SOLVING STEADY HEAT TRANSFER PROBLEMS VIA KASHURI FUNDO TRANSFORM

by

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Integral transforms provide us great convenience in finding exact and approximate solutions of many mathematical physics and engineering problems such as signals, wave equation, heat conduction, heat transfer. In this study, we consider the Kashuri Fundo transform, which is one of these integral transforms, and our aim is to show that this transform is an effective method in solving steady heat transfer problems and obtained results are compared with the results of the existing techniques.

Key words: *Kashuri Fundo transform, inverse Kashuri Fundo transform, steady heat transfer problem, integral transform*

Introduction

Integral transforms have been used in solving problems in many different fields such as physics, engineering, chemistry*, etc.* These transforms provide great convenience in reaching the solutions of equations by converting differential operators from the original domain to another domain. In particular, some equations can be quite difficult to solve in the original domain. In such equations, solving symbolically in the new domain obtained by integral transforms makes things much easier. The solution found as a result of integral transforms is converted back to the original domain with inverse integral transforms [1-5].

The most famous integral transforms are the Laplace transform introduced by the French mathematician Laplace (1747-1827) [6] and the Fourier integral transform introduced by another French mathematician Fourier (1768-1830) [7]. These transforms are very effective in finding precise and approximate solutions to mathematical physics and engineering problems such as signals, wave equation, transient and steady-state analysis of heat conduction in solids, vibrations of continuous mechanical systems [8-13]. There exist many different integral transforms such as Mellin transform [14], Sumudu transform [15], Laplace-Carson transform [16], z-transform [17], Hankel's transform [18], Weierstrass transform [19], natural transform [20], Yang transform [21, 22], NL-TI transform [23] which are used in mathematical physics and engineering problems. In this study, we consider one of these transforms, the Kashuri Fundo transform [24].

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The aim of this study is to show that Kashuri Fundo integral transform is an effective method for solving steady heat transfer problems.

Kashuri Fundo transform method

We consider functions in the set F defined:

$$F=\left\{f(t)\,|\,\exists M,k_1,k_2>0 \text{ such that } |f(t)|\le M\mathrm{e}^{\frac{|t|}{k_i^2}},\ \text{if}\ t\in(-1)^i\times[0,\infty)\right\} \tag{1}$$

For a function belonging to the set F, the constant M must be finite number. The k_1, k_2 may be finite or infinite. Kashuri Fundo transform denoted by the operator $K(.)$ is defined [24]:

$$K[f(t)](v)=A(v)=\frac{1}{v}\int_0^\infty \mathrm{e}^{\frac{-t}{v^2}} f(t)\mathrm{d}t,\ t\ge 0,\ -k_1<v<k_2 \tag{2}$$

Inverse Kashuri Fundo transform is denoted by $K^{-1}[A(v)]=f(t)$, $t\ge 0$.

Theorem (sufficient conditions for existence of Kashuri Fundo transform)

If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order $1/k^2$, then $K[f(t)](v)$ exists for $|v|<k$ [24].

Properties of the transform

Theorem (linearity property)

Let $f(t)$ and $g(t)$ be functions whose Kashuri Fundo integral transforms exists and c be a constant. Then [24]:

$$K[(f\pm g)(t)]=K[f(t)]\pm K[g(t)] \tag{3}$$

$$K[(cf)(t)]=cK[f(t)] \tag{4}$$

Theorem (Kashuri Fundo transform of the derivatives of the function f(t))

Let $A(v)$ be a Kashuri Fundo transform of $f(t)$. Then [24]:

$$K[f'(t)]=\frac{A(v)}{v^2}-\frac{f(0)}{v} \tag{5}$$

$$K[f''(t)]=\frac{A(v)}{v^4}-\frac{f(0)}{v^3}-\frac{f'(0)}{v} \tag{6}$$

$$K\left[f^{(n)}(t)\right][=\frac{A(v)}{v^{2n}}-\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{v^{2(n-k)-1}} \tag{7}$$

Theorem (Kashuri Fundo transform of the partial derivatives)

Let $A(x, v)$ be a Kashuri Fundo transform of $f(x, t)$. Then [25]:

$$K\left[\frac{\partial f(x,t)}{\partial t}\right]=\frac{A(x,v)}{v^2}-\frac{f(x,0)}{v} \tag{8}$$

$$K\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right]=\frac{A(x,v)}{v^4}-\frac{f(x,0)}{v^3}-\frac{1}{v}\frac{\partial f(x,0)}{\partial t} \tag{9}$$

$$K\left[\frac{\partial^n f(x,t)}{\partial t^n}\right]=\frac{A(x,v)}{v^{2n}}-\sum_{k=0}^{n-1}\frac{1}{v^{2(n-k)-1}}\frac{\partial^k f(x,t)}{\partial t^k} \tag{10}$$

$$K\left[\frac{\partial f(x,t)}{\partial x}\right]=\frac{\mathrm{d}}{\mathrm{d}x}\left[A(x,v)\right] \tag{11}$$

$$K\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right]=\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[A(x,v)\right] \tag{12}$$

$$K\left[\frac{\partial^n f(x,t)}{\partial x^n}\right]=\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[A(x,v)\right] \tag{13}$$

Kashuri Fundo Transform of some special functions

Kashuri Fundo transform of some special functions are listed in tab. 1 [24, 26].

Table 1.

$f(t)$	$K[f(t)] = A(v)$
1	v
t	v^3
t^n	$n!v^{2n+1}$
e^{at}	$\frac{v}{1-av^2}$
$\sin(at)$	$\frac{av^3}{1+a^2v^4}$
$\cos(at)$	$\frac{v}{1+a^2v^4}$
$\sinh(at)$	$\frac{av^3}{1-a^2v^4}$
$\cosh(at)$	$\frac{v}{1-a^2v^4}$
t^α	$\Gamma(\alpha+1)v^{2\alpha+1}$
$\sum_{k=0}^{n}a_k t^k$	$\sum_{k=0}^{n}k!a_k v^{2k+1}$

Applications

In this section, we show the applicability of the Kashuri Fundo integral transform to steady heat transfer problems.

Application 1

Consider the following steady heat transfer problem [5]:

$$-hA\theta(t)=\rho Vc_p\theta'(t) \tag{14}$$

subject to the initial condition:

$$\theta(0)=\theta_0 \tag{15}$$

where h is the convection heat transfer coefficient, A – the surface area of the body, ρ – the density of the body, V – the volume, c_p – the specific heat of the material, and $\theta(t)$ – the temperature.

Taking Kashuri Fundo transform of both sides of eq. (14), we get:

$$K[-hA\theta(t)]=K\left[\rho Vc_p\theta'(t)\right] \tag{16}$$

$$-hAK[\theta(t)]=\rho Vc_pK[\theta'(t)] \tag{17}$$

Let's rewrite eq. (17) using eq. (5) and initial condition, we found:

$$K[\theta(t)]=\frac{\rho Vc_p}{-hA}\left(\frac{K[\theta(t)]}{v^2}-\frac{\theta_0}{v}\right) \tag{18}$$

$$K[\theta(t)]=\frac{v}{v^2-\dfrac{\rho Vc_p}{-hA}}\left(\theta_0\frac{\rho Vc_p}{hA}\right) \tag{19}$$

$$K[\theta(t)]=\theta_0\frac{v}{1-\dfrac{v^2}{\dfrac{\rho Vc_p}{-hA}}} \tag{20}$$

Applying inverse Kashuri Fundo transform on both sides of eq. (20) and using tab. 1 leads to the solution of eq. (14):

$$\theta(t)=\theta_0\mathrm{e}^{\frac{-hA}{\rho Vc_p}t} \tag{21}$$

which is coincides with the results found in [4, 22, 23].

Application 2

Consider the following steady heat transfer problem [6]:

$$U_t(x,t)=2U_{xx}(x,t),\ 0<x<5,\ t>0 \tag{22}$$

subject to the boundary and initial conditions:

$$U(0,t)=0,\ U(5,t)=0,\ U(x,0)=10\sin(4\pi x)-5\sin(6\pi x) \tag{23}$$

Taking Kashuri Fundo transform of both sides of eq. (22), we have:

$$K[U_t(x,t)]=K[2U_{xx}(x,t)] \tag{24}$$

$$K[U_t(x,t)]=2K[U_{xx}(x,t)] \tag{25}$$

Let's rewrite eq. (25) using eqs. (8) and (12), we get:

$$\frac{A(x,v)}{v^2}-\frac{U(x,0)}{v}=2\frac{\mathrm{d}^2A(x,v)}{\mathrm{d}x^2} \tag{26}$$

Substituting the boundary and initial conditions in eq. (26):

$$\frac{2\mathrm{d}^2A(x,v)}{\mathrm{d}x^2}-\frac{A(x,v)}{v^2}=-\frac{1}{v}\left[10\sin(4\pi x)-5\sin(6\pi x)\right] \tag{27}$$

is obtained which represents an inhomogeneous linear differential equation. The general solution of eq. (27) can be expressed:

$$A(x,v)=A_h(x,v)+A_p(x,v) \tag{28}$$

where $A_h(x, v)$ is the solution of the homogeneous part of the eq. (27). If we calculate this solution, we find:

$$A_h(x,v)=c_1\mathrm{e}^{-\frac{1}{\sqrt{2v}}x}+c_2\mathrm{e}^{\frac{1}{\sqrt{2v}}x} \tag{29}$$

and $A_p(x, v)$ is the solution of the inhomogeneous part of the eq. (27). If we calculate this solution, we find:

$$A_p(x,v)=\frac{10v}{1+32\pi^2v^2}\sin(4\pi x)-\frac{5v}{1+72\pi^2v^2}\sin(6\pi x) \tag{30}$$

Substituting eqs. (29) and (30) in eq. (28), we get:

$$A(x,v)=c_1\mathrm{e}^{-\frac{1}{\sqrt{2v}}x}+c_2\mathrm{e}^{\frac{1}{\sqrt{2v}}x}+\frac{10v}{1+32\pi^2v^2}\sin(4\pi x)-\frac{5v}{1+72\pi^2v^2}\sin(6\pi x) \tag{31}$$

In order to obtain c_1 and c_2, we employ the boundary conditions:

$$U(0,t)=0 \Rightarrow K[U(0,t)]=A(0,v)=0 \tag{32}$$

$$U(5,t)=0 \Rightarrow K[U(5,t)]=A(5,v)=0 \tag{33}$$

Using eqs. (31) and (32) we get $c_1 = -c_2$. Using eqs. (31) and (33) we find $c_1 = 0 \Rightarrow c_2 = 0$. If these obtained values are substituted in eq. (31), we get:

$$A(x,v)=\frac{10v}{1+32\pi^2v^2}\sin(4\pi x)-\frac{5v}{1+72\pi^2v^2}\sin(6\pi x) \tag{34}$$

Applying inverse Kashuri Fundo transform on both sides of eq. (34) and using tab. 1:

$$U(x,t)=10\mathrm{e}^{-32\pi^2t}\sin(4\pi x)-5\mathrm{e}^{-72\pi^2t}\sin(6\pi x) \tag{35}$$

is obtained which is in excellent agreement with the result obtained in [6, 23].

Application 3

Consider the following steady heat transfer problem [6]:

$$Y_{tt}(x,t)=9Y_{xx}(x,t),\ 0<x<2,\ t>0 \tag{36}$$

subject to the boundary and initial conditions:

$$Y(0,t)=0,\ Y(2,t)=0,\ Y(x,0)=20\sin(2\pi x)-10\sin(5\pi x) \tag{37}$$

Taking Kashuri Fundo transform of both sides of eq. (36), we get:

$$K[Y_{tt}(x,t)]=K[9Y_{xx}(x,t)] \tag{38}$$

$$K[Y_{tt}(x,t)]=9K[Y_{xx}(x,t)] \tag{39}$$

Let us rewrite eq. (39) using eqs. (9) and (12), we have:

$$\frac{A(x,v)}{v^4}-\frac{Y(x,0)}{v^3}-\frac{1}{v}\frac{\partial Y(x,0)}{\partial t}=9\frac{\mathrm{d}^2A(x,v)}{\mathrm{d}x^2} \tag{40}$$

Substituting the boundary and initial conditions in eq. (40):

$$\frac{9\mathrm{d}^2A(x,v)}{\mathrm{d}x^2}-\frac{A(x,v)}{v^4}=-\frac{1}{v^3}\left[20\sin(2\pi x)-10\sin(5\pi x)\right] \tag{41}$$

is obtained which represents an inhomogeneous linear differential equation. The general solution of eq. (41) can be expressed:

$$A(x,v)=A_\mathrm{h}(x,v)+A_\mathrm{p}(x,v) \tag{42}$$

where $A_\mathrm{h}(x, v)$ is the solution of the homogeneous part of the eq. (41). If we calculate this solution, we find:

$$A_\mathrm{h}(x,v)=c_1\mathrm{e}^{-\frac{1}{3v^2}x}+c_2\mathrm{e}^{\frac{1}{3v^2}x} \tag{43}$$

and $A_\mathrm{p}(x, v)$ is the solution of the inhomogeneous part of the eq. (41). If we calculate this solution, we find:

$$A_\mathrm{p}(x,v)=\frac{20v}{1+36\pi^2v^4}\sin(2\pi x)-\frac{10v}{1+225\pi^2v^4}\sin(5\pi x) \tag{44}$$

Substituting eqs. (43) and (44) in eq. (42), we get:

$$A(x,v)=c_1\mathrm{e}^{-\frac{1}{3v^2}x}+c_2\mathrm{e}^{\frac{1}{3v^2}x}+\frac{20v}{1+36\pi^2v^4}\sin(2\pi x)-\frac{10v}{1+225\pi^2v^4}\sin(5\pi x) \tag{45}$$

In order to obtain c_1 and c_2, we employ the boundary conditions:

$$Y(0,t)=0\Rightarrow K\left[Y(0,t)\right]=A(0,v)=0 \tag{46}$$

$$Y(2,t)=0\Rightarrow K\left[Y(2,t)\right]=A(2,v)=0 \tag{47}$$

Using eqs. (45) and (46) we get $c_1 = -c_2$. Using eqs. (45) and (47) we find $c_1 = 0 \Rightarrow c_2 = 0$. If these obtained values are substituted in eq. (45), we get:

$$A(x,v)=\frac{20v}{1+36\pi^2v^4}\sin(2\pi x)-\frac{10v}{1+225\pi^2v^4}\sin(5\pi x) \tag{48}$$

Applying inverse Kashuri Fundo transform on both sides of eq. (48) and using tab. 1:

$$Y(x,t)=20\sin(2\pi x)\cos(6\pi t)-10\sin(5\pi x)\cos(15\pi t) \tag{49}$$

is found which is fully in good agreement with the result obtained in [6].

Nomenclature

c_p – specific heat of the material, [JKkg^{-1}]
h – convection heat transfer coefficient, [WKm^{-2}]
$U(x, t)$ – temperature at any plane x at any time t, [K]
V – volume, [m^3]
x – space co-ordinate, [m]

Greek symbols

$\theta(t)$ – temperature, [K]
ρ – density, [kgm^{-3}]

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