APPLICATION OF KASHURI FUNDO TRANSFORM AND HOMOTOPY PERTURBATION METHODS TO FRACTIONAL HEAT TRANSFER AND POROUS MEDIA EQUATIONS

by

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Integral transformations have been used for a long time in the solution of differential equations either solely or combined with other methods. These transforms provide a great advantage in reaching solutions in an easy way by transforming many seemingly complex problems into a more understandable format. In this study, we used an integral transform, namely Kashuri Fundo transform, by blending with the homotopy perturbation method for the solution of non-linear fractional porous media equation and time-fractional heat transfer equation with cubic non-linearity.

Key words*: Kashuri Fundo transform, homotopy perturbation method, Caputo fractional derivative, fractional heat transfer, porous media equation*

Introduction

Integral transforms are methods that facilitate us in solving complex problems encountered in many different fields for a long time. They transform the original domain of problems into another domain and makes complex problems more understandable. Then, the solution obtained by changing the domain with the inverse integral transform is mapped to the original domain [1].

Integral transforms are frequently used methods for solving linear and non-linear differential equations. Especially in equations that are difficult to find the exact analytical solution, they are sometimes used alone and sometimes by blending them with another method. One of the equations, finding whose analytical solution is difficult to find, is fractional differential equations. The fractional calculus was first introduced by Leibniz [see in 2]. Then, fractional differential equations attracted the attention of many researchers with their extensive applications in many different fields. Researchers have used many different methods such as homotopy perturbation method [3-6], sinc methods [7-9], variational iteration method [10-12], Adomian decomposition method [13], Laplace decomposition method [14], homotopy perturbation transformation method [15], *etc.* to solve these equations.

Non-linear heat equation called the porous media equation often occurs in non-linear problems of flows in porous media, heat and mass transfer, diffusion, boundary-layer theory, viscous fluids, biological systems, and other related fields. The aim of this study is to show that the solution using mixture of Kashuri Fundo transform and homotopy perturbation method [16,

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17] over fractional porous media equation and a non-linear time-fractional heat transfer equation with cubic non-linearity is simple and understandable.

Caputo fractional derivative

The Caputo fractional derivative of order $\alpha > 0$ of a function $f(x)$, $x > 0$ is defined [18-20]:

$$\mathrm{D}_t^\alpha f(x) = \begin{cases} \dfrac{\mathrm{d}^n}{\mathrm{d}t^n} f(t), & \alpha = n \in \mathbb{N} \\ \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n f(x,\tau)}{\partial \tau^n} \mathrm{d}\tau, & n-1 < \alpha \le n \in \mathbb{N} \end{cases} \tag{1}$$

where $\Gamma(.)$ is the Gamma function and D_t^α is called the Caputo derivative operator.

Kashuri Fundo transform method

We consider functions in the set F defined [21]:

$$F = \left\{ f(t) \mid \exists M, k_1, k_2 > 0 \quad \text{such that} \quad | f(t) | \le M \mathrm{e}^{\frac{|t|}{k_i^2}}, \quad \text{if} \quad t \in (-1)^i \times [0,\infty) \right\} \tag{2}$$

For a function belonging to the set F, the constant M must be finite number. The k_1, k_2 may be finite or infinite. Kashuri Fundo transform denoted by the operator $K(.)$ is defined [21]:

$$K[f(t)](v) = A(v) = \frac{1}{v} \int_0^\infty \mathrm{e}^{\frac{-t}{v^2}} f(t) \mathrm{d}t, \quad t \ge 0, \quad -k_1 < v < k_2 \tag{3}$$

Inverse Kashuri Fundo transform is denoted by [21]:

$$K^{-1}[A(v)] = f(t), \quad t \ge 0$$

Theorem (sufficient conditions for existence of Kashuri Fundo transform)

If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order $1/k^2$, then $K[f(t)](v)$ exists for $|v| < k$ [21].

Properties of the transform

Theorem (linearity property of Kashuri Fundo transform)

Let $f(x)$ and $g(x)$ be functions whose Kashuri Fundo integral transforms exists and a, b are constants. Then [21]:

$$K[af(x) \pm bg(x)] = aK[f(x)] \pm bK[g(x)] \tag{4}$$

Theorem (Kashuri Fundo transform of the partial derivatives)

Let $A(x, v)$ be Kashuri Fundo transform of $f(x, t)$. Then [16]:

$$K\left[\frac{\partial f(x,t)}{\partial t}\right] = \frac{A(x,v)}{v^2} - \frac{f(x,0)}{v} \tag{5}$$

$$K\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = \frac{A(x,v)}{v^4} - \frac{f(x,0)}{v^3} - \frac{1}{v}\frac{\partial f(x,0)}{\partial t} \tag{6}$$

$$K\left[\frac{\partial^n f(x,t)}{\partial t^n}\right]=\frac{A(x,v)}{v^{2n}}-\sum_{k=0}^{n-1}\frac{1}{v^{2(n-k)-1}}\frac{\partial^k f(x,t)}{\partial t^k} \tag{7}$$

$$K\left[\frac{\partial f(x,t)}{\partial x}\right]=\frac{\mathrm{d}}{\mathrm{d}x}\left[A(x,v)\right] \tag{8}$$

$$K\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right]=\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[A(x,v)\right] \tag{9}$$

$$K\left[\frac{\partial^n f(x,t)}{\partial x^n}\right]=\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[A(x,v)\right] \tag{10}$$

Kashuri Fundo trasform of some special functions

Kashuri Fundo transform of some special functions [21, 22] can be seen in tab. 1.

Table 1.

$f(t)$	$K[f(t)] = A(v)$
1	v
t	v^3
t^n	$n!v^{2n+1}$
e^{at}	$v/(1 - av^2)$
$\sin(at)$	$av^3/(1 + a^2v^4)$
$\cos(at)$	$v/(1 + a^2v^4)$
$\sinh(at)$	$av^3/(1 - a^2v^4)$
$\cosh(at)$	$v/(1 - a^2v^4)$
t^α	$\Gamma(\alpha+1)v^{2\alpha+1}$
$\sum_{k=0}^{n}a_k t^k$	$\sum_{k=0}^{n}k!a_k v^{2k+1}$

Kashuri Fundo transform of Caputo fractional derivative

Let $A(x, v)$ be Kashuri Fundo transform of $f(x, t)$. The Kashuri Fundo transform of Caputo fractional derivative is defined [20]:

$$K\left[\mathrm{D}_t^{n\alpha} f(x,t)\right]=\frac{A(x,v)}{v^{2n\alpha}}-\sum_{k=0}^{n-1}\frac{1}{v^{2(n\alpha-k)-1}}\frac{\partial^k f\left(x,0\right)}{\partial t^k} \tag{11}$$

where

$$\alpha>0,\ n-1<\alpha\le n\in\mathbb{N}$$

Mixture of Kashuri Fundo transform and homotopy perturbation method

Consider a time-fractional non-linear non-homogeneous partial differential equation of the form:

$$\mathrm{D}_t^\alpha f(x,t)=Rf(x,t)+Nf(x,t)+g(x,t) \tag{12}$$

with initial conditions:

$$f(x,0)=h(x),\ \ f_t(x,0)=u\left(x\right) \tag{13}$$

where $g(x, t)$ is the non-homogeneous term, N – the non-linear differential operator, R – the linear differential operator, and $D_t^\alpha f(x, t)$ is the Caputo fractional derivative.

The procedure is as follows [16, 17]:

Taking Kashuri Fundo transform on both sides of eq. (12) and by eq. (11), we get:

$$K\left[f(x,t)\right]=vh(x)+v^3u(x)+v^{2\alpha}K\left[Rf(x,t)+Nf(x,t)\right]+v^{2\alpha}K\left[g(x,t)\right] \tag{14}$$

Applying the inverse Kashuri Fundo transform on both sides of eq. (14) and using tab. 1, we find:

$$f(x,t)=G(x,t)+K^{-1}\left[v^{2\alpha}K\left[Rf(x,t)+Nf(x,t)\right]\right] \tag{15}$$

where $G(x, t)$ is the term resulting from the non-homogeneous term and given initial conditions.

Now, we apply the homotopy perturbation method. Assuming that the solution of eq. (12) can be written as a power series in p:

$$f(x,t)=\sum_{n=0}^{\infty}p^n f_n(x,t) \tag{16}$$

and the non-linear term $Nf(x, t)$ can be decomposed:

$$Nf(x,t)=\sum_{n=0}^{\infty}p^n H_n\left(f_0,f_1,...,f_n\right) \tag{17}$$

where $H_n(f_0, f_1,..., f_n)$ are the so-called He's polynomials that represents the non-linear terms and are given:

$$H_n\left(f_0,f_1,...,f_n\right)=\frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{n}p^i f_i\right)\right]_{p=0},\quad n=0,1,2,... \tag{18}$$

Substituting eqs. (16) and (17) in eq. (15), we find:

$$\sum_{n=0}^{\infty}p^n f_n=G(x,t)+p\left\{K^{-1}\left[v^{2\alpha}K\left[R\left(\sum_{n=0}^{\infty}p^n f_n(x,t)\right)+\sum_{n=0}^{\infty}p^n H_n\right]\right]\right\} \tag{19}$$

Comparing the coefficients of like powers of p, the following approximations are obtained:

$$p^0: f_0(x,t)=G(x,t) \tag{20}$$

$$p^1: f_1(x,t)=K^{-1}\left[v^{2\alpha}K\left[Rf_0(x,t)+H_0\right]\right] \tag{21}$$

$$p^2: f_2(x,t)=K^{-1}\left[v^{2\alpha}K\left[Rf_1(x,t)+H_1\right]\right] \tag{22}$$

$$\vdots$$

$$p^n: f_n(x,t)=K^{-1}\left[v^{2\alpha}K\left[Rf_{n-1}(x,t)+H_{n-1}\right]\right] \tag{23}$$

Therefore, the solution of eq. (12):

$$f(x,t)=f_0(x,t)+f_1(x,t)+f_2(x,t)+...+f_n(x,t)+... \tag{24}$$

Application to fractional porous media equation and non-linear heat transfer equation

Application to fractional porous media equation

Consider the following non-linear fractional porous media equation [23, 24]:

$$\mathrm{D}_t^\alpha u(x,t) = \mathrm{D}_x\left[u(x,t)\mathrm{D}_x u(x,t)\right], \quad 0<\alpha\le 1 \tag{25}$$

subject to the initial condition:

$$u(x,0) = x \tag{26}$$

Taking Kashuri Fundo transform on both sides of eq. (25) and by eq. (11), we get:

$$K\left[u(x,t)\right] = vu(x,0) + v^{2\alpha}K\left[\mathrm{D}_x\left[u(x,t)\mathrm{D}_x u(x,t)\right]\right] \tag{27}$$

Applying the inverse Kashuri Fundo transform on both sides of eq. (27), we find:

$$u(x,t) = u(x,0) + K^{-1}\left[v^{2\alpha}K\left[\mathrm{D}_x\left[u(x,t)\mathrm{D}_x u(x,t)\right]\right]\right] \tag{28}$$

Substituting eqs. (16) and (17) into eq. (28) and applying the Kashuri Fundo transform combined with homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n = u(x,0) + p\left\{K^{-1}\left[v^{2\alpha}K\left[\mathrm{D}_x\left(\sum_{n=0}^{\infty} p^n H_n\right)\right]\right]\right\} \tag{29}$$

where H_n is He's polynomial that represents the non-linear term $u(x, t)\mathrm{D}_x u(x, t)$. The few terms of H_n are computed:

$$H_0 = u_0 u_{0x}, \quad H_1 = u_0 u_{1x} + u_1 u_{0x}, \quad H_2 = u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x}, \ldots \tag{30}$$

Comparing the coefficients of like powers of p, the following approximations are obtained:

$$p^0 : u_0(x,t) = x \tag{31}$$

$$p^1 : u_1(x,t) = K^{-1}\left[v^{2\alpha}K\left[\mathrm{D}_x\left(H_0\right)\right]\right] = \frac{t^\alpha}{\Gamma(1+\alpha)} \tag{32}$$

$$p^2 : u_2(x,t) = K^{-1}\left[v^{2\alpha}K\left[\mathrm{D}_x\left(H_1\right)\right]\right] = 0 \tag{33}$$

$$\vdots$$

$$p^n : u_n(x,t) = K^{-1}\left[v^{2\alpha}K\left[\mathrm{D}_x\left(H_{n-1}\right)\right]\right] = 0, \ n\ge 2 \tag{34}$$

So the solution of eqs. (25) and (26) are given:

$$u(x,t) = x + \frac{t^\alpha}{\Gamma(1+\alpha)} \tag{35}$$

The result obtained in eq. (35) is the same as the result obtained in [23, 24]. If $\alpha = 1$, eq. (35) can be rearranged:

$$u(x,t) = x + t \tag{36}$$

which is exactly the same as the result given by [23, 24].

Application to time-fractional heat transfer equation

Consider the following non-linear time-fractional heat transfer equation with cubic non-linearity [23, 25, 26]:

$$\mathrm{D}_t^\alpha u(x,t) = u_{xx}(x,t) - 2u^3(x,t) \tag{37}$$

subject to the initial condition:

$$u(x,0) = \frac{1+2x}{x^2+x+1} \tag{38}$$

Taking Kashuri Fundo transform on both sides of eq. (37) and by eq. (11), we get:

$$K\left[u(x,t)\right] = vu(x,0) + v^{2\alpha} K\left[u_{xx}(x,t) - 2u^3(x,t)\right] \tag{39}$$

Applying the inverse Kashuri Fundo transform on both sides of eq. (39), we find:

$$u(x,t) = u(x,0) + K^{-1}\left[v^{2\alpha} K\left[u_{xx}(x,t) - 2u^3(x,t)\right]\right] \tag{40}$$

Substituting eqs. (16) and (17) into eq. (40) and applying the Kashuri Fundo transform combined with homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = u(x,0) + p\left\{K^{-1}\left[v^{2\alpha} K\left[\sum_{n=0}^{\infty} p^n u_{nxx}(x,t) - \sum_{n=0}^{\infty} H_n\right]\right]\right\} \tag{41}$$

where H_n is He's polynomial that represents the non-linear term $2u^3(x, t)$. The few terms of H_n are computed:

$$H_0 = 2u_0^3(x,t),\ H_1 = 6u_0^2(x,t)u_1(x,t),\ H_2 = 6u_0(x,t)u_1^2(x,t) + 6u_0^2(x,t)u_2(x,t),\ldots \tag{42}$$

Comparing the coefficients of like powers of p, the following approximations are obtained:

$$p^0 : u_0(x,t) = \frac{1+2x}{x^2+x+1} \tag{43}$$

$$p^1 : u_1(x,t) = K^{-1}\left[v^{2\alpha} K\left[u_{0xx}(x,t) - 2u_0^3(x,t)\right]\right] = \frac{-6(1+2x)}{\left(x^2+x+1\right)^2} \frac{t^\alpha}{\Gamma(1+\alpha)} \tag{44}$$

$$p^2 : u_2(x,t) = K^{-1}\left[v^{2\alpha} K\left[u_{1xx}(x,t) - 6u_0^2(x,t)u_1(x,t)\right]\right] = \frac{72(1+2x)}{\left(x^2+x+1\right)^3} \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \tag{45}$$

$$p^3 : u_3(x,t) = K^{-1}\left[v^{2\alpha} K\left[u_{2xx}(x,t) - 6u_0(x,t)u_1^2(x,t) - 6u_0^2(x,t)u_2(x,t)\right]\right] =$$
$$= \left(-\frac{1296(1+2x)}{\left(x^2+x+1\right)^4} + \frac{432(1+2x)^3}{\left(x^2+x+1\right)^5} - \frac{216(1+2x)^3}{\left(x^2+x+1\right)^5} \cdot \frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}\right) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \tag{46}$$

$$\vdots$$

So the solution of eqs. (37) and (38) are given:

$$u(x,t)=\frac{1+2x}{x^2+x+1}-\frac{6(1+2x)}{\left(x^2+x+1\right)^2}\frac{t^\alpha}{\Gamma(1+\alpha)}+\frac{72(1+2x)}{\left(x^2+x+1\right)^3}\cdot\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+$$

$$+\left(-\frac{1296(1+2x)}{\left(x^2+x+1\right)^4}+\frac{432(1+2x)^3}{\left(x^2+x+1\right)^5}-\frac{216(1+2x)^3}{\left(x^2+x+1\right)^5}\cdot\frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}\right)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+\ldots \qquad (47)$$

The result obtained in eq. (47) is the same as the result obtained in [23]. If $\alpha = 1$, eq. (47) can be rearranged:

$$u(x,t)=\frac{1+2x}{x^2+x+1}-\frac{6(1+2x)}{\left(x^2+x+1\right)^2}t+\frac{36(1+2x)}{\left(x^2+x+1\right)^3}t^2-\frac{216(1+2x)}{\left(x^2+x+1\right)^4}t^3+\ldots \qquad (48)$$

which is coincides with the result obtained in [25].

References

[1] Lokenath, D., Bhatta, D., *Integral Transform and Their Applications*, CRC Press, Boca Raton, Fla., USA, 2014
[2] Diethelm, K., *The Analysis of Fractional Differential Equations*, Springer-Verlag, Berlin, Germany, 2010
[3] He, J. H., Homotopy Perturbation Method: A New Non-Linear Analytical Technique, *Applied Mathematics and Computation*, *135* (2003), 1, pp. 73-79
[4] Roozi, A., *et al*., Homotopy Perturbation Method for Special Non-linear Partial Differential Equations, *Journal of King Saud University (Science)*, *23* (2011), 1, pp. 99-103
[5] Roul, P., Meyer, P., Numerical Solutions of Systems of Non-linear Integro-Differential Equations by Homotopy Perturbation Method, *Applied Mathematical Modellling*, *35* (2011), 9, pp. 4234-4242
[6] Yan, L.,M., Modified Homotopy Perturbation Method Coupled with Laplace Transform for Fractional Heat Transfer and Porous Media Equations, *Thermal Science*, *17* (2013), 5, pp. 1409-1414
[7] Alkan, S., A New Solution Method for Non-Linear Fractional Integro-Differential Equations, *Discrete and Continuous Dynamical Systems Series S*, *8* (2015), 6, pp. 1065-1077
[8] Alkan, S., Secer, A., Application of Sinc-Galerkin Method for Solving Space-Fractional Boundary Value Problems, *Mathematical Problems in Engineering*, *2015* (2015), ID217348
[9] Alkan, S., Secer, A., Solution of Non-Linear Fractional Boundary Value Problems with Non-Homogeneous Boundary Conditions, *Applied and Computational Mathematics*, *14* (2015), 3, pp. 284-295
[10] He, J. H., Variational Iteration Method- a Kind of Non-Linear Analytical Technique: Some Examples, *International Journal of Non-linear Mechanics*, *34* (1999), 4, pp. 609-708
[11] He, J. H., Wu, X. H., Variational Iteration Method: New Development and Applications, *Computers & Mathematics with Applications*, *54* (2007), 7-8, pp. 881-894
[12] Guo, S. M., *et al*., Fractional Variational Homotopy Perturbation Iteration Method and Its Application to a Fractional Diffusion Equation, *Applied Mathematics and Computation*, *219* (2013), 11, pp. 5909-5917
[13] Tatari, M., *et al*., Application of the Adomian Decomposition Method for the Fokker-Planck Equation, *Mathematical and Computer Modelling*, *45* (2007), 5-6, pp. 639-650
[14] Jafari, H., *et al*., Application of the Laplace Decomposition Method for Solving Linear and Non-Linear Fractional Diffusion-Wave Equations, *Applied Mathematics Letters*, *24* (2011), 11, pp. 1799-1805
[15] Liu, Y., Q., Approximate Solutions of Fractional Non-Linear Equations Using Homotopy Perturbation Transformation Method, *Abstract and Applied Analysis*, *2012* (2012), ID752869
[16] Kashuri, A., *et al*., Mixture of a New Integral Transform and Homotopy Perturbation Method for Solving Non-Linear Partial Differential Equations, *Advances in Pure Mathematics*, *3* (2013), 3, pp. 317-323
[17] Singh, B., K., Homotopy Perturbation New Integral Transform Method for Numeric Study of Space and Time- Fractional (n+1)-Dimensional Heat and Wave-Like Equations, *Waves*, *Wavelets and Fractals*, *4* (2018), 1, pp. 19-36
[18] Podlubny, I., *Fractional Differential Equations*, Academic Press, New York, USA, 1999
[19] Mathai, A. M., Haubold, H. J., *An Introduction to Fractional Calculus*, Nova Science Publishers, New York, USA, 2017

[20] Shah, K., *et al*., Combination of Integral and Projected Differential Transform Methods for Time-Fractional Gas Dynamics Equations, *Ain Shams Engineering Journal*, *9* (2018), 4, pp. 1683-1688
[21] Kashuri, A., Fundo, A., A New Integral Transform, *Advances in Theoretical and Applied Mathematics*, *8* (2013), 1, pp. 27-43
[22] Sumiati, I., *et al*., Adomian Decomposition Method and The New Integral Transform, *Proceedings*, 2nd African International Conference on Industrial Engineering and Operations Management, Harare, Zimbabwe, 2020, pp. 1882-1887
[23] Yan, L. M., Modified Homotopy Perturbation Method Coupled with Laplace Transform for Fractional Heat Transfer and Porous Media Equations, *Thermal Science*, *17* (2013), 5, pp. 1409-1414
[24] Pamuk, S., Solution of the Porous Media Equation by Adomian's Decomposition Method, *Physics Letters A*, *344* (2005), 2-4, pp. 184-188
[25] Ganji, D., D., Sadighi, A., Application of Homotopy Perturbation and Variational Iteration Methods to Non-Linear Heat Transfer and Porous Media Equations, *Journal of Computational and Applied Mathematics*, *207* (2007), 1, pp. 24-34
[26] Liu, J., Linear Stability Analysis and Homoclinic Orbit for a Generalized Non-Linear Heat Transfer, *Thermal Science*, *16* (2012), 5, pp. 1556-1559

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