APPLICATION OF JORDAN CANONICAL FORM AND SYMPLECTIC MATRIX IN FRACTIONAL DIFFERENTIAL MODELS

by

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Under consideration of this paper is the application of Jordan canonical form and symplectic matrix to two conformable fractional differential models. One is the new conformable fractional vector conduction equation which is reduced by using the Jordan canonical form of coefficient matrix and solved exactly, and the other is the new conformable fractional vector dynamical system with Hamilton matrix and symplectic matrix, which is derived by constructing the conformable fractional Euler-Lagrange equation and using fractional variational principle. It is shown that Jordan canonical form and symplectic matrix can be used to deal with some other conformable fractional differential systems in mathematical physics.

Key words: *conformable fractional vector conduction equation, conformable fractional vector dynamical system, Jordan canonical form, symplectic matrix, conformable fractional derivative*

Introduction

Fractional calculus has played an irreplaceable role in many fields and attracted more and more attention [1-13]. In this paper, we mainly have two contributions. One is to solve the following new conformable fractional vector conduction equation:

$$u_t^{(\alpha)} = (P^n + Q)u_x^{(2\alpha)}, \quad (0 < \alpha \le 1;\ x, t > 0;\ n \in N^+) \tag{1}$$

where $u = \left[u_1(x,t), u_2(x,t), u_3(x,t)\right]^T$, $u_x^{(2\alpha)} = \partial_x^\alpha(\partial_x^\alpha u)$, and $u_t^{(\alpha)}$ are the conformable fractional partial derivatives [9] of x and t, respectively, and P and Q are two 3 × 3 matrices:

$$P = \begin{pmatrix} 8 & -3 & 6 \\ 3 & -2 & 0 \\ -4 & 2 & -2 \end{pmatrix}, \quad Q = \begin{pmatrix} -49 & 3n & 18n-72 \\ 9n-36 & -1 & 18n-54 \\ 30-6n & -2n & 44 \end{pmatrix} \tag{2}$$

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The other is to derive the new conformable fractional vector dynamical system:

$$\frac{\mathrm{d}^\alpha v}{\mathrm{d}t^\alpha} = Hv = J\frac{\partial^\alpha H}{\partial v^\alpha}, \quad (0<\alpha\le 1;\ t>0) \tag{3}$$

with

$$H = \begin{pmatrix} A & D \\ -B & -A^T \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \quad v = \begin{pmatrix} q \\ p \end{pmatrix} \tag{4}$$

where

$$D = M^{-1}, \quad A = -M^{-1}G/2, \quad B = K + G^T M^{-1} G/4, \quad (M, G, K \in R^{n\times n}; G^T = G; K^T = -K)$$

M is a positive definite matrix, I_n is the n-order identity matrix, and the n-dimensional column vectors p and q satisfy the constrain:

$$p = M\frac{\mathrm{d}^\alpha q}{\mathrm{d}t^\alpha} + \frac{Gq}{2} \tag{5}$$

Some useful preparations

For the given arbitrary n-order quasi-diagonal matrix $P = \mathrm{diag}(P_1, P_2, \cdots, P_s)$, here $P_i = (p_{jl})_{n_i\times n_i} (j,l = 1,2,\cdots,n_i; i = 1,2,\cdots,s; n_1 + n_2 + \cdots + n_s = n)$ are upper or lower triangle matrices with equal main diagonal elements $p_{kl} = p_i (k = l = 1,2,\cdots,n_i; i = 1,2,\cdots,s)$, we have the following lemma and theorems.

Lemma 1 [10]. Suppose $\Delta_i = P_i - m_i I_i$ for $i = 1,2,\ldots,s$, then $\Delta_i^{n_i} = 0 (i = 1,2,\cdots,s)$.

Theorem 1. Suppose $k_i = m$ for $m < n_i$ while $k_i = n_i - 1$ for $m \ge n_i$ here it is assumed that $i = 1,2,\ldots,s$, then:

$$P_i^m = \sum_{s_i=0}^{k_i} C_m^{m-s_i} p_i^{m-s_i} \Delta_i^{s_i}, \quad \Delta_i^0 = I_i, \quad C_m^k = \frac{n!}{k!(m-k)!} \tag{6}$$

Proof. We write $P_i = p_i I_i + \Delta_i$. Clearly, the quantitative matrix $p_i I_i$ and the matrix Δ_i are commutative. Then one has:

$$P_i^m = (p_i I_i + \Delta_i)^m = \sum_{s_i=0}^{m} C_m^{m-s_i} p_i^{m-s_i} \Delta_i^{s_i} \tag{7}$$

If $k_i = m < n_i$, then eq. (6) is obviously true. Since *Lemma 1* tells that $\Delta_i^m = 0, (i = 1,2,\cdots,s)$ always hold for $m \ge n_i$, eq. (7) degenerates into eq. (6) when $k_i = m \ge n_i$. The proof is end.

Theorem 2. Suppose $k_i = m (m < n_i)$ or $k_i = n_i - 1 (m \ge n_i)$ for $i = 1,2,\ldots,s$, then P^m has expansion formula:

$$P^m = \begin{pmatrix} \sum_{j_1=0}^{k_1} C_m^{m-j_1} p_1^{m-j_1} \Delta_1^{j_1} & & & \\ & \sum_{j_2=0}^{k_2} C_m^{m-j_2} p_2^{m-j_2} \Delta_2^{j_2} & & \\ & & \ddots & \\ & & & \sum_{j_s=0}^{k_s} C_m^{m-j_s} p_s^{m-j_s} \Delta_s^{j_s} \end{pmatrix} \tag{8}$$

Proof. In view of $P=\operatorname{diag}(P_1,P_2,\cdots,P_s)$, we have $P^m=\operatorname{diag}(P_1^m,P_2^m,\cdots,P_s^m)$. Then one can arrive at eq. (8) by using eq. (7). We thus complete the proof.

Example 1. Calculate P^{100}, here:

$$P=\begin{pmatrix}2&1&-1&0&0\\0&2&4&0&0\\0&0&2&0&0\\0&0&0&3&0\\0&0&0&5&3\end{pmatrix} \tag{9}$$

It is easy to see from eq. (9) that we can write $P^m=\operatorname{diag}(P_1^m,P_2^m)$. Here m = 100, k_1 = 2, p_1 = 2, k_2 = 1, p_2 = 3 and:

$$P_1=\begin{pmatrix}2&1&-1\\0&2&4\\0&0&2\end{pmatrix},\ \Delta_1=\begin{pmatrix}0&1&-1\\0&0&4\\0&0&0\end{pmatrix},\ P_2=\begin{pmatrix}3&0\\5&3\end{pmatrix},\ \Delta_2=\begin{pmatrix}0&0\\5&0\end{pmatrix} \tag{10}$$

Since $m=100>k_i,\ (i=1,2)$, with the help of eq. (8) we have:

$$P^{100}=\begin{pmatrix}\sum_{j_1=0}^{2}C_{100}^{100-j_1}2^{100-j_1}\Delta_1^{j_1} & \\ & \sum_{j_2=0}^{1}C_{100}^{100-j_2}3^{100-j_2}\Delta_2^{j_2}\end{pmatrix}=\begin{pmatrix}2^{100}&100\times2^{99}&9800\times2^{99}&&\\&2^{100}&400\times2^{99}&&\\&&2^{100}&&\\&&&3^{100}&\\&&&500\times3^{99}&3^{100}\end{pmatrix} \tag{11}$$

Definition 1 [11]. The block matrix K is called a 2n-order symplectic matrix over the number field P, if there exist matrices $A,B,C,D\in P^{n\times n}$ which make:

$$K^TJK=J,\ K=\begin{pmatrix}A&B\\C&D\end{pmatrix},\ J=\begin{pmatrix}0&-I_n\\I_n&0\end{pmatrix} \tag{12}$$

Symplectic matrices are very important and have some applications and helpful properties [11-13]. Here we would like to recall and list several properties of symplectic matrices.

Property 1. Suppose V is a linear space on the number field P and f is a non-degenerate antisymmetric bilinear function, then necessary and sufficient condition for the matrix K of linear transformation κ under a set of symplectic orthogonal bases of symplectic space (V,f) to be a symplectic matrix is that κ must be a symplectic transformation on (V,f).

Proof. Taking a set of symplectic orthogonal bases of symplectic space (V,f) and recording them as $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n,\varepsilon_{-1},\varepsilon_{-2},\cdots,\varepsilon_{-n}$, then we can see that the metric matrix of f under this set of symplectic orthogonal bases is exactly J. We further let:

$$\begin{aligned}&(\eta_1,\eta_2,\cdots,\eta_n,\eta_{-1},\eta_{-2},\cdots,\eta_{-n})=\\&=(\kappa\varepsilon_1,\kappa\varepsilon_2,\cdots,\kappa\varepsilon_n,\kappa\varepsilon_{-1},\kappa\varepsilon_{-2},\cdots,\kappa\varepsilon_{-n})=\\&=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n,\varepsilon_{-1},\varepsilon_{-2},\cdots,\varepsilon_{-n})K\end{aligned} \tag{13}$$

When κ is a symplectic transformation, we have:

$$f(\eta_i,\eta_{-i})=f(\kappa\varepsilon_i,\kappa\varepsilon_{-i})=f(\varepsilon_i,\varepsilon_{-i})=1,\ \ (1\le i\le n) \tag{14}$$

$$f(\eta_i,\eta_j)=f(\kappa\varepsilon_i,\kappa\varepsilon_j)=f(\varepsilon_i,\varepsilon_j)=0,\quad(-n\le i,j\le n;i+j\ne 0)\tag{15}$$

Thus, $\eta_1,\eta_2,\cdots,\eta_n,\eta_{-1},\eta_{-2},\cdots,\eta_{-n}$ is also a set of symplectic orthogonal bases of symplectic space (V,f), and the metric matrix of f under this set of symplectic orthogonal bases is also J. In this case, we let the transition matrix from $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n,\varepsilon_{-1},\varepsilon_{-2},\cdots,\varepsilon_{-n}$ to $\eta_1,\eta_2,\cdots,\eta_n,\eta_{-1},\eta_{-2},\cdots,\eta_{-n}$ be K. Since the metric matrices under different sets of bases are congruent, we have $K^TJK=J$. This shows that K is a symplectic matrix.

Conversely, if K is a symplectic matrix, we take:

$$\alpha=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n,\varepsilon_{-1},\varepsilon_{-2},\cdots,\varepsilon_{-n})X,\beta=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n,\varepsilon_{-1},\varepsilon_{-2},\cdots,\varepsilon_{-n})Y\in V,\quad(X,Y\in P^{n\times 1})\tag{16}$$

Then one has:

$$\kappa\alpha=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n,\varepsilon_{-1},\varepsilon_{-2},\cdots,\varepsilon_{-n})KX,\quad\kappa\beta=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n,\varepsilon_{-1},\varepsilon_{-2},\cdots,\varepsilon_{-n})KY\tag{17}$$

and obtain

$$f(\kappa\alpha,\kappa\beta)=(KX)^TJ(KY)=X^T(K^TJK)Y=X^TJY=f(\alpha,\beta)\tag{18}$$

which hints κ is a symplectic transformation on symplectic space (V,f). The proof is end.

Property 2 [13]. Symplectic matrix K can be represented by the product of at most seven symplectic matrices of the following forms:

$$\begin{pmatrix}I_n & 0\\ T & I_n\end{pmatrix},\ \begin{pmatrix}S & 0\\ 0 & S^{-1}\end{pmatrix},\ \begin{pmatrix}I_n & R\\ 0 & I_n\end{pmatrix}\tag{19}$$

where T and R are n-order symmetric matrices while $S = S^T$ is a n-order reversible matrix.

Property 3. Suppose K is a symplectic matrix, then K is reversible and $\det K=1$, and K^{-1} and K^T are all symplectic matrices.

Proof. Since K is a symplectic matrix, we have $K^TJK=J$. Then it is easy to see that:

$$\det(K^TJK)=(\det K)^2=\det J=1\tag{20}$$

So, $\det K$ may be 1 or –1. However *Property 2* only supports $\det K = 1$. This also shows that K is reversible. Calculating out $(K^{-1})^TJK^{-1}=(K^{-1})^TJ(J^{-1}K^TJ)=J$ tells that K^{-1} is a symplectic matrix. Similarly, using $K^TJK=J$ and the reversibilities of K and J yields:

$$(K^T)^TJK^T=KJK^T=KJJK^{-1}J^{-1}=-KK^{-1}J^{-1}=-J^{-1}=J\tag{21}$$

from which we can conclude that K^T is a symplectic matrix. The proof is finished.

Property 4. The necessary and sufficient condition for K to be a symplectic matrix is equivalent to one of the following conditions:

$$A^TC=C^TA,\ \ B^TD=D^TB,\ \ A^TD-C^TB=I_n\tag{22}$$

$$K^{-1}=\begin{pmatrix}D^T & -B^T\\ -C^T & A^T\end{pmatrix}\tag{23}$$

$$AB^T=BA^T,\ \ CD^T=DC^T,\ \ AD^T-BC^T=I_n\tag{24}$$

$$KJK^T = J \tag{25}$$

Proof. Firstly, *Definition 1* shows that K is a symplectic matrix if and only if $K^T JK = J$. It is easy to see that:

$$K^T JK = \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A^T C - C^T A & A^T D - C^T B \\ B^T C - D^T A & D^T B - B^T D \end{pmatrix} \tag{26}$$

holds if and only if eq. (22) is true.

Secondly, it is easy to see that K and J are all reversible. So, $K^T JK = J$ holds if and only if $K^{-1} = J^{-1} K^T J$, namely:

$$K^{-1} = J^{-1} K^T J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix} \tag{27}$$

Thirdly, from eq. (23) we know that K is a symplectic matrix if and only if eq. (27) is true. This is equivalent to:

$$KK^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix} = \begin{pmatrix} AD^T - BC^T & BA^T - AB^T \\ CD^T - DC^T & DA^T - CB^T \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix} \tag{28}$$

which reaches eq. (24).

Finally, when K is a symplectic matrix, *Property 3* tells that K^T is also a symplectic matrix. Thus eq. (25) holds. Conversely, if eq. (25) is true, namely K^T is a symplectic matrix, then $(K^T)^T = K$ is also a symplectic matrix. The proof completes.

Property 5. Assuming that K is a symplectic matrix, it is not certain whether A, B, C, D are reversible. But when A or D is a zero matrix, then B and C are all reversible and $B^{-1} = C^T$, while when B or C is a zero matrix, A and D are all reversible and $D^{-1} = A^T$.

Proof. It can be seen from *Definition 1*, *Property 3* and eq. (22) that when K is a symplectic matrix, although K is reversible, it cannot determine whether A, B, C, and D are reversible. When A = 0 and/or D = 0 eq. (22) leads to $-C^T B = I_n$. This indicates that both B and C are reversible and $B^{-1} = -C^T$. By a similiar way, we can arrive at the conclusion that when B = 0 and/or C = 0, both A and D are all reversible and $D^{-1} = A^T$. We complete the proof.

Property 6. Suppose K_1 and K_2 are symplectic matrices, then K_1K_2 is also symplectic matrix, and $K_1 X = K_2$ has a unique solution X, here X is also a symplectic matrix.

Proof. Since $K_1$and K_2 are symplectic matrices, then we have $K_1^T JK_1 = J$ and $K_2^T JK_2 = J$. Thus, we further gain $K_2^T K_1^T JK_1K_2 = K_2^T JK_2 = J$, namely $(K_1K_2)^T J(K_1K_2) = J$. Therefore, K_1K_2 is a symplectic matrix. Using K_1 as the invertibility of symmetric matrix, from $K_1 X = K_2$ we can get $X = K_1^{-1} K_2$. Considering *Property 3*, we know that K_1^{-1} is a symplectic matrix. Therefore, $X = K_1^{-1} K_2$ is also a symplectic matrix. The proof is completed.

Property 7 . Suppose the characteristic polynomial of symplectic matrix K is:

$$f(\lambda) = |\lambda I_n - K| = \lambda^{2n} + a_1 \lambda^{2n-1} + \ldots + a_{2n-1}\lambda + a_{2n} \tag{29}$$

then $f(\lambda) = \lambda^{2n} f(\lambda^{-1})$, $a_{2n} = 1$ and $a_i = a_{2n-i} (i = 1, 2, \cdots, n)$. In addition, the number of positive and negative numbers in the eigenvalue of symplectic matrix K is even, and the number of –1 and +1 in the eigenvalues are also even.

Proof. Referring to the method of [11], we calculate $K = J^{-1}(K^{-1})^T J = -J(K^{-1})^T J$, det K = 1 and det J = 1. Then one has [11]:

$$f(\lambda) = |\lambda I_n - K| = |\lambda E + J(K^{-1})^T J| = |\lambda J I_n J - J(K^{-1})^T J| = |J||\lambda I_n - (K^{-1})^T||J| = \\ = |\lambda I_n - (K^{-1})^T J||K^T| = |\lambda K^T - I_n| = \lambda^{2n}|\lambda^{-1} I_n - K| = \lambda^{2n} f(\lambda^{-1}) \tag{30}$$

which leads to $a_{2n} = 1$ and $a_i = a_{2n-i} (i = 1, 2, \cdots, n)$ must be true. This also shows that in the eigenvalue of symplectic matrix K, λ and λ^{-1} occurs at the same time, so they have the same multiplicities [11]. Since det K = 1 the product of all eigenvalues of K is 1, so the multiplicity of eigenvalue –1 is even. Because the sum of multiplicities of eigenvalues which are not equal to ±1 and the order of K are all even numbers, the number of +1 is also even number.

Property 8. Assumed vector groups $\xi_1, \xi_2, \cdots, \xi_p$ and $\zeta_1, \zeta_2, \cdots, \zeta_q$ are two Jordan chains correspond to the same eigenvalue λ of the symplectic matrix K, and $\eta_1, \eta_2, \cdots, \eta_q$ composes a Jordan chain with eigenvalue λ^{-1}, here $\lambda \neq \pm 1$ and $p \leq q$, then there hold:

$$\xi_{i-1}^T J \xi_{j-1} + \lambda \xi_{i-1}^T J \xi_j + \lambda \xi_i^T J \xi_{j-1} + (\lambda^2 - 1)\xi_i^T J \xi_j = 0, \ (1 \leq i, j \leq p) \tag{31}$$

$$\xi_{i-1}{}^T J \eta_{j-1} + \lambda \xi_{i-1}{}^T J \eta_j + \lambda \xi_i^T J \eta_{j-1} = 0, \ (1 \leq i \leq p, 1 \leq j \leq q) \tag{32}$$

$$\xi_{i-1}{}^T J \zeta_{j-1} + \lambda \xi_{i-1}{}^T J \zeta_j + \lambda \xi_i^T J \zeta_{j-1} + (\lambda^2 - 1)\xi_i^T J \zeta_j = 0, \ (1 \leq i \leq p, 1 \leq j \leq q) \tag{33}$$

$$\xi_i^T J \xi_j = 0, \ (1 \leq i, j \leq p) \tag{34}$$

$$\xi_i^T J \eta_j = 0, \ (1 \leq i \leq p, 1 \leq j \leq q - i) \tag{35}$$

$$\xi_i^T J \zeta_j = 0, \ (1 \leq i \leq p, 1 \leq j \leq q) \tag{36}$$

Proof. Since $\xi_1, \xi_2, \cdots, \xi_p$, $\eta_1, \eta_2, \cdots, \eta_q$ and $\zeta_1, \zeta_2, \cdots, \zeta_q$ compose three Jordan chains, we have:

$$K\xi_i = \lambda \xi_i + \xi_{i-1} (1 \leq i \leq p), \ \ K\eta_j = \lambda^{-1}\eta_j + \xi_{j-1} (1 \leq j \leq q), \ K\zeta_j = \lambda \zeta_j + \zeta_{j-1} (1 \leq i \leq q) \tag{37}$$

Here $\xi_0 = \eta_0 = \zeta_0 = 0$ have been assumed. Then following to the method of [18], we have:

$$\xi_i^T J \xi_j = \xi_i^T (K^T J K) \xi_j = (K\xi_i)^T J (K\xi_j) = (\lambda \xi_i + \xi_{i-1})^T J (\lambda \xi_j + \xi_{j-1}) = \\ = \lambda^2 \xi_i^T J \xi_j + \lambda \xi_{i-1}^T J \xi_j + \lambda \xi_i^T J \xi_{j-1} + \xi_{i-1}^T J \xi_{j-1} \tag{38}$$

which can be rewritten as eq. (31). Starting from eq. (31), we finally gain eq. (34) by induction. Similarly, we can verify eqs. (32) and (35), and eqs. (33) and (36).

Reduction and solutions of fractional system (1)

To solve system (1), we first reduce P^n. We employ the method of [15] to determine a reversible matrix T such that $T^{-1}PT = J$, here J is the Jordan canonical form of P. Since:

$$f_A(\lambda) = |\lambda I - P| = (\lambda - 1)^2 (\lambda - 2) \tag{39}$$

Thus, the characteristic values of P are $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$, and then the elementary factors $(\lambda - 1)^2$ and $\lambda - 2$ of P can be obtained. In this case, we have $P_{1_1} = 2$ and $P_{1_2} = 1$. Since $P_{1_1} = 2 > 1$, we need to perform elementary row transformation on $(A - E \,\vdots\, b)$:

$$(A-E \;\vdots\; b)=\begin{pmatrix} 7 & -3 & 6 & \vdots & b_1 \\ 3 & -3 & 0 & \vdots & b_2 \\ -4 & 2 & -3 & \vdots & b_3 \end{pmatrix} \to \begin{pmatrix} -1 & -1 & -3 & \vdots & b_2+b_3 \\ 0 & 2 & 3 & \vdots & b_1-b_2+b_3 \\ 0 & 0 & 0 & \vdots & 3b_1+b_2+6b_3 \end{pmatrix} \tag{40}$$

Considering the condition $3b_1+b_2+6b_3=0$, we construct the homogeneous linear equations:

$$\begin{pmatrix} -1 & -1 & -3 \\ 0 & 2 & 3 \\ 3 & 1 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} -1 & -1 & -3 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{41}$$

Solve eq. (41) to obtain a basic solution system, that is, the first vector in the cyclic basis corresponding to $P_1=2$, which is recorded as $X_{(1)1}^{(1)}=(3,3,-2)^T$. Taking $b=X_{(1)1}^{(1)}$ and solving the nonhomogeneous linear equation system $(A-E)X_{(1)2}^{(1)}=X_{(1)1}^{(1)}$, namely:

$$\begin{pmatrix} -1 & -1 & -3 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \tag{42}$$

we obtain the second vector $X_{(1)2}^{(1)}=(3,2,-2)^T$ in the cyclic basis corresponding to $P_1=2$. Solving again the homogeneous linear equation system $(A-E)X_{(1)1}^{(2)}=0$ yields $X_{(1)1}^{(2)}=(8,6,-5)^T$. Then:

$$T=(X_{(1)1}^{(1)},X_{(1)2}^{(1)},X_{(1)1}^{(2)})=\begin{pmatrix} 3 & 3 & 8 \\ 3 & 2 & 6 \\ -2 & -2 & -5 \end{pmatrix},\; T^{-1}PT=J=\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{43}$$

Further using *Theorem 2* we have:

$$P^n=TJ^nT^{-1}=\begin{pmatrix} 3 & 3 & 8 \\ 3 & 2 & 6 \\ -2 & -2 & -5 \end{pmatrix}\begin{pmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix}\begin{pmatrix} -2 & 1 & -2 \\ -3 & -1 & -6 \\ 2 & 0 & 3 \end{pmatrix}=\begin{pmatrix} 49-9n & -3n & 18n-72 \\ 9n-36 & 1-3n & 54-18n \\ -30+6n & 2n & -44+12n \end{pmatrix} \tag{44}$$

Thus, system (1) is reduced to:

$$\begin{pmatrix} u_{1,t}^{(\alpha)} \\ u_{2,t}^{(\alpha)} \\ u_{3,t}^{(\alpha)} \end{pmatrix}=\begin{pmatrix} 9n & 0 & 0 \\ 0 & -3n & 0 \\ 0 & 0 & 12n \end{pmatrix}\begin{pmatrix} u_{1,x}^{(2\alpha)} \\ u_{2,x}^{(2\alpha)} \\ u_{3,x}^{(2\alpha)} \end{pmatrix} \tag{45}$$

Taking the traveling wave transformation $\xi_i=k_ix^\alpha/\alpha+c_it^\alpha/\alpha+w_i(i=1,2,3)$, here k_i, c_i, and w_i are constants, we have $c_1u_1'=9k_1^2nu_1''$, $c_2u_2'=-3k_2^2nu_2''$ and $c_3u_3'=12k_3^2nu_3''$. Integrating these three equations with respect to ξ_1, ξ_2, and ξ_3, respectively, taking the integral constants as zeros, then solving the resulting equations yields exact and explicit solutions of system (1):

$$u_1=a_1\mathrm{e}^{\frac{c_1}{9k_1^2n}\xi_1}+d_1,\; u_2=a_2\mathrm{e}^{-\frac{c_2}{3k_2^2n}\xi_2}+d_2,\; u_3=a_3\mathrm{e}^{\frac{c_3}{12k_3^2n}\xi_3}+d_3,\; \xi_i=k_i\frac{x^\alpha}{\alpha}+c_i\frac{t^\alpha}{\alpha}+w_i(i=1,2,3) \tag{46}$$

Derivation of fractional system (3)

Fixed two endpoints $0<t_1<t_2$, referring to [16] we find a sufficiently smooth solution $f(t)$ when the following conformable fractional functional is extremum:

$$\Pi_\alpha=I_\alpha^{x_1,x_2}(L_\alpha(f,f^{(\alpha)},t))(t),\; f=f(t),\; f^{(\alpha)}=\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} \tag{47}$$

where the variation of $f(t)$ satisfies $\delta f(t_1)=\delta f(t_2)=0$. Therefore, the variation of fractional functional Π_α is as:

$$\begin{gathered}\delta\Pi_\alpha=\Pi_\alpha(f+\delta f)-\Pi_\alpha(f)=\\ =I_\alpha^{x_1,x_2}(L_\alpha(f+\delta f,f^{(\alpha)}+\delta f^{(\alpha)},t)-L_\alpha(f,f^{(\alpha)},t))(t)=0\end{gathered}\tag{48}$$

Assume that $f(t)$ and $f(t)+\delta f(t)$ have the first-order approximation curve:

$$\begin{gathered}L_\alpha(f+\delta f,f^{(\alpha)}+\delta f^{(\alpha)},t)=L_\alpha(f,f^{(\alpha)},t)+\\ +\frac{\partial^\alpha L_\alpha(f,f^{(\alpha)},t)}{\partial f^\alpha}\delta f+\frac{\partial^\alpha L_\alpha(f,f^{(\alpha)},t)}{\partial(f^{(\alpha)})^\alpha}\delta f^{(\alpha)}\end{gathered}\tag{49}$$

Substituting eq. (49) into eq. (48), we have:

$$I_\alpha^{x_1,x_2}\left(\frac{\partial^\alpha L_\alpha(f,f^{(\alpha)},t)}{\partial f^\alpha}\delta f+\frac{\partial^\alpha L_\alpha(f,f^{(\alpha)},t)}{\partial(f^{(\alpha)})^\alpha}\delta f^{(\alpha)}\right)(t)=0\tag{50}$$

Through fractional integration by parts, from eq. (50) we get:

$$\begin{gathered}\frac{\partial^\alpha L_\alpha(f,f^{(\alpha)},t)}{\partial(f^{(\alpha)})^\alpha}\delta f(t)\Bigg|_{t_1}^{t_2}+\\ +I_\alpha^{t_1,t_2}\left(\frac{\partial^\alpha L_\alpha(f,f^{(\alpha)},t)}{\partial f^\alpha}-\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\frac{\partial^\alpha L_\alpha(f,f^{(\alpha)},t)}{\partial(f^{(\alpha)})^\alpha}\delta f\right)(t)=0\end{gathered}\tag{51}$$

Considering $\delta f(t_1)=\delta f(t_2)=0$, we simplify eq. (51) as:

$$I_\alpha^{t_1,t_2}\left[\left(\frac{\partial^\alpha L_\alpha(f,f^{(\alpha)},t)}{\partial f^\alpha}-\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\frac{\partial^\alpha L_\alpha(f,f^{(\alpha)},t)}{\partial(f^{(\alpha)})^\alpha}\right)\delta f\right](t)=0\tag{52}$$

We, therefore, derive the conformable fractional Euler-Lagrange equation:

$$\frac{\partial^\alpha L_\alpha(f,f^{(\alpha)},t)}{\partial f^\alpha}-\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\frac{\partial^\alpha L_\alpha(f,f^{(\alpha)},t)}{\partial(f^{(\alpha)})^\alpha}=0\tag{53}$$

Considering the homogeneous fractional vibration equation with n DoF:

$$Mq^{(2\alpha)}+Gq^{(\alpha)}+Kq=0\tag{54}$$

with M, q, G, and K described in eq. (3). The fractional Lagrange function of eq. (54) is:

$$L_\alpha(q,q^{(\alpha)},t)=\frac{1}{2}[(q^{(\alpha)})^T Mq^{(\alpha)}+(q^{(\alpha)})^T Gq-(q^{(\alpha)})^T Kq]=0\tag{55}$$

the corresponding action of which is the fractional functional of the displacement q to be solved:

$$S_\alpha=I_\alpha^{t_1,t_2}(L_\alpha(q,q^{(\alpha)},t))(t)\tag{56}$$

Introducing a dual momentum:

$$p=\frac{\partial^\alpha L_\alpha}{\partial(q^{(\alpha)})^\alpha}=Mq^{(\alpha)}+\frac{1}{2}Gq\tag{57}$$

namely:

$$q^{(\alpha)} = -\frac{1}{2}M^{-1}Gq + M^{-1}p \tag{58}$$

Performing fractional Legendre transformation with *A*, *B*, *D* described in eq. (3):

$$H_\alpha(q,p,t) = p^T q^{(\alpha)} - L_\alpha(q,q^{(\alpha)},t) = \frac{1}{2}p^T Dp + p^T Aq + \frac{1}{2}q^T Bq \tag{59}$$

Similar to eqs. (48) and (53), we have the fractional variational principle:

$$S_\alpha = I_\alpha^{t_1,t_2}(p^T q^{(\alpha)} - H_\alpha(q,p,t))(t) = 0, \quad \delta S_\alpha = 0 \tag{60}$$

and hence obtain a pair of fractional Hamilton canonical equations:

$$q^{(\alpha)} = \frac{\partial^\alpha H_\alpha}{\partial p^\alpha} = Aq + Dp, \quad p^{(\alpha)} = -\frac{\partial^\alpha H_\alpha}{\partial q^\alpha} = -Bq - A^T p \tag{61}$$

the vector form of which are exactly system (3). We would like to note that the fractional Hamilton function H_α of system (61) is conserved. In fact, using system (61) we have:

$$\frac{\mathrm{d}^\alpha H_\alpha}{\mathrm{d}t^\alpha} = \frac{\partial^\alpha H_\alpha}{\partial q^\alpha}q^{(\alpha)} + \frac{\partial^\alpha H_\alpha}{\partial p^\alpha}p^{(\alpha)} + \frac{\partial^\alpha H_\alpha}{\partial t^\alpha} \iff \frac{\partial^\alpha H_\alpha}{\partial t^\alpha} - \frac{\mathrm{d}^\alpha H_\alpha}{\mathrm{d}t^\alpha} = 0 \tag{62}$$

Besides, the matrix *H* of system (3) is a Hamilton matrix [12]. This is because that $JH = (JH)^T$ with J being a symplectic matrix. At the same time, there is a relationship between the Hamilton matrix *H* and the fractional Hamilton function H_α, that is, $H_\alpha = -v^T(JH)v/2$. With similar ideas, it is worthwhile to extend some existing models [17-20] to fractional-order cases.

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References

[1] Podlubny, I., *Fractional Differential Equations*, Academic Press, San Diego, Cal., USA, 1999
[2] Inc, M., *et al.*, Analysis of Novel Fractional COVID-19 Model with Real-Life Data Application, Results in Physics, *23* (2021), APR., ID 103968.
[3] Gulsen, S., Inc, M. Time Fractional Super KdV Equation: Lie Point Symmetries, Conservation Laws, Explicit Solutions with Convergence Analysis, *International Journal of Geometric Methods in Modern Physics*, *19* (2022), 8, ID 2250122
[4] Zhang, S., Zheng, X. W., Non-Differentiable Solutions for Non-Linear Local Fractional Heat Conduction Equation, *Thermal Science*, *25*, (2021), Special Issue 2, pp. S309-S314
[5] Xu, B., *et al.*, Analytical Methods for Non-Linear Fractional Kolmogorov-Petrovskii-Piskunov Equation: Soliton Solution and Operator Solution, *Thermal Science*, *25* (2021), 3B, pp. 2159-2166
[6] Xu, B., *et al.*, Line Soliton Interactions for Shallow Ocean-Waves and Novel Solutions with Peakon, Ring, Conical, Columnar and Lump Structures Based on Fractional KP Equation, *Advances in Mathematical Physics*, *2021* (2021), ID 6664039
[7] Xu, B., *et al.*, Variational Iteration Method for Two Fractional Systems with Boundary Conditions, *Thermal Science*, *26* (2022), 3B, pp. 2649-2657
[8] Xu, B., Zhang, S., Riemann-Hilbert Approach for Constructing Analytical Solutions and Conservation Laws of a Local Time-Fractional Non-Linear Schrödinger Equation, *Symmetry*, *13* (2021), 9, ID 13091593

[9] Khalil, R., *et al.*, A New Definition of Fractional Derivative, *Journal of Computational and Applied Mathematics*, *264* (2014), JUL., pp. 65-70
[10] Zhang, S., Li, C. H., Method of Calculating a Kind of Up-Triangular Matrix Power, *Journal of Jinzhou Normal College (Natural Science Edition)*, *23* (2001), 3, pp. 58-59. (in Chinese)
[11] ***, Former Algebra Group, Peking Univ., *Advanced Algebra* (in Chinese), 3rd ed., Higher Education Press, Beijing, CN, 2003
[12] Zhong W. X., Gao, Q., *Symplectic Breaking the Cocoon – Symplectic Expansion and New Level*, Dalian University of Technology Press, Dalian, CN, 2011. (in Chinese)
[13] Zhao, S. J., An Expression of Symplectic Matrix on Fields, *Journal of Heilongjiang Commercial College (Natural Sciences Edition)*, *15* (1999), 2, pp. 55-56. (in Chinese)
[14] Han, J., Long, Y., Normal Forms of Symplectic Matrices (II), *Acta Scientiarum Naturalium Universitatis Nankaiensis*, *31* (1999), 3, pp. 30-41
[15] Zhang, S., Notes and Modifications to the Calculation Method of Jordan Canonical Form (in Chinese), *Journal of Bohai Univeristy* (*Natural Science Edition*), *43* (2022), 2, pp. 155-16
[16] Qian, W. C., *Generalized Variational Principle*, Knowledge Publishing House, Shanghai, CN, 1985. (in Chinese)
[17] Ulutas, E., *et al*., Exact solutions of stochastic KdV equation with conformable derivatives in white noise environment, *Thermal Science, 25* (2021), Special Issue 2, pp. S143-S149
[18] Yildirim, E. N., *et al*., Reproducing kernel functions and homogenizing transforms, *Thermal Science, 25* (2021), Special Issue 2, pp. S9-S18
[19] Abdelrahman, M. A. E., *et al*., Exact solutions of the cubic Boussinesq and the coupled Higgs systems, *Thermal Science, 24* (2020), Suppl. 1, pp. S333-S342
[20] Menni, Y., et al., Heat and mass transfer of oils in baffled and finned ducts, *Thermal Science, 24* (2021), Suppl. 1, pp. S267-276

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