CANAL HYPERSURFACES ACCORDING TO ONE OF THE EXTENDED DARBOUX FRAME FIELD IN EUCLIDEAN 4-SPACE

by

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Original scientific paper
https://doi.org/10.2298/TSCI2204029K

In the present study, we deal with canal hypersurfaces according to extended Darboux frame field of second kind in Euclidean 4-space (E^4) and in this context, firstly we obtain the Gaussian, mean and principal curvatures of the canal hypersurface according to extended Darboux frame field of second kind and give some results for flatness and minimality of these hypersurfaces in E^4. Also, we give some results for Weingarten canal hypersurfaces according to extended Darboux frame field of second kind in E^4 and finally, we construct an example.

Key words: *canal hypersurface, extended Darboux frame field of second kind, tubular hypersurface*

Introduction

In this section, firstly we will give some general literature review about canal (hyper) surfaces and alternative frames to the Frenet frame and after that, we will recall some basic notions about hypersurfaces and extended Darboux frame field of second kind in E^4.

In 1850, Monge has firstly investigated the canal surfaces which are formed by sweeping a sphere. These surfaces may be generated either by sweeping a sphere along a path, or by sweeping a particular circular cross-section of the sphere along the same path and with the aid of these methods, the parametric expression of canal surfaces can be given:

$$\Omega(x,y)=\alpha(x)-r(x)r'(x)T(x)+r(x)\sqrt{1-r'^2(x)}\left(\cos yN(x)+\sin yB(x)\right)$$

where $\alpha(x)$ is called the spine curve or center curve which is a unit speed curve, $r(x)$ is called the radius function, and $\{T, N, B\}$ is called the Frenet frame of $\alpha(x)$. In case of a constant radius function, the canal surface is called tubular or pipe surface [1, 2]. Also for a canal surface, if the center curve is a straight line, then it becomes a surface of revolution. Canal surfaces (especially tubular surfaces) have been applied to many fields by mathematicians and engineers, such as the solid and the surface modelling for CAD/CAM, construction of blending surfaces, shape re-construction and so on. In this context, canal and tubular (hyper)surfaces have been studied by many scientists in Euclidean, Minkowskian, Galilean or pseudo-Galilean spaces, see [2-15 and *etc.*).

On the other hand, the Frenet frame has been used in many studies about curves and surfaces. But Frenet frame cannot be identified at the points where the curvature is zero and so, scienticists sometimes need alternative frames. Therefore, new alternative frames to the Frenet

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frame such as Bishop frame, Darboux frame or extended Darboux frames have been defined by geometers and the theories of curves and surfaces have been started to handle according to these alternative frames, [16-26].

For instance, the Darboux frame is a natural moving frame constructed on a surface. If M is a regular surface and $\alpha: I \subset R \rightarrow M$ is a unit speed curve on the surface M then the Darboux frame $\{e_1(x), B(x) = n_\alpha(x) \times e_1(x), n_\alpha(x)\}$ is well-defined along α, where $e_1(x)$ is the unit tangent vector of $\alpha(x)$ and $n_\alpha(\mathrm{x})$ is the unit normal vector of M along α. Also, Darboux equations of this frame:

$$
\begin{aligned}
e_1'(x) &= \kappa_g(x)B(x) + \kappa_n(x)n_\alpha(x) \\
B'(x) &= -\kappa_g(x)e_1(x) + \tau_g(x)n_\alpha(x) \\
n_\alpha'(x) &= -\kappa_n(x)e_1(x) - \tau_g(x)B(x)
\end{aligned} \tag{1}
$$

where $\kappa_n(x)$, $\kappa_g(x)$, and $\tau_g(x)$, are the normal curvature, the geodesic curvature, and the geodesic torsion of $\alpha(x)$, respectively. We can visualize a curve α which has Darboux frame on a surface M and a tubular surface $\mathcal{T}$ which is generated by the Darboux frame of the center curve α in Euclidean 3-space, fig. 1.

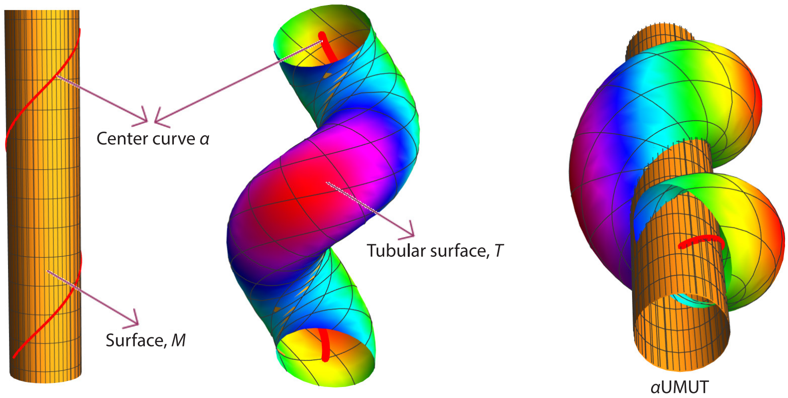

Figure 1.

Now, let we recall some basic notions about curves and hypersurfaces in E^4.

Let $\{e_1, e_2, e_3, e_4\}$ be the standart basis of E^4 and $\vec{x} = (x_1, x_2, x_3, x_4)$, $\vec{y} = (y_1, y_2, y_3, y_4)$, and $\vec{z} = (z_1, z_2, z_3, z_4)$ be three vectors in E^4. Then the inner product of two vectors is defined:

$$
\left\langle \vec{x}, \vec{y} \right\rangle = \sum_{i=1}^{4} x_i y_i
$$

and the vector product of three vectors is defined:

$$
\vec{x} \times \vec{y} \times \vec{z} = \begin{pmatrix} x_2y_3z_4 - x_2y_4z_3 - x_3y_2z_4 + x_3y_4z_2 + x_4y_2z_3 - x_4y_3z_2 \\ -x_1y_3z_4 + x_1y_4z_3 + x_3y_1z_4 - x_3y_4z_1 - x_4y_1z_3 + x_4y_3z_1 \\ x_1y_2z_4 - x_1y_4z_2 - x_2y_1z_4 + x_2y_4z_1 + x_4y_1z_2 - x_4y_2z_1 \\ -x_1y_2z_3 + x_1y_3z_2 + x_2y_1z_3 - x_2y_3z_1 - x_3y_1z_2 + x_3y_2z_1 \end{pmatrix} \tag{2}
$$

Let $M \subset E^4$ denote a regular hypersurface and α: $I \subset R \rightarrow M$ be a unit speed curve in E^4. If $\{T, n, b_1, b_2\}$ is the moving Frenet frame along α, then the Frenet formulas are given:

$$T' = k_1 n, n' = -k_1 T + k_2 b_1, \; b_1' = -k_2 n + k_3 b_2 \text{ and } b_2' = -k_3 b_1 \tag{3}$$

where T, n, b_1, and b_2 denote the unit tangent, the principal normal, the first binormal, and the second binormal vector fields, k_1, k_2, and k_3 are the curvature functions of the curve α [27].

Now, we will recall the extended Darboux frame field of second kind along a curve and for simplicity we'll call it ED^2-frame field. For details about the construction of ED^2-frame field, we refer to [22].

We consider an embedding Ω: $U \subset E^3 \rightarrow E^4$, where U is an open subset of E^3. Now, we denote $M = \Omega(U)$ and identify M and U through the embedding Ω. Let $\bar{\alpha}: I \rightarrow U$ be a regular curve and we have a curve α: $I \rightarrow M \subset E^4$ defined by $\alpha(x) = \Omega(\bar{\alpha}(x))$ and so, the curve α is on the hypersurface M. If M is an orientable hypersurface oriented by the unit normal vector field $\mathcal{N}$ in E^4 and α is a Frenet curve of class $C^n (n \geq 4)$ with arc-length parameter x lying on M, then we denote the unit tangent vector field of the curve by T and denote the hypersurface unit normal vector field restricted to the curve by N, *i.e.* $T(s) = \alpha'(x)$ and $N(s) = \mathcal{N}(\alpha(x))$.

If the set $\{N, T, \alpha''\}$ is linearly dependent, *i.e.* if α'' is in the direction of the normal vector N, applying the Gram-Schmidt orthonormalization method to $\{N, T, \alpha'''\}$ yields the orthonormal set $\{N, T, E\}$, where:

$$E = \frac{\alpha''' - \langle \alpha''', N \rangle N - \langle \alpha''', T \rangle T}{\left\| \alpha''' - \langle \alpha''', N \rangle N - \langle \alpha''', T \rangle T \right\|}$$

Here, if $D = N \times T \times E$ is defined, then four unit vector fields T, E, D, and N, which are mutually orthogonal at each point of α, have been obtained. So, the authors have obtained a new orthonormal frame field $\{T, E, D, N\}$ along the curve α instead of its Frenet frame field [22] and we will call it ED^2-frame field. The differential equations of ED^2-frame fields $\{T, E, D, N\}$ of the curve α in the E^4 can be given:

$$\begin{aligned} T' &= \kappa_n N \\ E' &= \kappa_g^2 D + \tau_g^1 N \\ D' &= -\kappa_g^2 E \\ N' &= -\kappa_n T - \tau_g^1 E \end{aligned} \tag{4}$$

where $\kappa_n = \langle T', N \rangle$ is the normal curvature of the hypersurface in the direction of the tangent vector T, $\kappa_g^2 = \langle E', D \rangle$ is the geodesic curvature of order 2, and $\tau_g^1 = \langle E', N \rangle$ is the geodesic torsion of order 1. Also, the relation matrix may be expressed:

$$\begin{bmatrix} n \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \cos\phi_1 & \cos\phi_2 & \cos\phi_3 \\ \cos\psi_1 & \cos\psi_2 & \cos\psi_3 \\ \cos\theta_1 & \cos\theta_2 & \cos\theta_3 \end{bmatrix} \begin{bmatrix} E \\ D \\ N \end{bmatrix} \tag{5}$$

as using the orthogonality of above 3×3 coefficient matrix:

$$\begin{bmatrix} E \\ D \\ N \end{bmatrix} = \begin{bmatrix} \cos\phi_1 & \cos\psi_1 & \cos\theta_1 \\ \cos\phi_2 & \cos\psi_2 & \cos\theta_2 \\ \cos\phi_3 & \cos\psi_3 & \cos\theta_3 \end{bmatrix} \begin{bmatrix} n \\ b_1 \\ b_2 \end{bmatrix} \tag{6}$$

where

$$\begin{gathered}\kappa_n = \langle T', N\rangle = k_1 \cos\phi_3 \\ \tau_g^1 = -\phi_1' \sin\phi_1 \cos\phi_3 - \psi_1' \sin\psi_1 \cos\psi_3 - \theta_1' \sin\theta_1 \cos\theta_3 + \\ +k_2(\cos\phi_1 \cos\psi_3 - \cos\psi_1 \cos\phi_3) + k_3(\cos\psi_1 \cos\theta_3 - \cos\theta_1 \cos\psi_3) \\ \kappa_g^2 = -\phi_1' \sin\phi_1 \cos\phi_2 - \psi_1' \sin\psi_1 \cos\psi_2 - \theta_1' \sin\theta_1 \cos\theta_2 + \\ +k_2(\cos\phi_1 \cos\psi_2 - \cos\psi_1 \cos\phi_2) + k_3(\cos\psi_1 \cos\theta_2 - \cos\theta_1 \cos\psi_2)\end{gathered} \tag{7}$$

Furthermore, the differential geometry of different types of (hyper)surfaces in 4-D spaces has been a popular topic for geometers, recently, [17, 28-35], and *etc*. If $\Omega: U \subset E^3 \to E^4$ is a hypersurface in E^4 parametrized by:

$$\Omega(x,y,z) = \big(\Omega_1(x,y,z), \Omega_2(x,y,z), \Omega_3(x,y,z), \Omega_4(x,y,z)\big)$$

then the unit normal vector field and the coefficients of the first and second fundamental forms:

$$\mathcal{N}_\Omega = \frac{\Omega_x \times \Omega_y \times \Omega_z}{\|\Omega_x \times \Omega_y \times \Omega_z\|}, \; g_{ij} = \left\langle \Omega_{x_i}, \Omega_{x_j} \right\rangle \text{ and } h_{ij} = \left\langle \Omega_{x_i x_j}, \mathcal{N}_\Omega \right\rangle \tag{8}$$

respectively, where:

$$\Omega_{x_i} = \frac{\partial \Omega(x_1,x_2,x_3)}{\partial x_i}, \; \Omega_{x_i x_j} = \frac{\partial^2 \Omega(x_1,x_2,x_3)}{\partial x_i x_j}, \; i,j = \{1,2,3\}$$

Also, if $[g_{ij}]^{-1}$ is inverse matrix of matrix form of the first fundamental form $[g_{ij}]$ and $[h_{ij}]$ is matrix form of the second fundamental form, then shape operator of the hypersurface Ω:

$$A = [a_{ij}] = [g_{ij}]^{-1}[h_{ij}] \tag{9}$$

With the aid of eqs. (8) and (9), the Gaussian curvature K and mean curvature H of a hypersurface in E^4 [36]:

$$K = \det(A) = \frac{\det[h_{ij}]}{\det[g_{ij}]} \text{ and } H = \frac{tr(A)}{3} \tag{10}$$

We say that a hypersurface is flat or minimal, if it has zero Gaussian or zero mean curvature, respectively.

Canal hypersurfaces according to *ED*²-frame field in *E*⁴

In this section, we study the canal hypersurfaces according to ED^2-frame field in E^4 and in this context, we will give some geometric characterizations about them.

Let $\alpha: I \to M$ be a unit speed curve on the regular hypersurface M and let we consider the canal hypersurface according to ED^2-frame field in E^4 parametrized:

$$\begin{gathered}\mathcal{C}(x,y,z) = \alpha(x) - r(x)r'(x)T(x) \pm \\ \pm r(x)\sqrt{1-r'(x)^2}\left[\big(\cos y \cos z\big)E(x) + \big(\sin y \cos z\big)D(x) + \big(\sin z\big)N(x)\right]\end{gathered} \tag{11}$$

where $\alpha(x)$ is center curve of the canal hypersurface, $r(x)$ – the radius function, $x \in [0, l]$ and $y, z \in [0, 2\pi)$. Also, from now on we state $\alpha = \alpha(x)$, $r = r(x)$, $r' = [\mathrm{d}r(x)]/\mathrm{d}x$, $T = T(x)$, $E = E(x)$, $D = D(x)$, $N = N(x)$, and we will consider the ± as +. One can obtain similar results by taking the sign as –.

Firstly from eq. (4), the first derivatives of the canal hypersurface eq. (11) are obtained:

$$\mathcal{C}_x = \left(1 - r'^2 - r\left(\kappa_n \sqrt{1-r'^2}\sin z + r''\right)\right)T +$$
$$+\left(\frac{\left(r'\cos y\cos z - r\left(\kappa_g^2 \sin y\cos z + \tau_g^1 \sin z\right)\right)\left(1-r'^2\right) - rr'r''\cos y\cos z}{\sqrt{1-r'^2}}\right)E +$$
$$+\left(\frac{\left(\left(r\kappa_g^2\cos y + r'\sin y\right)\left(1-r'^2\right) - rr'r''\sin y\right)\cos z}{\sqrt{1-r'^2}}\right)D +$$
$$+\left(r'\sqrt{1-r'^2}\sin z + r\left(-r'\kappa_n + \frac{\tau_g^1\left(1-r'^2\right)\cos y\cos z - r'r''\sin z}{\sqrt{1-r'^2}}\right)\right)N \tag{12}$$
$$\mathcal{C}_y = -r\sqrt{1-r'^2}\left(\left(\sin y\cos z\right)E - \left(\cos y\cos z\right)D\right)$$
$$\mathcal{C}_z = -r\sqrt{1-r'^2}\left(\left(\cos y\sin z\right)E + \left(\sin y\sin z\right)D - \left(\cos z\right)N\right)$$

From eq. (8) and eq. (12), the unit normal vector field of (11) in E^4:

$$\mathcal{N} = -r'T + \sqrt{1-r'^2}\left(\left(\cos y\cos z\right)E + \left(\sin y\cos z\right)D + \left(\sin z\right)N\right) \tag{13}$$

Also, the coefficients of the first fundamental form of eq (11) are given:

$$g_{11} = \frac{1}{1-r'^2}\left(\begin{array}{l}\left(1-r'^2\right)\left(1-r'^2 - r\left(\kappa_n\sqrt{1-r'^2}\sin z + r''\right)\right)^2 + \\ +\left(r\left(\kappa_g^2\sin y\cos z + \tau_g^1\sin z\right)\left(1-r'^2\right) - r'\left(1-r'^2 - rr''\right)\cos y\cos z\right)^2 + \\ +\cos^2 z\left(\kappa_g^2 r\left(1-r'^2\right)\cos y + r'\left(1-r'^2 - rr''\right)\sin y\right)^2 + \\ +\left\{r'\left(1-r'^2\right)\sin z + r\left(\tau_g^1\left(1-r'^2\right)\cos y\cos z - r'\left(\kappa_n\sqrt{1-r'^2} + r''\sin z\right)\right)\right\}^2\end{array}\right) \tag{14}$$
$$g_{12} = g_{21} = (1-r'^2)r^2\left(\kappa_g^2\cos z + \tau_g^1\sin y\sin z\right)\cos z$$
$$g_{13} = g_{31} = -r^2\left(\kappa_n r'\sqrt{1-r'^2}\cos z - \tau_g^1(1-r'^2)\cos y\right)$$
$$g_{22} = r^2(1-r'^2)\cos^2 z, g_{23} = g_{32} = 0, g_{33} = r^2(1-r'^2)$$

and it follows that:

$$\det[g_{ij}] = r^4(1-r'^2)\left(1-r'^2 - r\left(\kappa_n\sqrt{1-r'^2}\sin z + r''\right)\right)^2\cos^2 z \tag{15}$$

Now, for obtaining the coefficients of the second fundamental form, let we give the second derivatives:

$$\mathcal{C}_{x_i x_j} = \frac{\partial^2\mathcal{C}}{\partial x_i x_j}$$

of the canal hypersurface eq. (11):

$$\begin{aligned}
&\mathcal{C}_{xx} = \mathcal{C}_{xx}^1 T + \mathcal{C}_{xx}^2 E + \mathcal{C}_{xx}^3 D + \mathcal{C}_{xx}^4 N \\
&\mathcal{C}_{xy} = \mathcal{C}_{yx} = \frac{\left(-r'\left(1-r'^2\right)\sin y + r\left(-\kappa_g^2\left(1-r'^2\right)\cos y + r'r''\sin y\right)\right)\cos z}{\sqrt{1-r'^2}}E + \\
&+\frac{\left(r'\left(1-r'^2\right)\cos y - r\left(\kappa_g^2\left(1-r'^2\right)\sin y + r'r''\cos y\right)\right)\cos z}{\sqrt{1-r'^2}}D - \tau_g^1 r\sqrt{1-r'^2}\sin y\cos zN \\
&\mathcal{C}_{xz} = \mathcal{C}_{zx} = -\kappa_n r\sqrt{1-r'^2}\cos zT + \\
&+\frac{\left(r\left(\kappa_g^2\sin y\sin z - \tau_g^1\cos z\right) - r'\cos y\sin z\right)\left(1-r'^2\right) + rr'r''\cos y\sin z}{\sqrt{1-r'^2}}E + \\
&+\frac{\left(\left(-\kappa_g^2 r\cos y - r'\sin y\right)\left(1-r'^2\right) + rr'r''\sin y\right)\sin z}{\sqrt{1-r'^2}}D + \\
&+\frac{-\tau_g^1 r\left(1-r'^2\right)\cos y\sin z + r'\left(1-r'^2-rr''\right)\cos z}{\sqrt{1-r'^2}}N \qquad (16) \\
&\mathcal{C}_{yy} = -r\sqrt{1-r'^2}\left(\cos y\cos zE + \sin y\cos zD\right) \\
&\mathcal{C}_{yz} = \mathcal{C}_{zy} = r\sqrt{1-r'^2}\left(\sin y\sin zE - \cos y\sin zD\right) \\
&\mathcal{C}_{zz} = -r\sqrt{1-r'^2}\left(\cos y\cos zE + \sin y\cos zD + \sin zN\right)
\end{aligned}$$

where

$$\mathcal{C}_{xx}^1 = \frac{\begin{pmatrix} -r'\left(2\kappa_n\left(1-r'^2\right)\sin z + 3r''\sqrt{1-r'^2}\right) + \\ +r\begin{pmatrix} (\kappa_n)^2 r'\sqrt{1-r'^2} - \kappa_n\tau_g^1\left(1-r'^2\right)\cos y\cos z - \\ -\kappa_n'\left(1-r'^2\right)\sin z + 2\kappa_n r'r''\sin z - r'''\sqrt{1-r'^2} \end{pmatrix} \end{pmatrix}}{\sqrt{1-r'^2}}$$

$$\mathcal{C}_{xx}^2 = -\frac{\begin{pmatrix} \left(1-r'^2\right)\begin{pmatrix} 2\kappa_g^2 r'\left(1-r'^2\right)\sin y\cos z + \\ +2\tau_g^1 r'\left(1-r'^2\right)\sin z - r''\left(1-3r'^2\right)\cos y\cos z \end{pmatrix} + \\ +r\begin{pmatrix} \left[\left(\kappa_g^2\right)^2 + \left(\tau_g^1\right)^2\right]\left(1-r'^2\right)^2\cos y\cos z + \left(\kappa_g^2\right)'\left(1-r'^2\right)^2\sin y\cos z + \\ +\left(\tau_g^1\right)'\left(1-r'^2\right)^2\sin z - 2\kappa_g^2 r'r''\left(1-r'^2\right)\sin y\cos z - \\ -\tau_g^1 r'\left(1-r'^2\right)\left(\kappa_n\sqrt{1-r'^2} + 2r''\sin z\right) + \\ +\left(r'' + r'\left(1-r'^2\right)r'''\right)\cos y\cos z \end{pmatrix} \end{pmatrix}}{\left(1-r'^2\right)^{3/2}}$$

$$\mathcal{C}_{xx}^{3}=\frac{\begin{pmatrix}\left(1-r'^2\right)\left(2\kappa_g^2 r'\left(1-r'^2\right)\cos y+r''\left(1-3r'^2\right)\sin y\right)\cos z-\\ -r\begin{pmatrix}\left(\kappa_g^2\right)^2\left(1-r'^2\right)^2\sin y\cos z+\kappa_g^2\left(1-r'^2\right)\left(\tau_g^1\left(1-r'^2\right)\sin z+2r'r''\cos y\cos z+\right)\\ +\cos z\left(-(\kappa_g^2)'\left(1-r'^2\right)^2\cos y+\left(r''^2+r'r'''\left(1-r'^2\right)\right)\sin y\right)\end{pmatrix}\end{pmatrix}}{\left(1-r'^2\right)^{3/2}}$$

$$\mathcal{C}_{xx}^{4}=\frac{\begin{pmatrix}\left(1-r'^2\right)\left(2\tau_g^1 r'\left(1-r'^2\right)\cos y\cos z+\kappa_n\sqrt{1-r'^2}\left(1-2r'^2\right)+\left(1-3r'^2\right)r''\sin z\right)-\\ -r\begin{pmatrix}(\kappa_n)^2\sin z-(\kappa_n)^2(2-r'^2)r'^2\sin z+\kappa_g^2\tau_g^1\left(1-r'^2\right)^2\sin y\cos z+\\ +(\tau_g^1)^2\left(1-r'^2\right)^2\sin z-(\tau_g^1)'\cos y\cos z+(\tau_g^1)'(2-r'^2)r'^2\cos y\cos z+\\ +(\kappa_n)'(1-r'^2)^{3/2}r'+2\kappa_n(1-r'^2)^{3/2}r''+\\ +2\tau_g^1\left(1-r'^2\right)r'r''\cos y\cos z+r''^2\sin z+(1-r'^2)r'r'''\sin z\end{pmatrix}\end{pmatrix}}{\left(1-r'^2\right)^{3/2}}$$

Thus from eqs. (8), (13), and (16), the coefficients of the second fundamental form of eq. (11) are given:

$$h_{11}=-\frac{(1-r'^2)r}{4}\left(4\left(\kappa_g^2\right)^2\cos^2 z+4\kappa_g^2\tau_g^1\sin y\sin(2z)+(\tau_g^1)^2\left(3+\cos(2y)-2\sin^2 y\cos(2z)\right)\right)+$$
$$+\frac{\kappa_n\left(\left(\sin z+2\tau_g^1 rr'\cos y\cos z\right)\left(1-r'^2\right)-2rr''\sin z\right)}{\sqrt{1-r'^2}}-(\kappa_n)^2 r\left(\sin^2 z+r'^2\cos^2 z\right)+r''-\frac{rr''^2}{1-r'^2} \tag{17}$$
$$h_{12}=h_{21}=-r(1-r'^2)\left(\kappa_g^2\cos z+\tau_g^1\sin y\sin z\right)\cos z,\quad h_{22}=-r(1-r'^2)\cos^2 z$$
$$h_{13}=h_{31}=r\left(\kappa_n r'\sqrt{1-r'^2}\cos z-\tau_g^1(1-r'^2)\cos y\right),\quad h_{23}=h_{32}=0,\quad h_{33}=-r(1-r'^2)$$

and it implies:

$$\det[h_{ij}]=r^2(1-r'^2)\left(\kappa_n\sqrt{1-r'^2}\sin z+r''\right)\left(1-r'^2-r\left(\kappa_n\sqrt{1-r'^2}\sin z+r''\right)\right)\cos^2 z \tag{18}$$

So, from eqs. (10), (15), and (18), we have:

Proposition 1. The Gaussian curvature of the canal hypersurface eq. (11) according to ED^2-frame field in E^4:

$$K=\frac{\kappa_n\sqrt{1-r'^2}\sin z+r''}{r^2\left(1-r'^2-r\left(\kappa_n\sqrt{1-r'^2}\sin z+r''\right)\right)} \tag{19}$$

Corollary 1. The Gaussian curvature of the canal hypersurface eq. (11) according to ED^2-frame field in E^4 is only depend on the normal curvature, radius function and z.

Proposition 2. If the curve α is an asymptotic curve on M, then the Gaussian curvature of the canal hypersurface eq. (11) according to ED^2-frame field in E^4:

$$K=\frac{r''}{r^2\left(1-r'^2-rr''\right)}$$

Proof. We know that [22], a curve α is an asymptotic curve if and only if $\kappa_n = 0$ along α. Thus from eq. (19), the proof completes.

Corollary 2. The canal hypersurface (11) according to ED^2-frame field in E^4 is flat if the curve α is an asymptotic curve on M and the radius function r is linear such that $r(x) = ax + b$, $a \in (-1, 1)$, and $a, b \in \mathbb{R}$.

Proof. If the curve α is asymptotic and r is linear, then the Gaussian curvature vanishes and this completes the proof.

By taking $r(x) = r$ constant in eq. (11), we have:

$$\mathcal{T}(x,y,z) = \alpha(x) \pm r\left[(\cos y \cos z)E(x) + (\sin y \cos z)D(x) + (\sin z)N(x)\right] \tag{20}$$

which is a tubular hypersurface according to ED^2-frame field in E^4 and in this case, from eq. (19) we get:

Corollary 3. The Gaussian curvature of the tubular hypersurface (20) according to ED^2-frame field in E^4:

$$K = \frac{\kappa_n \sin z}{r^2(1-\kappa_n r \sin z)}$$

Also, after finding the inverse of the matrix of the first fundamental form and using this and eq. (17) in eq. (9), the shape operator of the canal hypersurface eq. (11) is obtained by $A = [A_{ij}]_{3\times3}$:

$$A_{11} = \frac{\kappa_n\sqrt{1-r'^2}\sin z + r''}{1-r'^2 - r\left(\kappa_n\sqrt{1-r'^2}\sin z + r''\right)},\quad A_{21} = -\frac{\left(1-r'^2\right)\left(\kappa_g^2 + \tau_g^1 \sin y \tan z\right)}{r\left(1-r'^2 - r\left(\kappa_n\sqrt{1-r'^2}\sin z + r''\right)\right)}$$

$$A_{31} = \frac{\kappa_n r'\sqrt{1-r'^2}\cos z - \tau_g^1(1-r'^2)\cos y}{r\left(1-r'^2 - r\left(\kappa_n\sqrt{1-r'^2}\sin z + r''\right)\right)},\quad A_{22} = A_{33} = -\frac{1}{r},\quad A_{12} = A_{13} = A_{23} = A_{32} = 0$$

Hence from eq. (10) and the shape operator A, we get:

Proposition 3. The mean curvature of the canal hypersurface (11) according to ED^2-frame field in E^4:

$$H = \frac{3r\left(\kappa_n\sqrt{1-r'^2}\sin z + r''\right) + 2r'^2 - 2}{3r\left(1-r'^2 - r\left(\kappa_n\sqrt{1-r'^2}\sin z + r''\right)\right)} \tag{21}$$

Corollary 4. The mean curvature of the canal hypersurface (11) according to ED^2-frame field in E^4 is only depend on the normal curvature, radius function and z.

Corollary 5. The mean curvature of the tubular hypersurface (20) according to ED^2-frame field in E^4:

$$H = \frac{3\kappa_n r \sin z - 2}{3r(1-\kappa_n r \sin z)}$$

Corollary 6. If the curve a is an asymptotic curve on M, then the mean curvature of the canal hypersurface (11) according to ED^2-frame field in E^4:

$$H = \frac{3rr'' + 2r'^2 - 2}{3r(1-r'^2 - rr'')} \tag{22}$$

Corollary 7 Let α be an asymptotic curve on M. Then, the canal hypersurface (11) according to ED^2-frame field in E^4 is minimal if and only if the radius function $r(x)$ is given:

$$\int \frac{\mathrm{d}r}{\sqrt{1-\left(\frac{\lambda}{r}\right)^{\frac{4}{3}}}} = \pm x + \mu, \lambda, \mu \in \mathbb{R}$$

Proof. If the curve α is an asymptotic curve lying on M and the canal hypersurface eq. (11) is minimal, then from eq. (22) we have:

$$3r(x)r''(x) + 2r'(x)^2 - 2 = 0 \tag{23}$$

The solution of the differential eq. (23) can be seen in [9] and so, the proof completes.

Here, from eqs. (19) and (21), we can state the following theorem which gives an important relation between Gaussian and mean curvatures:

Corollary 8. The Gaussian curvature K and the mean curvature H of the canal hypersurface (11) according to ED^2-frame field in E^4 satisfy:

$$3Hr - Kr^3 + 2 = 0 \tag{24}$$

Also, from the shape operator A, we have:

$$\det(A - \lambda I_3) = \frac{\left(\frac{\kappa_n \sqrt{1-r'^2}\sin z + r''}{1 - r'^2 - r\left(\kappa_n \sqrt{1-r'^2}\sin z + r''\right)} - \lambda\right)(1+\lambda r)^2}{r^2} \tag{25}$$

By solving the equation $\det(A - \lambda I_3) = 0$ from (25), we obtain the principal curvatures of the canal hypersurface eq. (11) and tubular hypersurface (20) in E^4:

Proposition 4. The principal curvatures of the canal hypersurface eq. (11) according to ED^2-frame field in E^4:

$$\lambda_1 = \lambda_2 = -\frac{1}{r},\ \lambda_3 = \frac{\kappa_n \sqrt{1-r'^2}\sin z + r''}{1 - r'^2 - r\left(\kappa_n \sqrt{1-r'^2}\sin z + r''\right)} \tag{26}$$

and so, the principal curvatures of the tubular hypersurface (20) according to ED^2-frame field in E^4:

$$\lambda_1 = \lambda_2 = -\frac{1}{r} \text{ and } \lambda_3 = \frac{\kappa_n \sin z}{1 - \kappa_n r \sin z} \tag{27}$$

Also we know that, if a curve α is a unit-speed geodesic curve parametrized by arc-length on an oriented hypersurface in E^4, then we have [22]:

$$\kappa_g^2 = k_3,\ \mathcal{T}_g^1 = -k_2,\ \kappa_n = k_1$$

So, from eqs. (19), (21), and (26) we get:

Corollary 9. Let the curve α be a unit-speed geodesic curve on M. Then the Gaussian curvature K, mean curvature H, and third principal curvature λ_3 of canal hypersurface eq. (11) according to ED^2-frame field in E^4:

$$K=\frac{k_1\sqrt{1-r'^2}\sin z+r''}{r^2\left(1-r'^2-r\left(k_1\sqrt{1-r'^2}\sin z+r''\right)\right)},\quad H=\frac{3r\left(k_1\sqrt{1-r'^2}\sin z+r''\right)+2r'^2-2}{3r\left(1-r'^2-r\left(k_1\sqrt{1-r'^2}\sin z+r''\right)\right)}$$

$$\lambda_3=\frac{k_1\sqrt{1-r'^2}\sin z+r''}{1-r'^2-r\left(k_1\sqrt{1-r'^2}\sin z+r''\right)}$$

Now, if

$$H_xK_y-H_yK_x=0,\ H_xK_z-H_zK_x=0,\ H_yK_z-H_zK_y=0$$

hold on a hypersurface, then we call the hypersurface as $(H, K)_{\{x, y\}}$, $(H, K)_{\{x, z\}}$, $(H, K)_{\{y, z\}}$-Weingarten hypersurface, respectively. So, from eqs. (19) and (21) we have:

Proposition 5. The canal hypersurface eq. (11) according to ED^2-frame field in E^4 is $(H, K)_{\{x, z\}}$ and $(H, K)_{\{y, z\}}$-Weingarten hypersurface. Also, the canal hypersurface eq. (11) according to ED-frame field in E^4 is $(H, K)_{\{x, z\}}$-Weingarten hypersurface, when the curve α is an asymptotic curve on M.

Proof. The canal hypersurface eq. (11) satisfies $H_xK_y - H_yK_x = H_yK_z - H_zK_y = 0$ and so, the first part of the Theorem is explicit. Also, from (19) and (21) we have:

$$H_xK_z-H_zK_x=\frac{2\kappa_n r'(1-r'^2)^{5/2}\cos z}{3r^4\left(1-r'^2-r\left(\kappa_n\sqrt{1-r'^2}\sin z+r''\right)\right)^3}$$

and so, if the curve α is an asymptotic curve, using $\kappa_n = 0$ in the last equation, the proof completes.

Also we know that, a hypersurface is called a linear Weingarten hypersurface, if it satisfies $aH + bK = c$, where a, b, c are not all zero constants. Thus, supposing r = constant in eq. (24), we have:

Proposition 6. The tubular hypersurface eq. (20) according to ED^2-frame field in E^4 is a linear Weingarten hypersurface.

Example for canal hypersurface according to ED^2-frame field in E^4

In this section, we construct a canal hypersurface according to ED^2-frame field in E^4, obtain its curvatures and draw its projections into 3-spaces. Let we consider the unit speed curve:

$$\alpha(x)=\left(2\sqrt{3}\cos\left(\frac{x}{5}\right),\cos\left(\frac{x}{5}\right),\sqrt{13}\sin\left(\frac{x}{5}\right),\frac{2\sqrt{3}x}{5}\right)\tag{28}$$

on the hypercylinder $M\ldots x^2 + y^2 + z^2 = 13$ in E^4. The unit tangent vector feld of α:

$$T(x)=\frac{1}{5}\left(-2\sqrt{3}\sin\left(\frac{x}{5}\right),-\sin\left(\frac{x}{5}\right),\sqrt{13}\cos\left(\frac{x}{5}\right),2\sqrt{3}\right)\tag{29}$$

Also, the unit normal vector of the hypercylinder is

$$\mathcal{N}=\frac{(x,y,z,0)}{\sqrt{13}}$$

and so:

$$N(x)=\mathcal{N}\left(\alpha(x)\right)=\left(2\sqrt{\frac{3}{13}}\cos\left(\frac{x}{5}\right),\frac{1}{\sqrt{13}}\cos\left(\frac{x}{5}\right),\sin\left(\frac{x}{5}\right),0\right) \tag{30}$$

Since α'' is linear dependent with N, we can construct ED^2-frame field along the curve α. Thus we obtain the other vectors of ED^2-frame field along the curve α:

$$\begin{gathered}E=\frac{1}{5}\left(\frac{12}{\sqrt{13}}\sin\left(\frac{x}{5}\right),\frac{2\sqrt{3}}{\sqrt{13}}\sin\left(\frac{x}{5}\right),-2\sqrt{3}\cos\left(\frac{x}{5}\right),\sqrt{13}\right)\\ D=\frac{1}{\sqrt{13}}\left(1,-2\sqrt{3},0,0\right)\end{gathered} \tag{31}$$

and the normal curvature, geodesic curvature of order 2 and geodesic torsion of order 1:

$$\kappa_n=-\frac{\sqrt{13}}{25},\ \kappa_g^2=0,\tau_g^1=\frac{2\sqrt{3}}{25} \tag{32}$$

Hence using eqs. (29)-(31) in eq. (11), we get the canal hypersurface according to ED^2-frame field in E^4:

$$C(x,y,x)=$$

$$=\left(\begin{array}{l}2\sqrt{3}\cos\left(\frac{x}{5}\right)+\frac{r}{65}\left(26\sqrt{3}r'\sin\left(\frac{x}{5}\right)+\sqrt{13(1-r'^2)}\left(\begin{array}{l}\cos z\left(12\sin\left(\frac{x}{5}\right)\cos y+5\sin y+\right)\\ +10\sqrt{3}\cos\left(\frac{x}{5}\right)\sin z\end{array}\right)\right),\\ \cos\left(\frac{x}{5}\right)+\frac{r}{65}\left(13r'\sin\left(\frac{x}{5}\right)+\sqrt{13(1-r'^2)}\left(\begin{array}{l}2\sqrt{3}\cos z\left(\sin\left(\frac{x}{5}\right)\cos y-5\sin y+\right)\\ +5\cos\left(\frac{x}{5}\right)\sin z\end{array}\right)\right),\\ \sqrt{13}\sin\left(\frac{x}{5}\right)-\frac{r}{5}\left(\sqrt{13}r'\cos\left(\frac{x}{5}\right)+\sqrt{1-r'^2}\left(2\sqrt{3}\cos\left(\frac{x}{5}\right)\cos y\cos z-5\sin\left(\frac{x}{5}\right)\sin z\right)\right),\\ \frac{1}{5}\left(2\sqrt{3}x-2\sqrt{3}rr'+\sqrt{13(1-r'^2)}r\cos y\cos z\right)\end{array}\right) \tag{33}$$

and from eqs. (19), (21), and eq. (32), we obtain the Gaussian and mean curvatures of the canal hypersurface eq. (33):

$$K=\frac{-\sqrt{13(1-r'^2)}\sin z+25r''}{r^2\left(\sqrt{13(1-r'^2)}r\sin z+25\left(1-r'^2-rr''\right)\right)},\ H=\frac{r\left(3r''-\frac{3\sqrt{13(1-r'^2)}}{25}\sin z\right)+2r'^2-2}{3r\left(1-r'^2-r\left(r''-\frac{\sqrt{13(1-r'^2)}}{25}\sin z\right)\right)} \tag{34}$$

respectively. In fig. 2, one can see the projections of the hypersurface eq. (33) for $z=\pi/3$ and $r(x)=x/3$ into $x_1x_2x_3$-, $x_1x_2x_4$-, $x_1x_3x_4$-, and $x_2x_3x_4$-spaces in figs. 2(a)-2(d), respectively.

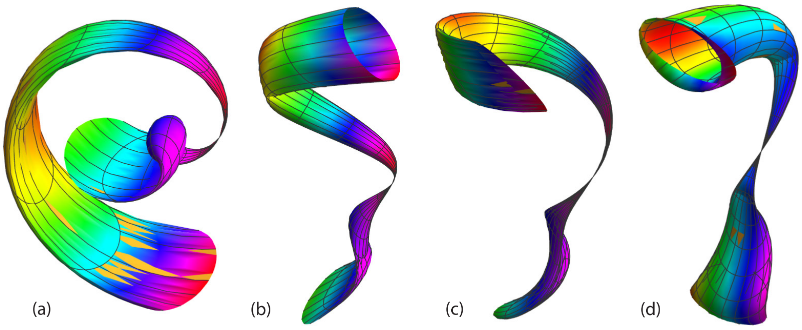

Figure 2.

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Paper submitted: June 19, 2021
Paper revised: November 1, 2021
Paper accepted: May 6, 2022