NON-FICKIAN TRANSPORT MODELS FOR CHARACTERISING THE SEDIMENT SUSPENSION IN UNSTEADY FLOWS

by

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Non-Fickian transport of suspended sediment has been observed at field and laboratory scales. Such as turbulent bursting, resulting in complex dynamics for the sediment particles movement. The erosion and deposition of sediment have an impact on the hydraulic engineering and environment. This study makes an attempt to develop the variable-order fractional advection-diffusion equation (VOFADE) and variable-order Hausdorff fractal derivative advection-diffusion equation (VOHADE) models to describe the vertical distribution of suspended sediment in unsteady turbulent flows. From a classical viewpoint, the distribution of the concentration in sediment-laden flows is determined based on Fick's first law. However, the vertical diffusion of suspended particles exhibits the non-locality/space scale dependency and history memory/time scale dependency properties due to turbulent bursting. Moreover, previous literatures have indicated that turbulence structure changes with the water depth. Hereby, we employ the space-dependent VOFADE and VOHADE models to describe the vertical diffusion of suspended sediment in unsteady flows, and further test its applicability with the experimental data. Numerical simulation results confirm that the VOFADE and VOHADE models give a better agreement with the experimental data and can well characterise the space-dependent anomalous transport. Hence, the models proposed by this study may help to provide a powerful mathematical physical model in the quantification of suspended sediment transport.

Key words*: non-Fickian, suspended sediment, vertical distribution, turbulence, anomalous diffusion*

Introduction

For river sediment control, the vertical distribution of suspended sediment concentration in unsteady flows is an important topic in sediment transport and a critical consideration [1-4]. In the past few decades, different theoretical models and empirical formulas based on Fick's law have been applied to describe the sediment transport in unsteady flow [1-3, 5-8]. However, there often have insufficient information accurately characterise preferential turbulent bursting and non-local jump of particles. Hence, the diffusion of suspended sediment does not satisfy the classical Fick's law, and the classical advection-diffusion equation (ADE) is unable to accurately simulate the vertical transport of suspended sediment. The fractional and fractal models can characterise the anomalous behavior which does not abide by Fick's law.

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Recently, the fractional advection-diffusion equation (FADE) and Haudorff fractal derivative advection-diffusion equation (HADE) models have advantages to describe the heavy-tailed and early arrived [9-12]. However, there are differences between the two models:

- the FADE mode derived from the assumption of the long-term memory and non-local transport is a non-local model, while the HADE based on the metric transform between normal and fractal structures is a local model,
- the FADE model characterises the non-Fickian transport behavior via the memory and non-locality of the process, while the HADE model describes the non-Fickian transport via the temporal variation of the fractal time or space scale, and
- from the statistical viewpoint, the FADE model produces power-law distribution, while the HADE model yields the stretched exponential distribution in the process of capturing anomalous transport.

Therefore, the FADE and HADE models are effective methods to describe anomalous transport.

For unsteady open channels flow with suspended sediment, the distribution of the suspended load has been often proposed from the theory. Numerous studies related to sediment-laden flows have been used to examine the vertical distribution of the suspended sediment concentration [13, 14]. The problems of sediment-laden flows have affected the river and environment [15, 16]. The vertical distribution of suspended sediment concentration reflects the sediment transport mechanism in unsteady flow.

The FADE and HADE models have been used to describe the anomalous diffusion of suspended sediment [17, 18]. The vertical distribution of suspended sediment concentration is based on the gravity and turbulent diffusion [4]. However, due to the turbulent bursting, the particles could display the influence of temporal memory and spatial dependency on the mechanics of sediment transport, and the vertical transport of the suspended load is a complex diffusion process. The FADE and HADE models are the diffusion models that are not governed by Fick's law. In these cases, the literature indicated that the constant-order diffusion models are not suitable to describe time- and space-dependent diffusion processes [19, 20].

There is fundamental significance to study the diffusion behavior in turbulence. Based on the turbulent bursting, the sediment-laden flows present anisotropic characteristics. Simultaneously, the particles have a chance to move upwards any distance from their original location the water surface in a given time interval. The vertical movement of suspended particles is non-Fickian transport. Non-Fickian diffusion is currently modeled to statistical mechanics and non-locality [21]. Due to the complexity of turbulence, the turbulent bursting causes the variations of the suspended sediment concentration with the spatial position. Thus, it is difficult to be accurately characterised by a constant-order FADE and HADE models. The variable-order fractional operator depended on the non-stationary power-law kernel can characterise the memory and non-locality of many physical processes [20]. The Hausdorff fractal derivative is based on the spatiotemporal scale transform [22]. The fractal structure usually changes with time or space, resulting in anomalous diffusion behaviors [23, 24]. Therefore, the variable-order fractional and Hausdorff fractal calculus are the effective method to accurately characterise complex physical behaviors.

The VOFADE and VOHADE models are an extension of the constant-order advection-diffusion models, which have been applied to describe complex dynamics (anomalous diffusion) in science and engineering fields [25-28]. However, the comprehensive investigation of the VOFADE and VOHADE models about the anomalous sediment diffusion has not been reported in the literature. In this situation, the space-dependent VOFADE and VOHADE models are the available approaches that can correctly describe location-dependent super-diffusion behaviors.

Therefore, we can characterise the memory effect and non-local/fast displacements based on turbulent bursting of sediment suspension. We solve the variable-order fractional and Hausdorff fractal derivative advection-diffusion equations to obtain numerical solutions, in which the time fractional/fractal derivative corresponds to history memory/time scale dependence, and the space fractional/fractal derivative refers to non-locality/spatial scale dependency properties of suspended load transport. The different fractional/fractal order connects the complexity of temporal and spatial changes in sediment-laden flows.

Materials and methods

As aforementioned, some extensions of the fractional and fractal derivative models can simulate the non-Fickian diffusion (*i.e.*, super-diffusion, sub-diffusion, and normal-diffusion). Therefore, here we focus on the VOFADE and VOHADE models which are proposed to characterise anomalous diffusion phenomenon.

Based on the theory of turbulent diffusion, the suspended sediment transport equation is derived. This equation includes turbulent diffusion and gravity in the suspension of sediment particles [29]. The physical mechanism of the unsteady sediment suspension distribution is the vertical dynamic evolution of sediment concentration between downward sediment settlement and upward turbulent diffusion.

In unsteady-state, the ADE model has been undertaken to describe the transport of the suspended sediment [1, 3]. Many previous investigations on sediment diffusion in unsteady sediment-laden flows were based on the traditional advection-diffusion equation:

$$\frac{\partial S(y,t)}{\partial t} = \omega \frac{\partial S(y,t)}{\partial y} + \frac{\partial}{\partial y}\left[\varepsilon_{sy} \frac{\partial S(y,t)}{\partial y}\right] \tag{1}$$

where S is the sediment volumetric concentration, y – the vertical distance from the bed L, ω – the sediment settling velocity L/T, and ε_{sy} – the vertical component of sediment turbulent diffusion coefficient. When ε_{sy} is a constant, eq. (1) can be written as:

$$\frac{\partial S(y,t)}{\partial t} = \omega \frac{\partial S(y,t)}{\partial y} + \varepsilon_{sy} \frac{\partial^2 S(y,t)}{\partial y^2} \tag{2}$$

Equation (1) indicates that the time rate of sediment concentration at the position y is due to the difference between two opposite movements (*i.e.*, downward settling and upward diffusion).

The previous investigations indicate that the turbulent diffusion of suspended sediment transport is an anomalous diffusion behavior [30, 31]. The fractional and fractal derivative are powerful in characterising spatial non-locality/scale-dependency and history memory processes. The variable-order fractional/fractal derivative means that the order of the derivative is a time- or space-dependent function. In recent decades, the variable fractional/fractal theory and applications of variable fractional/fractal calculus have drawn quickly increasing attentions [32-34].

Space variable-order fractional model for sediment suspension

The vertical distribution of suspended sediment is an anomalous diffusion phenomenon with spatial dependence caused by turbulent bursting. Thus, in this study, we developed a VOFADE as the governing equation for sediment suspension. Accordingly, the 1-D variable fractional suspended sediment transport equation can be expressed:

$$\frac{\partial^{\alpha} S(y,t)}{\partial t^{\alpha}} = \omega \frac{\partial S(y,t)}{\partial y} + \varepsilon_{sy} \frac{\partial^{\beta(y)} S(y,t)}{\partial y^{\beta(y)}} \tag{3}$$

where α is the order of time fractional derivative and $\beta(y)$ – the variable order of the space fractional derivative ($0 < \alpha < 1$, $1 < \beta(y) < 2$). The time fractional derivative term $\partial S/\partial t^{\alpha}$ captures the time scale dependency of the sediment transport, and the variable space fractional derivative term ($\partial^{\beta(y)} S/\partial t^{\beta(y)}$) describes the vertical heavy-tailed transport of sediment. The sediment transport along the flume can be simulated by solving the governing eq. (3). The ε_{sy} is the diffusion coefficient. The Caputo time fractional derivative is expressed:

$$\frac{\partial^{\alpha} f(t)}{\partial t^{\alpha}} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^{\alpha}} \mathrm{d}\tau, \ 0 < \alpha \le 1 \tag{4}$$

The left-side variable-order Riemann-Liouville space fractional derivative can be written [35]:

$$\frac{\partial^{\beta(y)} g(y)}{\partial y^{\beta(y)}} = \frac{1}{\Gamma[2-\beta(y)]} \frac{\mathrm{d}^2}{\mathrm{d}y^2} \int_0^y \frac{g(\xi)}{(y-\xi)^{\beta(y)-1}} \mathrm{d}\xi, \ 1 < \beta(y) \le 2 \tag{5}$$

Space variable-order fractal model for sediment suspension

Hausdorff fractal derivative is proposed based on the fractal structure of medium, which is used to characterise anomalous diffusion phenomena [22, 36]. The turbulent flow structure over rough beds, which have fractal features, can cause anomalous transport behavior of the suspended sediment varying in space location. Thus, in this paper, we attempted to develop VOHADE model to describe anomalous transport of sediment by space-time transform- Hence, the 1-D variable fractal suspended sediment transport equation can be written:

$$\frac{\partial S(y,t)}{\partial t^{\alpha}} = \omega \frac{\partial S(y,t)}{\partial y^{\beta(y)}} + \frac{\partial}{\partial y^{\beta(y)}} \left[\varepsilon_{sy} \frac{\partial S(y,t)}{\partial y^{\beta(y)}} \right] \tag{6}$$

where α is the order of the time fractal derivative, $\beta(y)$ – the variable order of the space fractal derivative ($0 < \alpha < 1$, $0 < \beta(y) < 1$). The time fractal derivative term $\partial S/\partial t^{\alpha}$ captures the time scale dependency of sediment transport, and the variable space fractal derivative term $\partial S/\partial y^{\beta(y)}$ describes the space-scale dependent transport of sediment. The sediment transport along the flume can be simulated by solving the governing eq. (6), ε_{sy} is a constant which denotes the diffusion coefficient.

The definitions of Hausdorff fractal derivative in space and time are written:

$$\begin{aligned} \frac{\mathrm{d}f(t)}{\mathrm{d}t^{\alpha}} &= \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{(t+\Delta t)^{\alpha} - t^{\alpha}} \\ \frac{\mathrm{d}g(y)}{\mathrm{d}y^{\beta(y)}} &= \lim_{\Delta y \to 0} \frac{g(y+\Delta y) - g(y)}{(y+\Delta y)^{\beta(y+\Delta y)} - y^{\beta(y)}} \end{aligned} \tag{7}$$

Initialization and boundary conditions

Initialization condition

In unsteady turbulent flows, we employ the initial conditions and boundary conditions as follows.

The initial condition:

$$S(y,0) = C \tag{8}$$

where C is a constant.

Boundary conditions

At the water surface, there is no net transfer sediment:

$$\omega S+\varepsilon_{sy}\frac{\partial S}{\partial y}=0,\ \left(y=h\right) \tag{9}$$

At the bottom, we assume the concentration S is equal to the reference concentration:

$$S=S_a,\ \left(y=a\right) \tag{10}$$

Numerical results and model analysis

In this study, we used MATLAB to simulate the transport of suspended sediment. In figs. 1(a) and 1(b), with the decreasing of the space variable-order index, the sediment diffusion velocity increases. The concentration curve changes from slow to steep. It shows that the non-locality/scale dependency of space is enhanced, and the anomalous diffusion is faster than the normal diffusion. As shown in figs. 1(c) and 1(d), with the decreasing of time order index, the sediment diffusion velocity decreases. And the concentration curve changes from steep to slow. It shows that the history memory/time scale dependency is enhanced, and the anomalous diffusion is slower than the normal diffusion.

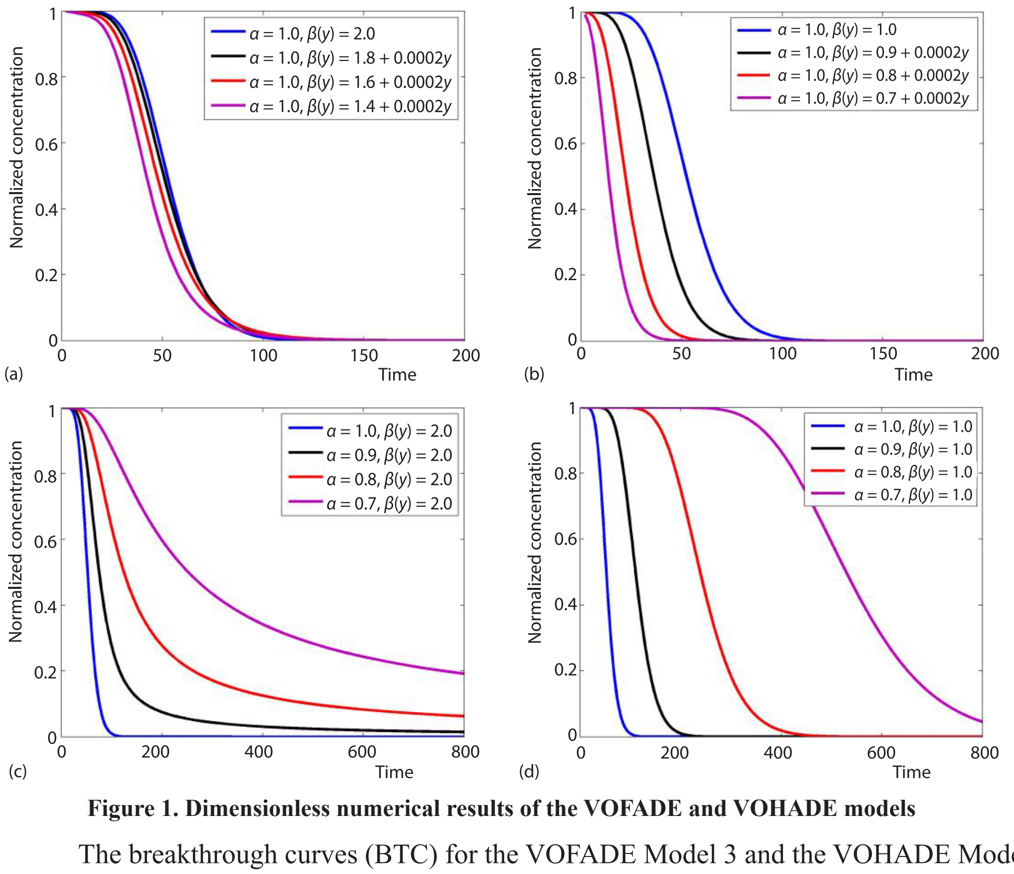

Figure 1. Dimensionless numerical results of the VOFADE and VOHADE models

The breakthrough curves (BTC) for the VOFADE Model 3 and the VOHADE Model 6 are shown in fig. 2. These two models describe anomalous transport of sediment using the space-dependent index $\beta(y)$. For these two models, the diffusion rate of the VOHADE model

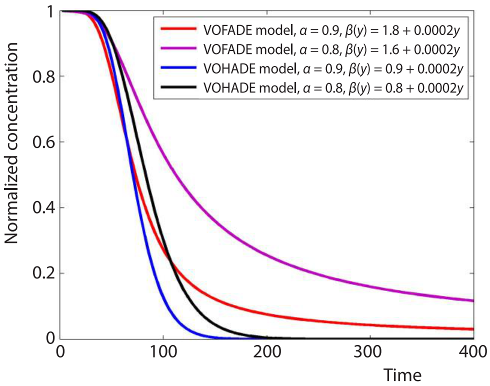

Figure 2. Dimensionless numerical results calculated by the VOFADE Model 3 and the VOHADE Model 6

is faster than that of the VOFADE model under the same order of the time derivative except α = 0.9 in the early stage. Statistically speaking, the VOFADE model describes the BTC of sediment transport with the power-law decay, while the VOHADE model yields the stretched exponential decay. These two models define a space-dependent scaling index to characterise diffusion rates, and could capture the anomalous transport behavior of suspended sediment in turbulence.

Here, we use a set of reported experimental data of Nguyen to test the efficiency of the VOFADE and VOHADE models in describing the vertical distribution of suspended sediment [8]. The experiments are divided into two groups, the first group is the quartz flour, and the average concentration of sediment in the experiment is 3 g/L and 20 g/L, respectively. The second group is the alumina.

To estimate the difference between the experimental data and simulation results quantitatively, the root mean square error (RMSE) is introduced:

$$RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(S_{i0} - S_{ie}\right)^2} \tag{11}$$

where S_{i0} is the measured sediment concentration, S_{ie} – simulated sediment concentration, and N – the number of individual observation points.

Case 1: Figures 3 and 4 show the best-fit results for two different concentration (S = 3 g/L, 20 g/L). The results show that the VOFADE model is more efficiently than the VOHADE model in capturing the observed vertical distribution of suspended sediment transport. Figures 3 and 4 use the experimental data from the Laboratory of Institute of Hydraulic Engineering and Water Resources Management (IWK) to calibrate the variable fractional/fractal derivative orders α and $\beta(y)$. The numerical simulation results indicate that the VOFADE model can capture the sediment BTC observed in experiments reliably, by fitting the parameters α and $\beta(y)$.

In order to calculate conveniently, this study only investigates variable-orders with linear function of space. The BTC obtained by the VOFADE and VOHADE models with the space-dependent index are presented in figs. 3-6 and tabs. 1 and 2. The space-dependent index means that the vertical transport of suspended sediment is space-location dependent. As shown in figs. 3 and 4, the VOFADE model gives a better agreement with experimental data, except for 3 g/L in $t \in (1200, 1500]$ in the BTC. The inconsistency could be caused by the error of the measurement. The VOHADE model captures the general trend of the BTC, which cannot fit as well as the VOFADE model.

In figs. 3 and 4 and tab. 1, the time fractional/fractal derivative order of 3 g/L is larger than 20 g/L, and the space variable fractional/fractal derivative order of 3 g/L is smaller than 20 g/L, because the non-locality/space scale dependency is strong compare with the history-dependency in low concentration. The effect of the space variable order is larger than the time order, which shows that the simulated sediment BTC is sensitive to the space index $\beta(y)$. It exhibits that low concentration displays more obvious super-diffusion process than high concentration, indicating that the VOFADE model can describe sediment transport in unsteady flows reliably, by fitting the parameters α and $\beta(y)$.

As the concentration increasing, the viscosity of the water flow increases. More sediment particles gather in the vortex body manufacturing field near the side wall, and the average size of the vortex body increases. The masking effect of sediment particles on the rough side walls is strengthened, and the relative motion between particles and water on the interference of the water flow is enhanced, and the turbulence of the water flow is weakened. Therefore, the turbulent diffusion of the low concentration is stronger than the high concentration. The super-diffusion phenomenon of the low concentration is stronger, the diffusion rate is faster.

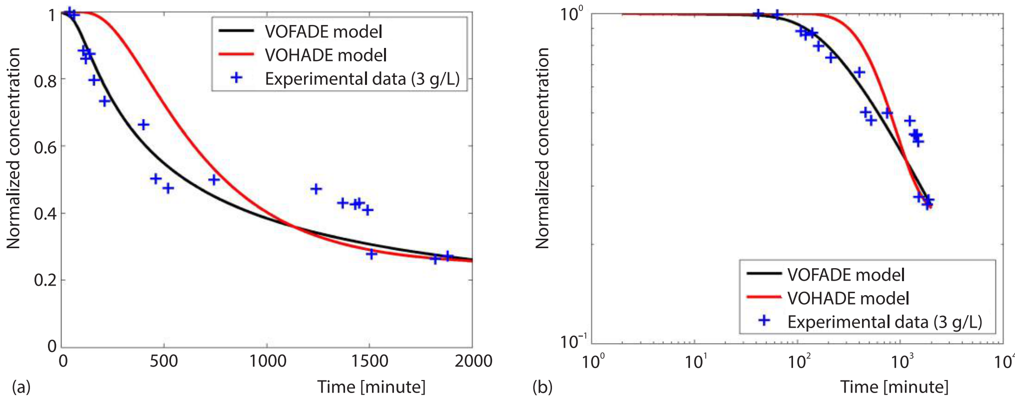

Figure 3. Comparison of the best-fit BTC obtained by the VOFADE Model 3 and VOHADE Model 6 for the experimental data; *S* = 3g/L; (a) linear plot and (b) double-log plot

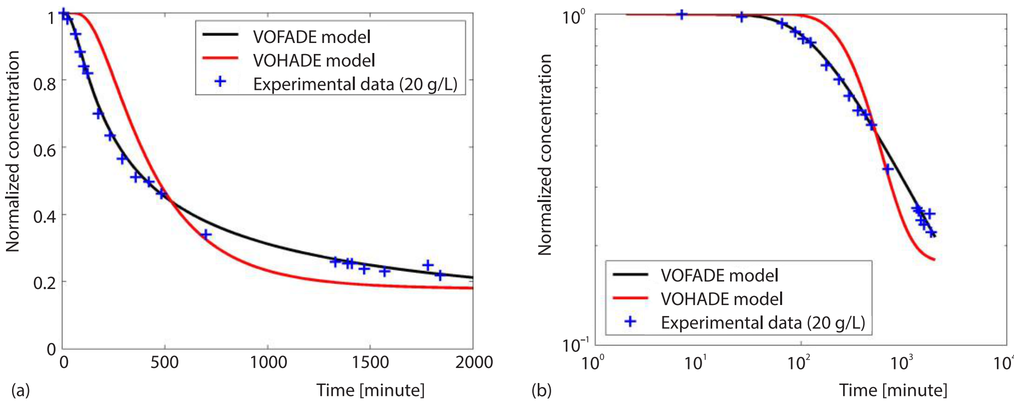

Figure 4. Comparison between the best-fit BTC obtained by the VOFADE Model 3 and VOHADE Model 6 and the experimental data; *S* = 20g/L; (a) linear plot and (b) double-log plot

Table 1. Model parameters for experimental data with the quartz flour

Run number	Time fractional order α	Space fractional order $\beta(y)$	Time fractal order α	Space fractal order $\beta(y)$
3 g/L	0.57	1.30+0.00002y	0.39	0.51+0.00002y
20 g/L	0.56	1.45+0.00002y	0.37	0.52 + 0.00002y

Case 2: Figures 5 and 6 show that the best-fit results for two different pH (pH = 6.6, 9.0) about the alumina. In figs. 5 and 6 and tab. 2, the time fractional/fractal derivative order of pH = 6.6 is smaller than that of pH = 9.0, and the space variable fractional/fractal derivative

order of pH = 6.6 is larger than that of pH = 9.0. As shown in figs. 5 and 6, the VOFADE model provides a better agreement with experimental data in BTC. And the VOHADE model captures the general trend of the BTC, which has the deviation with the VOFADE model in describing pH = 6.6.

The alumina is an amphoteric oxide, when the pH value is less than 7, the absolute value of the ζ potential of the particle surface is smaller. The electrostatic repulsive force between particles is not enough to compete with the attractive force between particles. The Brown's motion of the particles causes the particles to collide and settle with each other, and the diffusion stability is poor. As the pH value increasing, the absolute value of the ζ potential on the particle surface continues to increase. The electrostatic repulsive force formed between the particles is sufficient to prevent the particles from attracting and colliding with each other due to Brown's motion. The larger the electrostatic repulsive force, this also makes the particles relatively independent, and the distance between particles increases, further reducing the opportunities for particles to agglomerate and settle, and improving the diffusion stability and diffusion rate of the alumina. When the pH value is about 9, the negative charge on the particle surface increases, and the viscosity of the suspension decreases. The electrostatic repulsion between particles in the suspension is strong, consequently, the suspension particles have better diffusivity, and the diffusion rate is faster.

In figs. 5 and 6, the space derivative value of pH = 9.0 is smaller than that of pH = 6.6, because the non-locality/space scale dependency is strong compared with the history-dependency in pH = 9.0, it exhibits super-diffusion phenomenon. In contrast, the time derivative value of pH = 6.6 is smaller than that of pH = 9.0, because the history-dependency is stronger than non-locality in pH = 6.6, it displays sub-diffusion behaviors.

The diffusion rate of alumina in alkaline condition is higher than that in acid condition. Thus, the alumina displays stronger super-diffusive behavior in pH = 9.0 than pH = 6.6. Figures 5 and 6 show that the simulated sediment BTC is sensitive to the index α and $\beta(y)$.

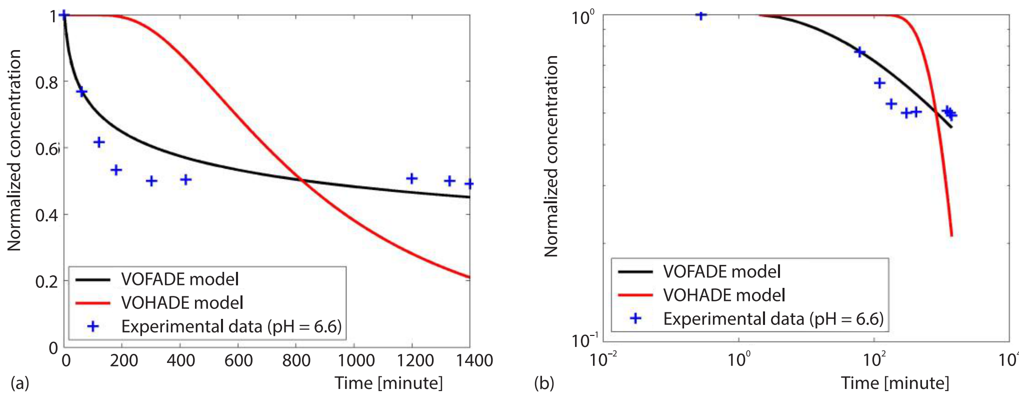

Figure 5. Comparison between the best-fit BTC obtained by the VOFADE Model 3 and VOHADE Model 6 and the experimental data; pH = 6.6; (a) linear plot and (b) double-log plot

Table 2. Model parameters for experimental data with the alumina

Run number	Time fractional order α	Space fractional order $\beta(y)$	Time fractional order α	Space fractional order $\beta(y)$
pH = 6.6	0.27	1.52+0.00002y	0.20	0.57+0.00002y
pH = 9.0	0.72	1.33+0.00002y	0.46	0.56+0.00002y

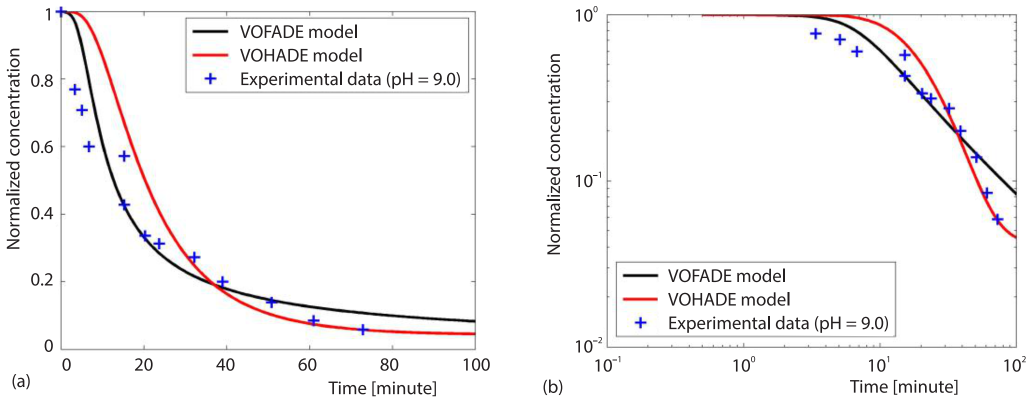

Figure 6. Comparison between the best-fit BTC obtained by the VOFADE Model 3 and VOHADE Model 6 and the experimental data; pH = 9.0; (a) linear plot and (b) double-log plot

To compare the fitting effect of the two models, tab. 3 provides the root mean square error (RMSE) corresponding to the fitting results of the experimental data. It is shown that the VOFADE model is more suitable in describing anomalous transport of suspended sediment.

Table 3. The RMSE of numerical results using the VOFADE model and VOHADE model for the measurement data

Models	RMSE (VOFADE)	RMSE (VOHADE)
$S = 3$ g/L	0.0671	0.1405
$S = 20$ g/L	0.0143	0.1044
pH = 6.6	0.0654	0.3254
pH = 9.0	0.1014	0.1634

Discussion

Here, two stochastic models are briefly compared in characterising suspended sediment transport. First, the VOFADE model is based on the assumption of non-local transport and long-term memory for the suspended sediment, while the VOHADE model is established on the basis of the metric transformation between the normal and fractal structures. It should be pointed out that both non-locality/long-memory and metric transformation are used to characterise the influence of the anomalous transport mechanics of the suspended sediment. Second, statistically speaking, the VOFADE model produces the power-law decay for the sediment BTC, while the VOHADE model yields the stretched exponential decay. Third, the variable fractal derivative is a local operator and the variable fractional derivative is a global one, the numerical results indicate that the solution of the VOHADE model displays a faster decay than the VOFADE model.

The super-diffusion may also be space-dependent due to the spatial variation of the sediment suspension. Moreover, the suspension changes apparently in vertical space corresponding to the river. Therefore, it indicates that the VOFADE model can reliably describe the sediment transport in experiments, via fitting parameters α and $\beta(y)$.

In summary, the previous analysis shows that the anomalous dynamics of suspended sediment transport is sensitive to the index embedded in the VOFADE and VOHADE models. The VOFADE and VOHADE models capture the trapping effect of suspended sediment transport using the time fractional/fractal derivative index α, and describe the vertical fast displacement of the sediment particles using the space variable fractional/fractal derivative index $\beta(y)$. By adjusting the two indexes in the VOFADE and VOHADE models, it is conveniently captured a wide range of the anomalous behavior for suspended sediment transport.

Conclusion

The VOFADE and VOHADE models were proposed in this study to capture the history memory and non-local dependence/space scale dependency of the vertical sediment diffusion due to the turbulent bursting observed in laboratorial experiments. According to the comparison of results, the VOFADE model provides a better fit to the measured data than VOHADE model. The VOFADE and VOHADE models can capture the overall anomalous transport of suspended sediment along vertical line. Two models, however, introduce a new parameter (*i.e.*, fractional/fractal derivative order α, $\beta(y)$. Time and space fractional/fractal derivative orders α and $\beta(y)$ are two key parameters to characterise the history-dependency and non-locality of anomalous sediment transport. However, the advantages and application potential of variable fractional/fractal advection-diffusion equation model on suspended sediment transport in unsteady flows should be further studied by theoretical and experimental verification.

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