AIR TEMPERATURE MEASUREMENT BASED ON LIE GROUP SO(3)

by

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This study aims to analyze the behaviors of air temperature during the period from 1895(5) to 2021(12) using Lie algebras method. We proposed an alternative method to model air temperature, in which the non-linear structure of temperature is evolved by a stochastic differential equation captured on a curved state space. After expressing stochastic differential equations based on Lie algebras and Lie groups, we tested the non-linear and random behavior of air temperature. This method allow a rich geometric structure. Moreover diffusion processes can easily be built without needing the machinery of stochastic calculus on manifolds.

Key words*: Lie groups, Lie algebras, air temperature, stochastic equation*

Introduction

Lie groups and Lie algebras have very important in mathematical physics and mathematics [1, 2] and it has been used as a mathematical tool in the solutions of differential equations since 1881. Therefore, Lie groups and algebras, it has been used in many fields from engineering problems to economic problems involving stochastic differential equations and stochastic processes [3-13].

In the effect of a recent activity in the atmosphere-ocean sciences, we wanted to make the stochastic air temperature prediction by employing Lie method. Air temperature are a serious threat to the whole world. Increasing temperatures, changing rainfall patterns and, storms have effect on the humanity. According to the National Oceanic and AtmosphericAdministration (NOAA), the surface temperature of the world has been rising for the last 100 years [14, 15]. A few papers used Lie group and algebras to analyze climate change and air temperature [16, 17]. As differentation from these papers, in our study, we will analyse the non-linear and random behavior of air temperature by a stochastic differential equation covering a curved state space and developed the model on S^2 manifolds using matrix representations instead of differential operator representations of Lie algebras. But the purpose of these articles was not only air temperature estimation but also to predict the future air temperature. As similar to our method, Park *et al.* [10] analysed the structure of interest rates by a stochastic differential equation covering a curved state space and they also developed models on S^1 and S^2 manifolds using matrix representations instead of differential operator representations of Lie algebras. Bildirici *et al*. [13] tested the non-linear and random structure of spot oil price by a stochastic differential equation covering a curved state space.

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We will use air temperature for California, Illinois, Iowa, Minnesota, Nebraska, New York, North Carolina,Texas, Washington and USA from 1895(5)-2021(12). The drift and noise volatility terms of stochastic state equations are carefully crafted to reflect various observed phenomena and the drift and volatility terms are kept simple instead of choosing an underlying state space that is curved. As it was preferred by this paper, Park *et al*. [10] and Bildirici *et al.* [13] used the ordinary least square (OLS) estimation method for parameter estimation. Lastly, the RMSE and MAE values were obtained to explore their forecast accuracy for $T+1$ months, $T+1$ years, $T+5$ years, and $T+10$ years.

Brief information on orthogonal Lie groups and algebras

A geometrical Lie group is a differentiable manifold and its algebra is the tangent space in the unit to the manifold. This section, The explanations of orthogonal matrix Lie groups, their algebras and the relationships of stochastic dynamics between these groups are presented [1, 10]. Usually, the algebra is denoted with a lowercase letter and the group with a capital letter. The orthogonal Lie groups $O(n)$ are defined:

$$O(n)=\left\{A\in GL(n): A^T A=I\right\} \tag{1}$$

Special orthogonal matrix groups are denoted by $SO(n)$ and defined:

$$O(n)=\left\{A\in GL(n): A^T A=I \text{ and } \det A=1\right\} \tag{2}$$

For $n = 2$:

$$A=\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$$

the typical element of the group $SO(2)$, is a rotation matrix. The manifold of this group is identified with the unit circle S^2 and defined:

$$S^1=\left\{(x_1,x_2): x_1^2+x_2^2=1\right\} \text{ with paramrteization } x_1=\cos\theta,\ x_2=\sin\theta$$

For $n = 3$:

$$A=\begin{bmatrix} c\theta & -s\theta s\varphi & -s\theta c\varphi \\ -s\varnothing s\theta & c\varnothing c\varphi - s\varnothing c\theta s\varphi & -c\varnothing s\varphi - s\varnothing c\theta c\varphi \\ c\varnothing s\theta & s\varnothing c\varphi + c\varnothing c\theta s\varphi & -s\varphi s\varnothing + c\theta c\varnothing c\varphi \end{bmatrix}$$

the typical element of the group $SO(3)$, is a rotation matrix. The manifold of this group is identified with the unit sphere S^2 and defined:

$$S^2=\left\{(x_1,x_2,x_3): x_1^2+x_2^2+x_3^2=1\right\}$$

with parametrization $x_1 = \cos\theta$, $x_2 = \sin\theta\sin\varphi$, and $x_3 = \sin\theta\cos\varphi$.

Continuing in a similar way, the Lie group $SO(\mathrm{n})$ is the manifold S^{n-1} with $(n - 1)$ dimensional.

Lie algebras of these groups are denoted by $so(n)$ and the elements of the algebra satisfy the condition $B^T = -B$ for $B \in so(n)$. The relationship between this algebra and the group is expressed:

$$\exp: so(3)\to SO(3),\ \exp B = A\in SO(3) \tag{3}$$

Proposition 1. State equation $\mathrm{d}A = AB\mathrm{d}t + A\mathrm{d}W$, where $A \in G$, B, $\mathrm{d}W \in g$, A is a constant and $\mathrm{d}W$ diffusion process is given. Quadratic function

$$f(A) = \frac{1}{2}Tr\left(A^T QAN\right) \text{ where } Q, N \in R^{n\times n}$$

is a symmetric matrix and the dynamics for f are given:

$$\mathrm{d}f = Tr\left[A^T QA\left(BN\mathrm{d}t + \mathrm{d}WN + \frac{1}{2}\mathrm{d}WN\mathrm{d}W^T + \frac{1}{2}\mathrm{d}W\mathrm{d}WN\right)\right] \tag{4}$$

where $\mathrm{d}t\mathrm{d}t = \mathrm{d}w_i\mathrm{d}t = \mathrm{d}t\mathrm{d}w_i = 0$, $\mathrm{d}w_i\mathrm{d}w_j = \rho_{ij}\mathrm{d}t$, and ρ_{ij} is the correlation coefficient between w_i and w_j.

Proposition 2. If $f(A) = Tr(MA)$ under the conditions given in *Proposition 1* then the dynamics for are given:

$$\mathrm{d}f = Tr\left[MA\left(B\mathrm{d}t + \mathrm{d}W + \frac{1}{2}\mathrm{d}W\mathrm{d}W\right)\right] \tag{5}$$

Air temperature models on the *SO*(2) and the *SO*(3)

The Lie group $SO(2)$ is a differentiable manifold and this manifold can be identified with the unit circle S^1 and the model is defined: $s(A) = Tr(MA)$ where $A \in SO(2)$ and M positive-definite symmetric matrix.

In this case, the state equation is given:

$$\mathrm{d}A = AB\mathrm{d}t + A\mathrm{d}W \text{ and } B, \mathrm{d}W \in so(2) \tag{6}$$

Indded, for

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} > 0,\ m_{12} = m_{21} \text{ and } A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},\ \theta \in [0, 2\pi]$$

$$B = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix},\ \mathrm{d}W = \begin{bmatrix} 0 & -\mathrm{d}w \\ \mathrm{d}w & 0 \end{bmatrix} \in so(2)$$

$$s(A) = (m_{11} + m_{22})\cos\theta = \frac{s}{\gamma},\ \sin\theta = \frac{\sqrt{\gamma^2 - s^2}}{\gamma}$$

Using eq. (5) for (s, θ) stochastic dynamics:

$$\mathrm{d}s = \left(-b\gamma\sin\theta - \frac{b}{2}\cos\theta\right)\mathrm{d}t + \left(-\gamma\sin\theta\right)\mathrm{d}w$$

and for the dynamics s, using the model relation

$$\mathrm{d}s = \left(-b\sqrt{\gamma^2 - s^2} - \frac{s}{2}\right)\mathrm{d}t - \sqrt{\gamma^2 - s^2}\,\mathrm{d}w \tag{7}$$

The Lie group $SO(3)$ is a differential manifold and can be identified with the unit sphere S^2. In this manifold, the state equation are given:

$$s(A) = \frac{1}{2}Tr\left(QANA^T\right)\ A \in SO(3)$$

$$\mathrm{d}A = \mathrm{A}B\mathrm{d}t + A\mathrm{d}W \text{ and } B, \mathrm{d}W \in so(3) \text{ where } Q, N \in R^{3\times 3}$$

positive symmetric matrices.

Thus, the dynamics f:

$$\mathrm{d}f = Tr\left\{A^T QA\left[\left(B-I\right)N\mathrm{d}t + \mathrm{d}WN + \frac{1}{2}\mathrm{d}WN\mathrm{d}W^T\right]\right\} \tag{8}$$

where

$$B=\begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix} \quad \mathrm{d}W=\begin{bmatrix} 0 & -\mathrm{d}w_3 & \mathrm{d}w_2 \\ \mathrm{d}w_3 & 0 & -\mathrm{d}w_1 \\ -\mathrm{d}w_2 & \mathrm{d}w_1 & 0 \end{bmatrix} \in so(3)$$
$$Q=\begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix}, \quad N=\begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{12} & n_{22} & n_{23} \\ n_{13} & n_{23} & n_{33} \end{bmatrix} \tag{9}$$

and we can use one of the following forms for the matrix element of $SO(3)$:

After these operations, the model s is obtained:

$$s\left(A\right)=\frac{1}{2}\left[c_{11}n_{11}+c_{22}n_{22}+c_{33}n_{33}+\left(c_{12}+c_{21}\right)n_{12}+\left(c_{13}+c_{31}\right)n_{13}+\left(c_{23}+c_{32}\right)n_{23}\right]$$

where

$$\begin{gathered} c_{11}=b_{11}a_{11}+b_{12}a_{21}+b_{13}a_{31},\ c_{22}=b_{22}a_{11}+b_{22}a_{22}+b_{23}a_{32} \\ c_{33}=b_{31}a_{13}+b_{32}a_{23}+b_{33}a_{33},\ c_{12}=b_{11}a_{12}+b_{12}a_{22}+b_{13}a_{32} \\ c_{21}=b_{21}a_{11}+b_{22}a_{21}+b_{23}a_{31},\ c_{13}=b_{11}a_{13}+b_{12}a_{23}+b_{13}a_{33} \\ c_{31}=b_{31}a_{11}+b_{32}a_{21}+b_{33}a_{31},\ c_{23}=b_{21}a_{13}+b_{22}a_{23}+b_{23}a_{33} \\ c_{32}=b_{31}a_{2}+b_{32}a_{22}+b_{33}a_{32} \\ \text{and} \\ b_{11}=a_{11}q_{11}+a_{21}q_{21}+a_{31}q_{31},\ b_{12}=a_{11}q_{12}+a_{21}q+a_{31}q_{32} \\ b_{21}=a_{12}q_{11}+a_{22}q_{21}+a_{32}q_{31},\ b_{22}=a_{12}q_{12}+a_{22}q_{22}+a_{23}q_{32} \\ b_{31}=a_{13}q_{11}+a_{22}q_{21}+a_{33}q_{31},\ b_{32}=a_{13}q_{12}+a_{23}q_{22}+a_{33}q_{32} \\ b_{13}=a_{11}q_{13}+a_{21}q_{23}+a_{31}q_{33},\ b_{23}=a_{12}q_{13}+a_{22}q_{23}+a_{32}q_{33} \\ b_{33}=a_{13}q_{13}+a_{23}q_{23}+a_{33},\ a_{ij}\in A \end{gathered} \tag{10}$$

Data and results

We used monthly air temperature for California, Illinois, Iowa, Minnesota, Nebraska, New York, North Carolina, Texas, Washington, and USA from 1895(5) to 2021(12). Data was taken from NOAA. It was applied the logarithmic transformations for data. The air temperature via the expectation can be obtained:

$$A\left(t,t+\tau\right)=E\left[\frac{1}{\tau}\left(\int\limits_{\tau}^{t+\tau} s\left(r\right)\mathrm{d}r\right)\right]$$

In tab. 1, it was given descriptive statistics, and unit root test. The ADF unit root test determined as stationary at the levels for the variables.

Table 1. Statistics results

	California	Illinois	Iowa	USA	Minnesota	Nebraska	New York	North Carolina	Texas	Washington
Kurtosis	2.78	3.07	2.89	2.93	2.91	2.88	3.07	3.12	3.19	2.98
Skewness	–0.94	1.19	0.89	-0.9	–0.9	1.46	0.93	1.42	1.38	1.48
ADF unit root test										
	–4. 6	–4. 4	–3.4	–3.8	–3.6	–3.7	–3.9	–4.1	–4.69	–4.5

The results of the Lie algebras are presented in tab. 2. We considered a three-factor model on the group $SO(3)$. The model s is always positive.

The model parameters in this example are $B \in so(3)$ in the underlying state equations, the symmetric positive definite matrices Q, $N \in \mathbb{R}^{3\times3}$ used, and the covariance matrix S associated with the $\mathrm{d}W \in so(3)$.

The Monte-Carlo simulation employed to directly evaluate the aforementioned expectation. The following OLS estimation procedure is used for parameter estimation. It is chosen an initial set of parameters $B \in so(3)$ and the symmetric positive definite matrices Q, $N \in \mathbb{R}^{3\times3}$. The initial value of $A(0) \in SO(3)$ is chosen for convenience.

It is determined if the model selection parameters are optimizers for the objective function. The objective function is optimized for the given model parameters:

$$L(B,Q,N,S) = \sqrt{\Sigma\left[\hat{A}(t,t+\tau) - A(t,t+\tau)\right]^2}$$

where $A(t, t + \tau)$ is the historical time series. To examine the goodness of fit of the model to the actual data, we calculated the skewness and kurtosis of the error between the estimated and actual values, using the Newyork as a proxy (the figures were not presented). In all period, the kurtosis and skewness of the errors are determined to be 3.2 and –0.5516. Table 2 shows estimations of Lie parameters.

Table 2. Estimations of Lie parameters with OLS method*

Estimations of Lie parameters with OLS method
$B = \begin{pmatrix} 0 & -0.56 & 0.84 \\ 0.56 & 0 & -0.32 \\ -0.84 & 0.32 & 0 \end{pmatrix}, \; S = \begin{pmatrix} 1.22 & 0 & 0 \\ 0 & 0.298 & 0 \\ 0 & 0 & 1.22 \end{pmatrix}$
$Q = \begin{pmatrix} 1.44 & 1.48 & 1.08 \\ 1.48 & 2.053 & 1.21 \\ 1.08 & 1.21 & 1.66 \end{pmatrix}, \; N = \begin{pmatrix} 0.985 & 1.28 & 0.938 \\ 1.28 & 1.19 & 1.02 \\ 0.938 & 1.02 & 0.929 \end{pmatrix}$

In order to determine which of the results was more appropriate, the results of the in-sample and out-of-sample were compared. Since the lowest RMSE and MAE coefficients in the in-sample and out-of-sample results will be considered as the best result, the model that gives these results will be the most successful.

Forecast results

Table 3 shows the results of in-sample forecast results determined by Lie and traditional methods. Moreover, the results of traditional models using OLS method are presented.

Table 3. In-sample forecast results

	Lie	Traditional
	Lie OLS	OLS
RMSE	1.062	6.96
MAE	0.96	5.73

Lie method found more successful results than traditional OLS. Then, Out-of-Sample forecast results were obtained. The MAE and RMSE for Lie methods with OLS were presented to determine their forecast accuracy for $T + 1$ months, $T + 1$ years, $T + 5$ years, and $T + 10$ years as shown in tab. 4.

Table 4. The out-of-sample performances of compared methods

	$T + 1$ months	$T + 1$ years	$T + 5$ years	$T + 10$ years
RMSE	0.73	2.709	2.88	3.95
MAE	0.67	2.65	2.81	3.76

Conclusion

This paper aims to analyze the non-linear behavior of air temperature for the period 1895(5) to 2021(12) using Lie algebras method with OLS estimators for California, Illinois, Iowa, Minnesota, Nebraska, New York, North Carolina, Texas, Washington, and USA. In this paper, our aim is not to examine the factors that cause air temperature with Lie method, but to estimate air temperature using Lie method. In this context, we suggested $\mathrm{Lie}_{\mathrm{OLS,}}$ models in the context of the drift and noise volatility terms of stochastic state equations. In our basic model, the Lie group $SO(3)$ is a differentiable manifold and it can be identified with unit sphere S^2. On the other hand, we applied the Lie algebras method with OLS, estimators and determined the forecast accuracies. Our results show that the air temperature will continue to increase over the next 10 years. The USA should take measures to reverse the temperature rise and determine environmental policies in the fact that air temperature rise will be a major problem for the world in the future. We hope that this model we propose can be developed for fractional supergroup $SU(2)$ [18], the real form of fractional supergroup $SL(2, C)$.

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