ROTATIONAL HYPERSURFACES GENERATED BY WEIGHTED MEAN CURVATURE IN E_1^4 WITH DENSITY

by

Mustafa ALTIN*

Technical Sciences Vocational School, Bingol University, Bingol, Turkey

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In this paper, we study rotational hypersurfaces in 4-D Lorentz-Minkowski space with density. We give the weighted mean curvature of a rotational hypersurface about spacelike (timelike) axis in E_1^4 with densities $e^{x^2-y^2-z^2}$ and e^{x-y-z} ($e^{-y^2-z^2-t^2}$ and e^{-y-z-t}). We obtain the parametric expressions of the rotational hypersurfaces about spacelike (timelike) axis in E_1^4 with density $e^{x^2-y^2-z^2}$ ($e^{-y^2-z^2-t^2}$) with respect to the weighted mean curvature and give some examples for them. Also, we give some results about rotational hypersurfaces about spacelike (timelike) axis in E_1^4 with density e^{x-y-z} (e^{-y-z-t}) to be with constant or non-constant weighted mean curvature.

Key words: *rotational hypersurface, weighted mean curvature, density*

Introduction

The rotational surfaces have many application areas in many fields of physics and engineering. When certain objects are designed digitally, revolutions like these can be used to determine surface area without the use of measuring the length and radius of the object being designed [1]. So, let α: $I \subset \mathbb{R} \to \pi$ be a curve in a plane π of space and l be a straight line in this space. Then a rotational (hyper)surface is defined by a (hyper)surface rotating the profile curve α around the axis l. According to this definition, lots of studies have been done by mathematicians about rotational (hyper)surfaces in different spaces. Moore [2] has studied surfaces of rotation in 4-D space and in Chen and Ishikawa [3], the authors have classified all finite type surfaces in E^3. The rotation surfaces with finite type Gauss map in E^4 have been given in [4]. Also, the extrinsic differential geometry of submanifolds in 4-D Lorentz-Minkowski space is special interest in Relativity Theory. In this context, the explicit parameterizations of rotation hypersurfaces with constant mean curvature in Lorentz-Minkowski space and flat lightlike hypersurfaces in Lorentz-Minkowski 4-space have been studied in [5, 6], respectively. Furthermore, different studies about rotational (hyper)surfaces in four dimensional Euclidean and Minkowskian 4-spaces have been done in [7-11], and, *etc*.

On the other hand, manifolds with density arise in physics when considering surfaces or regions with differing physical density. An example of an important 2-D surface with density is the Gauss plane, a Euclidean plane with volume and length weighted by $(2\pi)^{-1}e^{-r^2/2}$, where r is the distance from the origin. Also, in physics, an object may have differing internal densities so in order to determine the object's mass it is necessary to integrate volume weighted with density. In general, for a manifold with density, in terms of the underlying Riemannian volume dV and perimeter dP, the new weighted volume and area are given [12]:

* Author's e-mail: maltin@bingol.edu.tr

$$\mathrm{d}V_\varphi = \mathrm{e}^\varphi \mathrm{d}V \text{ and } \mathrm{d}P_\varphi = \mathrm{e}^\varphi \mathrm{d}P \tag{1}$$

General relativity models the physical universe as a 4-D C^∞ Hausdorff differentiable space-time manifold, M, with a Lorentzian metric, g, of signature (–, +, +, +) which is topologically connected, paracompact and space-time orientable. These properties are suitable when we consider for local physics. As soon as we investigate global features then we face various pathological difficulties such as, the violation of time orientation, possible non-Hausdorff or non-papacompactness, disconnected components of space-time, *etc*. However, we think that the space-time is causally well-behaved in this context [13]. If H is the mean curvature and N is the unit normal vector field of an n-dimensional hypersurfaces, the κ is the curvature and $\mathcal{N}$ is the principal normal vector of a curve, then the notions of weighted mean curvature of an n-dimensional hypersurface and weighted curvature of a curve on manifolds with density e^φ have been introduced:

$$H_\varphi = H - \frac{1}{n-1}\frac{\mathrm{d}\varphi}{\mathrm{d}N} \text{ and } \kappa_\varphi = \kappa - \frac{\mathrm{d}\varphi}{\mathrm{d}\mathcal{N}} \tag{2}$$

respectively, [14]. The weighted mean curvature is a natural generalization of the mean curvature of a surface and a surface with $H_\varphi = 0$ is called a weighted minimal surface.

Also in [12], the authors have introduced the notion of weighted Gaussian curvature of a surface which is a generalization of the Gaussian curvature of a surface in a manifold with density e^φ and defined:

$$K_\varphi = K - \Delta\varphi \tag{3}$$

where K is the Gaussian curvature of a surface and Δ is the Laplacian operator. If a surface's weighted Gaussian curvature is zero everywhere, then we call it a weighted flat surface. According to these definitions, some characterizations of weighted curves and surfaces in Euclidean, Minkowskian and Galilean spaces with different densities have been studied by geometers, physicists and, *etc*., see [12, 15-29], and, *etc*.

Now, let we recall some fundamental notions for hypersurfaces in Lorentz-Minkowski 4-space.

If $\vec{a} = (a_1, a_2, a_3, a_4)$, $\vec{b} = (b_1, b_2, b_3, b_4)$, $\vec{c} = (c_1, c_2, c_3, c_4)$

are three vectors in E_1^4, then the inner product and vector product are defined:

$$\langle \vec{a}, \vec{b} \rangle = -a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4$$

and

$$\vec{a} \times \vec{b} \times \vec{c} = \det \begin{bmatrix} -e_1 & e_2 & e_3 & e_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{bmatrix} \tag{4}$$

respectively. Also, the norm of the vector $\vec{a}$ is

$$\|\vec{a}\| = \sqrt{|\langle \vec{a}, \vec{a} \rangle|}$$

If $\Lambda : E^3 \to E_1^4, \Lambda(x_1, x_2, x_3) = \big(\Lambda_1(x_1, x_2, x_3), \Lambda_2(x_1, x_2, x_3), \Lambda_3(x_1, x_2, x_3), \Lambda_4(x_1, x_2, x_3)\big)$

is a hypersurface in Lorentz-Minkowski 4-space E_1^4, then the Gauss map (*i.e.*, the unit normal vector field), the matrix forms of the first and second fundamental forms:

$$N=\frac{\Lambda_{x_1}\times\Lambda_{x_2}\times\Lambda_{x_3}}{\left\|\Lambda_{x_1}\times\Lambda_{x_2}\times\Lambda_{x_3}\right\|},\ g_{ij}=\left\langle\Lambda_{x_i},\Lambda_{x_j}\right\rangle,h_{ij}=\left\langle\Lambda_{x_ix_j},N\right\rangle \tag{5}$$

respectively. Here

$$\Lambda_{x_i}=\frac{\partial\Lambda}{\partial x_i},\ \Lambda_{x_ix_j}=\frac{\partial^2\Lambda}{\partial x_ix_j},\ i,j\in\{1,2,3\}$$

If $[g^{ij}]$ is the inverse matrix of $[g_{ij}]$, then the matrix of shape operator of the hypersurface Λ:

$$\mathcal{S}=[a_{ij}]=[g^{ij}][h_{ij}] \tag{6}$$

With the aid of eqs. (5) and (6), the Gaussian curvature and mean curvature of a hypersurface in E_1^4 are given:

$$K=\varepsilon\frac{\det[h_{ij}]}{\det[g_{ij}]}\ \text{and}\ 3\varepsilon H=tr(\mathcal{S}) \tag{7}$$

respectively [30]. Here, $\varepsilon=\langle N,N\rangle$.

Rotational hypersurfaces about spacelike axis in E_1^4 with density

In this section, we give the weighted mean curvature of a rotational hypersurface about spacelike axis in E_1^4 with densities $e^{x2-y2-z2}$ and e^{x-y-z} and also, we obtain the parametric expressions of the rotational hypersurfaces with respect to (wrt) the weighted mean curvature. Firstly, we recall the curvatures of the rotational hypersurfaces about spacelike axis which can be obtained by taking $a=b=0$ in [11].

For a differentiable function $\omega(x)$: $I\subset\mathbb{R}\to\mathbb{R}$, the rotational hypersurface generated by rotating the profile curve $\gamma_1(x)=(x,0,0,\omega(x))$ about spacelike axis (0, 0, 0, 1):

$$S(x,y,z)=\begin{bmatrix}\cosh y\cosh z & \sinh y\cosh z & \sinh z & 0\\ \sinh y & \cosh y & 0 & 0\\ \cosh y\sinh z & \sinh y\sinh z & \cosh z & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\0\\0\\\omega(x)\end{bmatrix}= \\ =\left(x\cosh y\cosh z,x\sinh y,x\cosh y\sinh z,\omega(x)\right) \tag{8}$$

where $x\in\mathbb{R}-\{0\}$. The unit normal vector of the rotational hypersurface eq. (8) is given:

$$N^S=-\frac{1}{\sqrt{\varepsilon(1-\omega'^2)}}\left(\omega'\cosh y\cosh z,\omega'\sinh y,\omega'\cosh y\sinh z,1\right) \tag{9}$$

where $\varepsilon=\mp1$. So, we have $\langle N^S,N^S\rangle=1/\varepsilon$. If $\varepsilon=-1$, then rotational hypersurface eq. (8) is spacelike hypersurface (if $\varepsilon=1$, then rotational hypersurface eq. (8) is timelike hypersurface).

The Gaussian curvature and the mean curvature of the rotational hypersurface eq. (8) are given:

$$K^S=\varepsilon\frac{\omega'^2\omega''}{x^2\left(\varepsilon(1-\omega'^2)\right)^{5/2}}\ \text{and}\ H^S=\varepsilon\frac{2\omega'(1-\omega'^2)+x\omega''}{3x\left[\varepsilon(1-\omega'^2)\right]^{3/2}} \tag{10}$$

respectively. Here, we state

$$\omega=\omega(x),\ \omega'=\frac{\mathrm{d}\omega(x)}{\mathrm{d}x},\ \text{and}\ \omega''=\frac{\mathrm{d}^2\omega(x)}{\mathrm{d}x^2}$$

Rotational hypersurfaces about spacelike axis generated by weighted mean curvature in E_1^4 with density $e^{x^2-y^2-z^2}$

From eqs. (2), (9), and (10), the weighted mean curvature of the rotational hypersurface eq. (8) in E_1^4 with density $e^{x^2-y^2-z^2}$ is obtained:

$$H_\varphi^S = \varepsilon \frac{2(1+x^2)\omega' - 2(1+x^2)\omega'^3 + x\omega''}{3x\left(\varepsilon(1-\omega'^2)\right)^{3/2}} \tag{11}$$

Now, our aim is to obtain the function ω wrt the weighted mean curvature H_φ^S by solving the eq. (11). Hence, let us take the function F_1:

$$F_1 = \frac{\omega'(x)}{x^2\sqrt{\varepsilon\left(1-\omega'^2(x)\right)}} \tag{12}$$

By differentiting eq. (12), from eq. (10) we have:

$$H_\varphi^S = \frac{x^2 F_1' + 2x(2+x^2)F_1}{3} \tag{13}$$

The solution of the first ODE (13) wrt F_1 is obtained:

$$F_1 = \frac{\mathrm{e}^{-x^2}\left(c_1 + 3\int_1^x t^2 \mathrm{e}^{t^2} H_\varphi^S(t)\mathrm{d}t\right)}{x^4}, \quad c_1 \in \mathbb{R} \tag{14}$$

Using eqs. (12) and (14), we reach:

$$\omega(x) = \pm\int \sqrt{\frac{\varepsilon \mathrm{e}^{-2x^2}\left(c_1 + 3\int_1^x t^2 \mathrm{e}^{t^2} H_\varphi^S(t)\mathrm{d}t\right)^2}{x^4 + \varepsilon \mathrm{e}^{-2x^2}\left(c_1 + 3\int_1^x t^2 \mathrm{e}^{t^2} H_\varphi^S(t)\mathrm{d}t\right)^2}}\mathrm{d}x \tag{15}$$

Therefore, we have:

Theorem 1. The rotational hypersurface eq. (8) about spacelike axis in E_1^4 with density $e^{x^2-y^2-z^2}$ can be parametrized wrt the weighted mean curvature:

$$S(x,y,z) = \left(x\cosh y\cosh z, x\sinh y, x\cosh y\sinh z, \pm\int \sqrt{\frac{\varepsilon \mathrm{e}^{-2x^2}\left(c_1 + 3\int_1^x t^2 \mathrm{e}^{t^2} H_\varphi^S(t)\mathrm{d}t\right)^2}{x^4 + \varepsilon \mathrm{e}^{-2x^2}\left(c_1 + 3\int_1^x t^2 \mathrm{e}^{t^2} H_\varphi^S(t)\mathrm{d}t\right)^2}}\mathrm{d}x\right) \tag{16}$$

where

$$c_1 \in \mathbb{R} \quad \text{and if} \quad \varepsilon = 1, \text{ then } x^4 < \mathrm{e}^{-2x^2}\left(c_1 + 3\int_1^x t^2 \mathrm{e}^{t^2} H_\varphi^S(t)\mathrm{d}t\right)^2$$

Example 1. Taking

$$H_\varphi^S(x) = \frac{1+2x^2}{3x^2}$$

and $c_1 = e$ in eq. (16), then the rotational hypersurface is obtained:

$$S(x,y,z) = \left(x\cosh y\cosh z, x\sinh y, x\cosh y\sinh z, \arcsin h(x)\right), \text{ if } \varepsilon = 1 \tag{17}$$

$$S(x,y,z) = \left(x\cosh y\cosh z, x\sinh y, x\cosh y\sinh z, \arcsin(x)\right), \text{ if } \varepsilon = -1 \tag{18}$$

In figs. 1 and 2, the projections of the rotational hypersurfaces eqs. (17) and (18) Gaussian and mean curvatures' graphics and the variations of them on hypersurfaces can be seen for $z = 0.5$ into $x_1x_2x_4$-space, respectively.

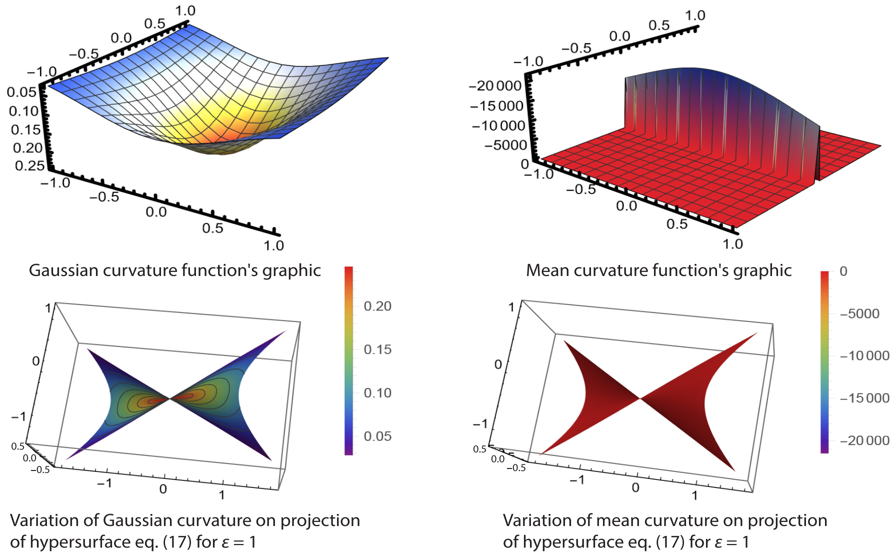

Figure 1. Graphics of curvatures and their variations on eq. (17)

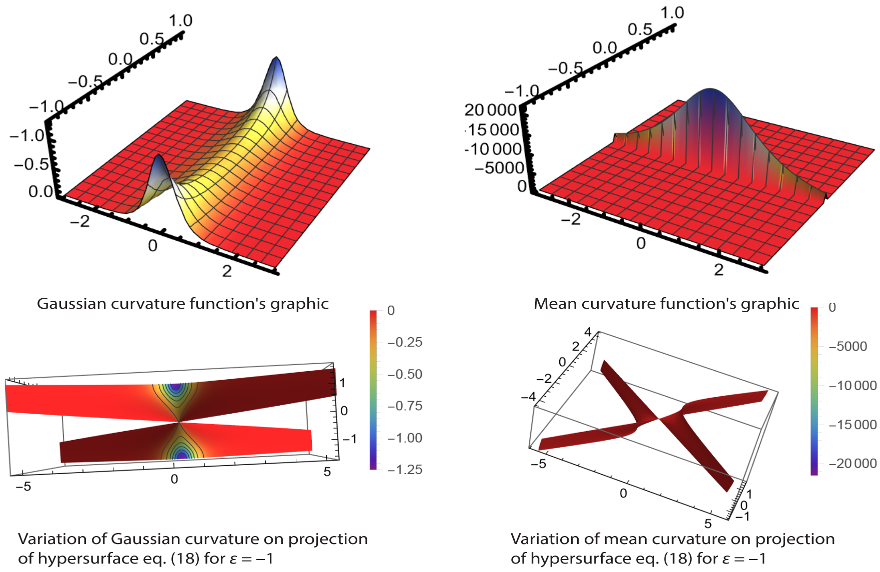

Figure 2. Graphics of curvatures and their variations on eq. (18)

Weighted mean curvature of rotational hypersurfaces about spacelike axis in E_1^4 with density e^{x-y-z}

From eqs. (2), (9), and (10), the weighted mean curvature of the rotational hypersurface eq. (8) in E_1^4 with density e^{x-y-z} is obtained:

$$H_\varphi^S = \varepsilon \frac{2 - x\sinh y - x\cosh y(\sinh z - \cosh z)\omega'(1-\omega'^2) + x\omega''}{3x\left(\varepsilon(1-\omega'^2)\right)^{3/2}} \tag{19}$$

Let the weighted mean curvature be constant, *i.e.*, $H_\varphi^S = c$. From eq. (19), we have:

$$\varepsilon x\omega'(1-\omega'^2)(\sinh y - e^{-z}\cosh y) + 3cx\left(\varepsilon(1-\omega'^2)\right)^{3/2} - \varepsilon x\omega'' - 2\varepsilon\omega'(1-\omega'^2) = 0 \tag{20}$$

If the eq. (20) is solved, the following two equations must hold:

$$x\omega'(1-\omega'^2) = 0 \;\text{ and }\; 3cx\left(\varepsilon(1-\omega'^2)\right)^{3/2} - \varepsilon x\omega'' - 2\varepsilon\omega'(1-\omega'^2) = 0$$

Since $(1 - \omega'^2) \neq 0$, the joint solution of these two equations is possible only with $\omega' = 0$ and $c = 0$.

Therefore, we have:

Theorem 2. The rotational hypersurface eq. (8) about spacelike axis in E_1^4 with density e^{x-y-z} can never have a non-zero constant weighted mean curvature.

Theorem 3. Weighted minimal rotational hypersurface eq. (8) about spacelike axis in E_1^4 with density e^{x-y-z} can be parametrized:

$$S(x,y,z) = \left(\left(x\cosh y\cosh z, x\sinh y, x\cosh y\sinh z, k\right)\right), \; k \in \mathbb{R} \tag{21}$$

In fig. 3, the projection of the spacelike rotational hypersurface eq. (21) can be seen for $z = 0, 5$, $k = 3$ into $x_1x_2x_3$ and $x_1x_2x_4$-spaces, respectively.

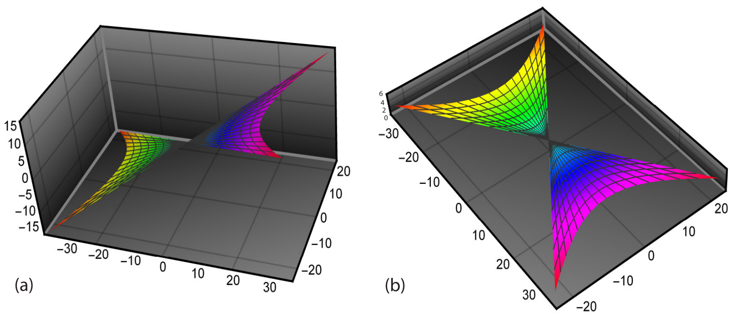

Figure 3. Rotational hypersurfaces (21) into $x_1x_2x_3$-spaces (a) and $x_1x_2x_3$-spaces (b)

Rotational hypersurfaces about timelike axis in E_1^4 with density

In this section, we give the weighted mean curvature of a rotational hypersurface about timelike axis in E_1^4 with densities $e^{-y^2-z^2-t^2}$ and e^{-y-z-t} also, we obtain the parametric expressions of them wrt the weighted mean curvature. Firstly, let we recall the curvatures of the rotational hypersurface about timelike axis which can be obtained by taking $a = b = 0$ in [11].

For a differentiable function $\phi(x)$: $I \subset \mathbb{R} \to \mathbb{R}$, the rotational hypersurface generated by rotating the profile curve $\gamma(x) = (\phi(x), 0, 0, x)$ about timelike axis (1, 0, 0, 0):

$$T(x,y,z)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos z & -\sin y\sin z & -\cos y\sin z\\ 0 & 0 & \cos y & -\sin y\\ 0 & \sin z & \sin y\cos z & \cos y\cos z\end{bmatrix}\begin{bmatrix}\phi(x)\\ 0\\ 0\\ x\end{bmatrix}= \\ =\left(\phi(x),-x\cos y\sin z,-x\sin y,x\cos y\cos z\right) \tag{22}$$

where $x \in \mathbb{R} - \{0\}$ and $0 \leq y,\ z \leq 2\pi$. The unit normal vector, Gaussian curvature and mean curvature of the rotational hypersurface eq. (22) are given:

$$N^T=\frac{1}{\sqrt{\varepsilon(\phi'^2-1)}}\left(1,-\phi'\cos y\sin z,-\phi'\sin y,\phi'\cos y\cos z\right) \tag{23}$$

$$K^T=\varepsilon\frac{\phi'^2\phi''}{x^2\left(\varepsilon(\phi'^2-1)\right)^{5/2}}\ \text{ and }\ H^T=\varepsilon\frac{2\phi'(1-\phi'^2)+x\phi''}{3x\left(\varepsilon(\phi'^2-1)\right)^{3/2}} \tag{24}$$

respectively. Here, we state

$$\phi=\phi(x),\ \phi'=\frac{\mathrm{d}\phi(x)}{\mathrm{d}x},\ \text{and}\ \phi''=\frac{\mathrm{d}^2\phi(x)}{\mathrm{d}x^2}$$

Rotational fypersurfaces about timelike axis generated by weighted mean curvature in E_1^4 with density $e^{-y^2-z^2-t^2}$

From eqs. (2), (23), and (24), the weighted mean curvature of the rotational hypersurface in E_1^4 with density $e^{y^2-z^2-t^2}$:

$$H_\varphi^T=\varepsilon\frac{2(1-x^2)\phi'-2(1-x^2)\phi'^3+x\phi''}{3x\left(\varepsilon(\phi'^2-1)\right)^{3/2}} \tag{25}$$

For solving eq. (25) wrt the weighted mean curvature H_φ^S, let us put:

$$F_2=\frac{\phi'(x)}{x^2\sqrt{\varepsilon(\phi'^2-1)}} \tag{26}$$

From eqs. (25) and (26), we have:

$$H_\varphi^T=\frac{-x^2F_2'+2x(x^2-2)F_2}{3} \tag{27}$$

and the solution of the first order differential eq. (27) *wrt* F_4 is obtained:

$$F_2=\frac{e^{x^2}\left(d_1-3\int_1^x t^2e^{-t^2}H_\varphi^T(t)\mathrm{d}t\right)}{x^4}\mathrm{d}t,\ d_1\in\mathbb{R} \tag{28}$$

Using eqs. (26) and (28), we reach that:

$$\phi(x)=\pm\int\sqrt{\frac{-\varepsilon e^{2x^2}\left(d_1-3\int_1^x t^2e^{-t^2}H_\varphi^T(t)\mathrm{d}t\right)^2}{x^4-\varepsilon e^{2x^2}\left(d_1-3\int_1^x t^2e^{-t^2}H_\varphi^T(t)\mathrm{d}t\right)^2}}\mathrm{d}x \tag{29}$$

So, we can state the following theorem:

Theorem 4. The rotational hypersurface eq. (22) about timelike axis in E_1^4 with density $e^{-y^2-z^2-t^2}$ can be parametrized wrt the weighted mean curvature:

$$T(x,y,z)=\left(\pm\int\sqrt{\frac{-\varepsilon e^{2x^2}\left(d_1-3\int_1^x t^2 e^{-t^2}H_\varphi^T(t)dt\right)^2}{x^4-\varepsilon e^{2x^2}\left(d_1-3\int_1^x t^2 e^{-t^2}H_\varphi^T(t)dt\right)^2}}dx, -x\cos y\sin z, -x\sin y, x\cos y\cos z\right) \tag{30}$$

where $d_1 \in \mathbb{R}$ and if $\varepsilon = 1$ then

$$x^4 < e^{2x^2}\left(d_1-3\int_1^x t^2 e^{-t^2}H_\varphi^T(t)dt\right)^2$$

Example 2. If we put

$$H_\varphi^T = \frac{2x^2-3}{3}$$

and $d_1 = 1/e$ in eq. (30), then the rotational hypersurface is given:

$$T(x,y,z)=\left(\sqrt{x^2-\varepsilon}, -x\cos y\sin z, -x\sin y, x\cos y\cos z\right) \tag{31}$$

In fig. 4 (for $\varepsilon = 1$) and fig. 5 (for $\varepsilon = -1$), the projections of the rotational hypersurfaces eq. (31) Gaussian and mean curvatures' graphics and the variations of them on hypersurfaces can be seen for $z = 0.5$ into $x_1 x_3 x_4$-space, respectively.

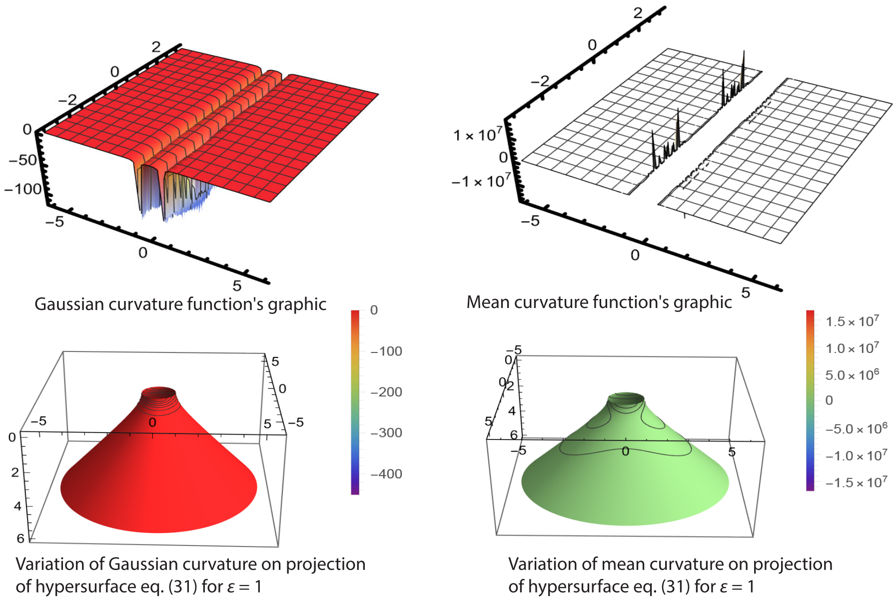

Figure 4. Graphics of curvatures and their variations on eq. (31) for $\varepsilon = 1$

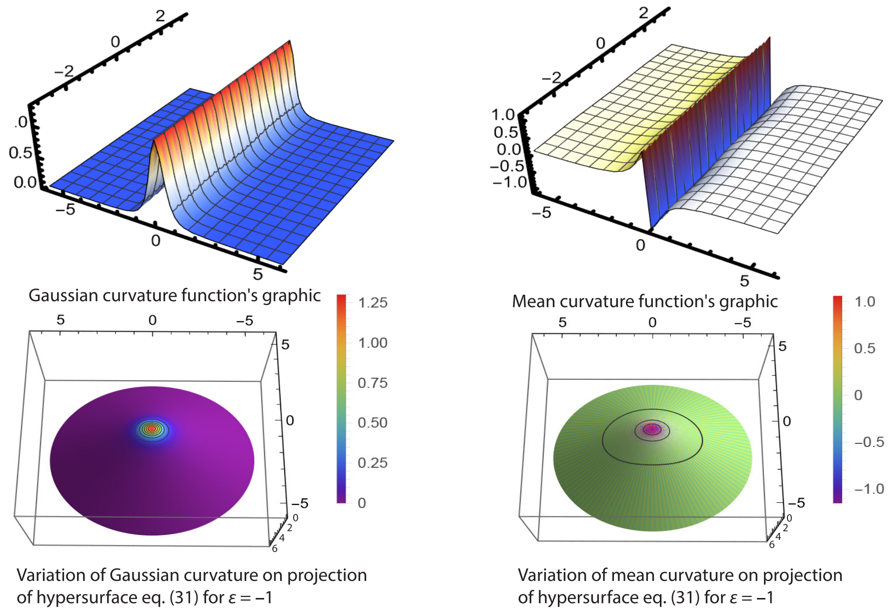

Figure 5. Graphics of curvatures and their variations on eq. (31) for $\varepsilon = -1$

Weighted mean curvature of rotational hypersurfaces about timelike axis in E_1^4 with Density e^{-y-z-t}

From eqs. (2), (23) and (24), the weighted mean curvature of the rotational hypersurface in E_1^4 with density e^{-y-z-t}:

$$H_\varphi^T = -\varepsilon \frac{\left(2 + x\sin y - x\cos y(\cos z - \sin z)\right)\phi'(\phi'^2 - 1) + x\phi''}{3x\left(\varepsilon(\phi'^2 - 1)\right)^{3/2}} \tag{32}$$

Let the weighted mean curvature be constant (*i.e.*, $H_\varphi^S = c$). From eq. (32), we have:

$$\varepsilon x\phi'(1-\phi'^2)\left(\sin y - \cos y(\cos z - \sin z)\right) - 3cx\left(\varepsilon(\phi'^2 - 1)\right)^{3/2} - \varepsilon x\phi'' + 2\varepsilon\phi'(1-\phi'^2) = 0 \tag{33}$$

If the eq. (33) is solved, it is seen:

$$x\phi'(1-\phi'^2) = 0 \ \text{ and } \ 3cx\left(\varepsilon(\phi'^2 - 1)\right)^{3/2} - \varepsilon x\phi'' + 2\phi'(1-\phi'^2) = 0$$

Since $\phi'^2 - 1 \neq 0$, the joint solution of these two equations is possible only with $\phi' = 0$, $c = 0$, and $\varepsilon = -1$.

Therefore, we have:

Theorem 5. The timelike rotational hypersurface eq. (22) about timelike axis in E_1^4 with density e^{x-y-z} can never have a constant weighted mean curvature.

Theorem 6. The spacelike rotational hypersurface eq. (22) about timelike axis in E_1^4 with density e^{x-y-z} can never have a non-zero constant weighted mean curvature.

Theorem 7. Weighted minimal spacelike rotational hypersurface (8) about spacelike axis in E_1^4 with density e^{x-y-z} can be parametrized:

$$S(x,y,z) = \left(k, -x\cos y\sin z, -x\sin y, x\cos y\cos z\right), \ k \in \mathbb{R} \tag{34}$$

In fig. 6, the projection of the spacelike rotational hypersurface eq. (34) can be seen for $z = 0.5$, $k = 3$ into $x_1x_2x_3$-spaces and $x_2x_3x_4$-spaces, respectively.

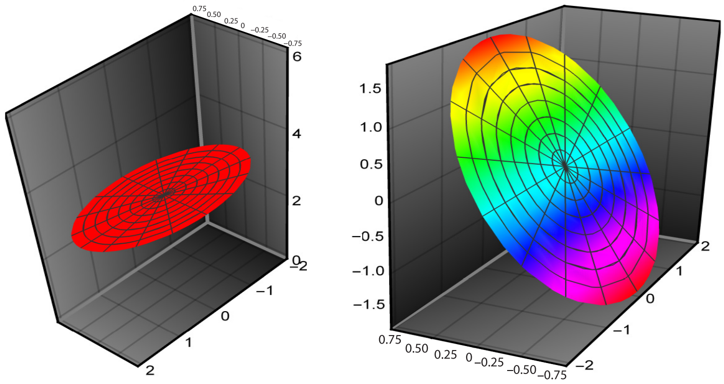

Figure 6. Rotational hypersurface eq. (34) into $x_1x_2x_3$-spaces and $x_2x_3x_4$-spaces

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