$, (b) the 2-D surfaces of the aforementioned solution for $\beta = 4$, $\gamma = 1$, $\sigma = 3$, $\alpha = 2$, $a_0 = 1$, $y = 1$, $z = 1$, $b_0 = 2$, $\delta = -2$, $\rho = 3$, $t = 1$, $t = 2$, $t = 3$, and (c) the contour plot of the aforementioned solution for $\beta = 4$, $\gamma = 1$, $\sigma = 3$, $\alpha = 2$, $a_0 = 1$, $y = 1$, $z = 1$, $b_0 = 2$, $\delta = -2$, $\rho = 3$

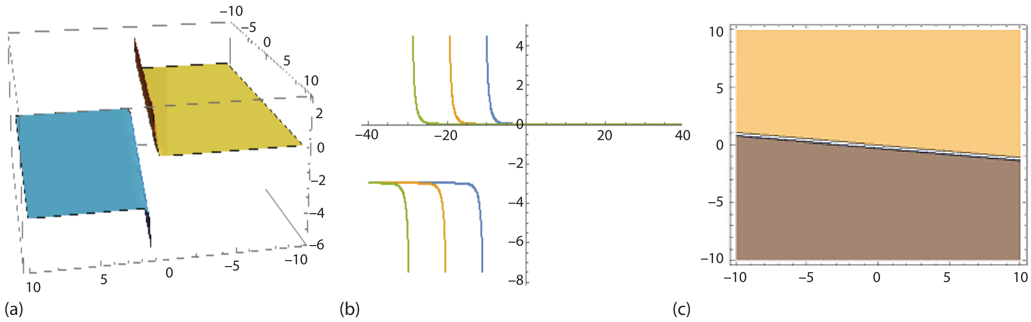

Figure 4. (a) The 3-D surfaces of the aforementioned solution for $\beta = 4$, $\gamma = 1$, $\sigma = 3$, $\alpha = 2$, $a_0 = 1$, $y = 1$, $z = 1$, $b_0 = 2$, $\delta = -2$, $\rho = 3$, (b) the 2-D surfaces of the aforementioned solution for $\beta = 4$, $\gamma = 1$, $\sigma = 3$, $\alpha = 2$, $a_0 = 1$, $y = 1$, $z = 1$, $b_0 = 2$, $\delta = -2$, $\rho = 3$, $t = 1$, $t = 2$, $t = 3$ (c) the contour plot of the aforementioned solution for $\beta = 4$, $\gamma = 1$, $\sigma = 3$, $\alpha = 2$, $a_0 = 1$, $y = 1$, $z = 1$, $b_0 = 2$, $\delta = -2$, $\rho = 3$

$$u_2(x,y,z,t) = \frac{6\mathrm{e}^{t+t\delta\rho+y\sigma} a_0}{6\mathrm{e}^{x+z\rho+\frac{t\beta}{\sigma}+t\gamma\sigma} - \mathrm{e}^{t+t\delta\rho+y\sigma}\alpha a_0}$$

$$u_3(x,y,z,t) = \frac{12\mathrm{e}^{2(x+z\rho)}}{\alpha\left(\mathrm{e}^{2(x+z\rho)} + \mathrm{e}^{2y\sigma+t\left(8+2\delta\rho-\frac{2\beta}{\sigma}-2\gamma\sigma\right)} b_{-1}\right)}$$

$$u_4(x,y,z,t) = \frac{a_0}{\mathrm{e}^{x+z\rho-y\sigma+t\left(-1-\delta\rho+\frac{\beta}{\sigma}+\gamma\sigma\right)} - \frac{\alpha a_0}{6}}$$

$$u_5(x,y,z,t) = \frac{\left[\mathrm{e}^{2y\sigma+t\left(8+2\delta\rho-\frac{2\beta}{\sigma}-2\gamma\sigma\right)} a_{-1} + \mathrm{e}^{2(x+z\rho)} a_1\right](-12+\alpha a_1)}{\mathrm{e}^{2y\sigma+t\left(8+2\delta\rho-\frac{2\beta}{\sigma}-2\gamma\sigma\right)} \alpha a_{-1} + \mathrm{e}^{2(x+z\rho)}(-12+\alpha a_1)}$$

Conclusion

In this study, we have obtained the exact solutions of (2+1) and (3+1)-D constant coefficient KdV equations by applying the exponential function method. These exact solutions we find are in the form of an exponential function. In addition, we have seen that these solutions provide the equations by using MATHEMATICA 11.3 program. Apart from that, we have shown the graphics performance of some of the solutions found. This method can be applied to many non-linear PDE and systems of equations.

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