SOLUTION OF ABEL'S INTEGRAL EQUATION BY KASHURI FUNDO TRANSFORM

by

Fatma Aybike CUHA [a] ***and Haldun Alpaslan PEKER*** [b*]

[a] Graduate School of Natural and Applied Sciences, Selcuk University, Konya, Turkey
[b] Department of Mathematics, Faculty of Science, Selcuk University, Konya, Turkey

Original scientific paper
https://doi.org/10.2298/TSCI2204003C

Integral equations can be defined as equations in which unknown function to be determined appears under the integral sign. These equations have been used in many problems occurring in different fields due to the connection they establish with differential equations. Abel's integral equation is an important singular integral equation and Abel found this equation from a problem of mechanics, namely the tautochrone problem. This equation and some variants of it found applications in heat transfer between solids and gases under non-linear boundary conditions, theory of superfluidity, subsolutions of a non-linear diffusion problem, propagation of shock-waves in gas field tubes, microscopy, seismology, radio astronomy, satellite photometry of airglows, electron emission, atomic scattering, radar ranging, optical fiber evaluation, X-ray radiography, flame and plasma diagnostics. Integral transforms are widely used mathematical techniques for solving advanced problems of applied sciences. One of these transforms is the Kashuri Fundo transform. This transform was derived by Kashuri and Fundo to facilitate the solution processes of ODE and PDE. In some works, it has been seen that it provides great convenience in finding the unknown function in integral equations. In this work, our aim is to solve Abel's integral equation by Kashuri Fundo transform and some applications are made to explain the solution procedure of Abel's integral equation by Kashuri Fundo transform.

Key words: *Abel's integral equation, convolution theorem, integral transforms, Kashuri Fundo transform, inverse Kashuri Fundo transform*

Introduction

Integral equations can be defined as equations that connect the unknown function $u(x)$ and the definite integral in which this function is found [1-3]. These equations one of the most useful mathematical tools used in fields such as engineering, applied mathematics and mathematical physics. They have enormous applications especially in obtaining mathematical solutions to complex boundary value problems of mathematical physics. Integral equations have been used in many problems in different fields due to their close relationship with differential equations, which have a wide application area [1-5].

Abel's integral equation is an important singular integral equation and Abel found this equation from a problem of mechanics, namely the tautochrone problem, which is considered to be the first application of fractional calculus to an engineering problem [6]. This equation and some variants of it found applications in heat transfer between solids and gases under non-linear

[*] Corresponding author, e-mail: pekera@gmail.com

boundary conditions [7], theory of superfluidity [8], percolation of water [9], subsolutions of a non-linear diffusion problem, propagation of shock-waves in gas fields tubes [10], microscopy, seismology, radio astronomy, satellite photometry of airglows, electron emission, atomic scattering, radar ranging, optical fiber evaluation, X-ray radiography, flame and plasma diagnostics [11]. In 1823, Abel [6] used:

$$f(x)=\int_0^x \frac{1}{\sqrt{x-t}}u(t)\mathrm{d}t \tag{1}$$

while investigating the motion of a particle sliding down along an unknown curve in a vertical plane [5, 6]. Here the kernel of the integral equation is the bivariate function

$$k(x,t)=\frac{1}{\sqrt{x-t}} \text{ and } k(x,t)=\frac{1}{\sqrt{x-t}}\to\infty \text{ while } t\to x$$

The aim of Abel's integral equation is to determine unknown $u(t)$ function. There are numerous integral transforms that can be used to determine $u(t)$ function. Aggarwal and Gupta [12-15] and Aggarwal and Sarma [16, 17] used some integral transforms for solving Abel's integral equations. The aim of this work is to show that $u(t)$ function can be determined by using Kashuri Fundo transform, without complex calculations.

Kashuri Fundo integral transform is a transform that has its origins in the Fourier integral and has a deep connection with the Laplace transform. This transformation was derived by Kashuri and Fundo [18] to facilitate the solution processes of ODE and PDE. Kashuri *et al*. [19] investigated the solution of non-linear PDE by mixing Kashuri Fundo transform and homotopy perturbation method. Kashuri *et al*. [20] applied Kashuri Fundo transform to solve some families of fractional differential equations. Gungor [21] investigated the solution of the convolution type linear Volterra integral equation with Kashuri Fundo transform.

Kashuri Fundo transform method

We consider functions in the set F defined:

$$F=\left\{ f(t)\,|\,\exists M,k_1,k_2>0 \text{ such that } |f(t)|\le M\mathrm{e}^{\frac{|t|}{k_i^2}},\text{ if } t\in(-1)^i\times[0,\infty)\right\} \tag{2}$$

For a function belonging to the set F, the constant M must be finite number. The k_1, k_2 may be finite or infinite. Kashuri Fundo transform denoted by the operator $K(.)$ is defined [18]:

$$K[f(t)](v)=A(v)=\frac{1}{v}\int_0^\infty \mathrm{e}^{\frac{-t}{v^2}}f(t)\mathrm{d}t,\ t\ge 0,\ -k_1<v<k_2 \tag{3}$$

Theorem (sufficient conditions for existence of Kashuri Fundo transform)

If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order $1/k^2$, then $\mathrm{K}[f(t)](v)$ exists for $|v| < k$ [18].

Properties of the transform

Theorem (linearity property)

Let $f(t)$ and $g(t)$ be functions whose Kashuri Fundo integral transforms exists and c be a constant. Then [18]:

$$K[(f\pm g)(t)]=K[f(t)]\pm K[g(t)] \tag{4}$$

$$K[(cf)(t)] = cK[f(t)] \tag{5}$$

Theorem (Kashuri Fundo transform of the derivatives of the function f(t))

Let $A(v)$ be Kashuri Fundo transform of $f(t)$. Then [18]:

$$K[f'(t)] = \frac{A(v)}{v^2} - \frac{f(0)}{v} \tag{6}$$

$$K[f''(t)] = \frac{A(v)}{v^4} - \frac{f(0)}{v^3} - \frac{f'(0)}{v} \tag{7}$$

$$K[f^{(n)}(t)] = \frac{A(v)}{v^{2n}} - \sum_{0}^{n-1} \frac{f^{(k)}(0)}{v^{2(n-k)-1}} \tag{8}$$

Kashuri Fundo transform of some special functions

Kashuri Fundo transform of some special functions [18] are listed in tab. 1.

Table 1.

$f(t)$	$K[f(t)] = A(v)$
1	v
t	v^3
t^n	$n!v^{2n+1}$
e^{at}	$\frac{v}{1-av^2}$
$\sin(at)$	$\frac{av^3}{1+a^2v^4}$
$\cos(at)$	$\frac{v}{1+a^2v^4}$
$\sinh(at)$	$\frac{av^3}{1-a^2v^4}$
$\cosh(at)$	$\frac{v}{1-a^2v^4}$
t^α	$\Gamma(\alpha+1)v^{2\alpha+1}$
$\sum_{k=0}^{n} a_k t^k$	$\sum_{k=0}^{n} k!a_k v^{2k+1}$

Theorem (convolution theorem)

Let $f(t)$ and $g(t)$ be defined in F having Kashuri Fundo integral transforms $M(v)$ and $N(v)$, respectively. Then, Kashuri Fundo integral transform of convolution of $f(t)$ and $g(t)$ is given [18]:

$$K[(f*g)(t)] = K\left[\int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau\right] = vM(v)N(v) \tag{9}$$

Main result

In this section, we show the solution of Abel's integral equation by Kashuri Fundo transform. Taking Kashuri Fundo transform of both sides of eq. (1), we have:

$$K[f(x)] = K\left[\int_0^x \frac{1}{\sqrt{x-t}} u(t)\mathrm{d}t\right] \tag{10}$$

$$K[f(x)] = K\left[x^{-1/2} * u(x)\right] \tag{11}$$

Applying convolution theorem of Kashuri Fundo transform in eq. (11), we find:

$$K[f(x)] = vK\left[x^{-1/2}\right]K[u(x)] \tag{12}$$

$$K[u(x)] = \frac{K[f(x)]}{vK\left[x^{-1/2}\right]} \tag{13}$$

If we calculate $K[x^{-1/2}]$ we find:

$$K\left[x^{-1/2}\right] = \frac{1}{v}\int_0^\infty \mathrm{e}^{\frac{-x}{v^2}} \frac{1}{\sqrt{x}}\mathrm{d}x \tag{14}$$

Applying the change of variable $x = t^2$, ($\mathrm{d}x = 2t\mathrm{d}t$) to eq. (14):

$$K\left[x^{-1/2}\right] = \frac{1}{v}\int_0^\infty \mathrm{e}^{-\frac{t^2}{v^2}} \frac{1}{t}2t\mathrm{d}t \tag{15}$$

Applying the change of variable $t/v = u$, ($\mathrm{d}t = v\mathrm{d}u$) to eq. (15):

$$K\left[x^{-1/2}\right] = \frac{2}{v}\int_0^\infty \mathrm{e}^{-u^2} v\mathrm{d}u = 2\int_0^\infty \mathrm{e}^{-u^2}\mathrm{d}u = \sqrt{\pi} \tag{16}$$

Substituting the result from eq. (16) into eq. (13), we have:

$$K[u(x)] = \frac{1}{v}\frac{K[f(x)]}{\sqrt{\pi}} = \frac{1}{v\sqrt{\pi}}K[f(x)] = \frac{1}{\pi v^2}\left(v\sqrt{\pi}K[f(x)]\right) \tag{17}$$

$$K[u(x)] = \frac{1}{\pi v^2}\left(vK\left[x^{-1/2}\right]K[f(x)]\right) = \frac{1}{\pi v^2}\left[K\left[\int_0^x \frac{1}{\sqrt{x-t}} f(t)\mathrm{d}t\right]\right] = \frac{1}{\pi v^2}K[F(x)] \tag{18}$$

where

$$F(x) = \int_0^x \frac{1}{\sqrt{x-t}} f(t)\mathrm{d}t \tag{19}$$

Now applying Kashuri Fundo transform to derivative of the function on eq. (19), we have:

$$K\left[F'(x)\right] = \frac{K\left[F(x)\right]}{v^2} - \frac{F(0)}{v} \tag{20}$$

$$K\left[F'(x)\right] = \frac{K\left[F(x)\right]}{v^2} \tag{21}$$

$$K\left[F(x)\right] = v^2 K\left[F'(x)\right] \tag{22}$$

Substituting eq. (22) into eq. (18), we obtain:

$$K[u(x)] = \frac{1}{\pi v^2} K[F(x)] \tag{23}$$

$$\pi v^2 K[u(x)] = v^2 K[F'(x)] \tag{24}$$

$$K[u(x)] = \frac{1}{\pi} K[F'(x)] \tag{25}$$

Applying inverse Kashuri Fundo transform on both sides of eq. (25) and using eq. (19), we find:

$$u(x) = \frac{1}{\pi} F'(x) = \frac{1}{\pi}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^x \frac{1}{\sqrt{x-t}} f(t)\mathrm{d}t\right)\right] \tag{26}$$

which is the required solution of eq. (1).

Applications

In this section, some applications are made to explain the solution procedure of the Abel's integral equation with Kashuri Fundo transform.

Aplication 1

Consider the following Abel's integral equation [5]:

$$x = \int_0^x \frac{1}{\sqrt{x-t}} u(t)\mathrm{d}t \tag{27}$$

Taking Kashuri Fundo transform of both sides of eq. (27):

$$K[x] = K\left[\int_0^x \frac{1}{\sqrt{x-t}} u(t)\mathrm{d}t\right] \tag{28}$$

$$v^3 = K\left[x^{-1/2} * u(x)\right] \tag{29}$$

Using convolution theorem of Kashuri Fundo transform on eq. (29):

$$v^3 = vK\left[x^{-1/2}\right]K\left[u(x)\right] \tag{30}$$

$$v^3 = v\sqrt{\pi}K[u(x)] \tag{31}$$

$$K[u(x)] = \frac{v^2}{\sqrt{\pi}} \tag{32}$$

Applying inverse Kashuri Fundo transform on both sides of eq. (32):

$$u(x) = \frac{1}{\sqrt{\pi}} K^{-1}\left[v^2\right] \tag{33}$$

$$u(x) = \frac{2}{\pi}\sqrt{x} \tag{34}$$

which is the required solution of eq. (27).

Application 2

Consider the following Abel's integral equation [5]:

$$x + x^3 = \int_0^x \frac{1}{\sqrt{x-t}} u(t)\mathrm{d}t \tag{35}$$

Taking Kashuri Fundo transform of both sides of eq. (35):

$$K\left[x + x^3\right] = K\left[\int_0^x \frac{1}{\sqrt{x-t}} u(t)\mathrm{d}t\right] \tag{36}$$

$$K[x] + K\left[x^3\right] = K\left[x^{-1/2} * u(x)\right] \tag{37}$$

$$v^3 + 6v^7 = K\left[x^{-1/2} * u(x)\right] \tag{38}$$

Using convolution theorem of Kashuri Fundo transform on eq. (38), we write:

$$v^3 + 6v^7 = vK\left[x^{-1/2}\right]K\left[u(x)\right] \tag{39}$$

$$v^3 + 6v^7 = v\sqrt{\pi}K\left[u(x)\right] \tag{40}$$

$$K\left[u(x)\right] = \frac{v^2 + 6v^6}{\sqrt{\pi}} \tag{41}$$

Applying inverse Kashuri Fundo transform on both sides of eq. (41):

$$u(x) = \frac{1}{\sqrt{\pi}} K^{-1}\left[v^2 + 6v^6\right] \tag{42}$$

$$u(x) = \frac{1}{\sqrt{\pi}}\left(K^{-1}\left[v^2\right] + 6K^{-1}\left[v^6\right]\right) \tag{43}$$

$$u(x) = \frac{2\sqrt{x}}{\pi}\left(1 + \frac{8}{5}x^2\right) \tag{44}$$

which is the required solution of eq. (35).

Application 3

Consider the following Abel's integral equation [5]:

$$\pi(x+1) = \int_0^x \frac{1}{\sqrt{x-t}} u(t)\mathrm{d}t \tag{45}$$

Taking Kashuri Fundo transform of both sides of eq. (45):

$$K\left[\pi(x+1)\right] = K\left[\int_0^x \frac{1}{\sqrt{x-t}} u(t)\mathrm{d}t\right] \tag{46}$$

$$\pi\left(K[x] + K[1]\right) = K\left[x^{-1/2} * u(x)\right] \tag{47}$$

$$\pi\left(v^3 + v\right) = K\left[x^{-1/2} * u(x)\right] \tag{48}$$

Using convolution theorem of Kashuri Fundo transform on eq. (48), we write:

$$\pi\left(v^3+v\right)=vK\left[x^{-1/2}\right]K\left[u(x)\right] \tag{49}$$

$$\pi\left(v^3+v\right)=v\sqrt{\pi}K\left[u(x)\right] \tag{50}$$

$$K\left[u(x)\right]=\sqrt{\pi}\left(v^2+1\right) \tag{51}$$

Applying inverse Kashuri Fundo transform on both sides of eq. (51):

$$u(x)=\sqrt{\pi}K^{-1}\left[v^2+v\right] \tag{52}$$

$$u(x)=2\sqrt{x}+\frac{1}{\sqrt{x}} \tag{53}$$

which is the required solution of eq. (45).

Application 4

Consider the following Abel's integral equation [5]:

$$\frac{3\pi}{8}x^2=\int_0^x\frac{1}{\sqrt{x-t}}u(t)\mathrm{d}t \tag{54}$$

Taking Kashuri Fundo transform of both sides of eq. (54):

$$K\left[\frac{3\pi}{2}x^2\right]=K\left[\int_0^x\frac{1}{\sqrt{x-t}}u(t)\mathrm{d}t\right] \tag{55}$$

$$\frac{3\pi}{8}K\left[x^2\right]=K\left[x^{-1/2}*u(x)\right] \tag{56}$$

$$\frac{3\pi}{2}\left(2v^5\right)=K\left[x^{-1/2}*u(x)\right] \tag{57}$$

Using convolution theorem of Kashuri Fundo transform on eq. (57), we write:

$$\frac{3\pi}{8}\left(2v^5\right)=vK\left[x^{-1/2}\right]K\left[u(x)\right] \tag{58}$$

$$\frac{3\pi}{8}\left(2v^5\right)=v\sqrt{\pi}K\left[u(x)\right] \tag{59}$$

$$K\left[u(x)\right]=\frac{3\sqrt{\pi}}{4}v^4 \tag{60}$$

Applying inverse Kashuri Fundo transform on both sides of eq. (60):

$$u(x)=\frac{3\sqrt{\pi}}{4}K^{-1}\left[v^4\right] \tag{61}$$

$$u(x)=x^{3/2} \tag{62}$$

which is the required solution of eq. (54).

References

[1] Rahman, M., *Integral Equation and Their Applications*, Wit Press, Southampton, Boston, UK, 2007
[2] Wazwaz, A. M., Abel's Integral Equation and Singular Integral Equations, in: *Linear and Non-linear Integral Equations: Methods and Applications*, Higher Education Press, Beijing, China, 2011, pp. 237-242
[3] Masujima, M., *Applied Mathematical Methods in Theoretical Physics*, Wiley-Vch Verlag, Weinheim, Germany, 2009
[4] Mandal, B. N., Chakrabarti, A., Some Elementary Methods of Solution of Singular Integral Equations, in: *Applied Singular Integral Equations*, Science Publishers, New York, USA, 2011, pp. 20-26
[5] Wazwaz, A. M., *A First Course in Integral Equations*, World Scientific, Wiley, New York, USA, 2015
[6] Abel, N. H., Solution de quelques problèmes à l'aide d'intégrales définies (in French), *Magazin for Naturvidenskabernc*, *1* (1823), 2, pp. 1-27
[7] Mann, W. R., Wolf, F., Heat Transfer between Solids and Gases under Non-linear Boundary Conditions, *Quarterly of Applied Mathematics*, *9* (1951), 2, pp. 163-184
[8] Levinson, N., A Non-Linear Volterra Equation Arising in the Theory of Superfluidity, *Journal of Mathematical Analysis and Applications*, *1* (1960), 1, pp. 1-11
[9] Goncerzewicz, J., *et.al.*, On the Percolation of Water from a Cylindrical Reservoir Into the Surrounding Soil, *Zastosowania Matematyki*, *16* (1978), pp. 249-261
[10] Okrasinski, W., Non-Linear Volterra Equations and Physical Applications, *Extracta Mathematicae*, *4* (1989), 2, pp. 51-80
[11] Singh, V. K., *et al.*, New Stable Numerical Solutions of Singular Integral Equations of Abel Type by Using Normalized Bernstein Polynomials, *Applied Mathematical Sciences, 3* (2009), 5, pp. 241-255
[12] Aggarwal, S., Gupta, A. R., A New Application of Mohand Transform for Handling Abel's Integral Equation, *Journal of Emerging Technologies and Innovative Research*, *6* (2019), 3, pp. 600-608
[13] Aggarwal, S., Gupta, A. R., Solution of Abel's Integral Equation Using Mahgoub Transform Method, *Journal of Emerging Technologies and Innovative Research*, *6* (2019), 4, pp. 252-260
[14] Aggarwal, S., Gupta, A. R., Sumudu Transform for the Solution of Abel's Integral Equation, *Journal of Emerging Technologies and Innovative Research*, *6* (2019), 4, pp. 423-431
[15] Aggarwal, S., Gupta, A. R., Shehu Transform for Solving Abel's Integral Equation, *Journal of Emerging Technologies and Innovative Research*, *6* (2019), 5, pp. 101-110
[16] Aggarwal, S., Sharma, S. D., Solution of Abel's Integral Equation by Aboodh Transform Method, *Journal of Emerging Technologies and Innovative Research*, *6* (2019), pp. 317-325
[17] Aggarwal, S., Sharma, N., Laplace Transform for the Solution of Abel's Integral Equation, *Journal of Advanced Research in Applied Mathematics and Statistics*, *4* (2019), 3-4, pp. 8-15
[18] Kashuri, A., Fundo, A., A New Integral Transform, *Advances in Theoretical and Applied Mathematics*, *8* (2013), 1, pp. 27-43
[19] Kashuri, A., *et al.*, Mixture of a New Integral Transform and Homotopy Perturbation Method for Solving Non-Linear Partial Differential Equations, *Advances in Pure Mathematics*, *3* (2013), 3, pp. 317-323
[20] Kashuri, A., *et al.*, New Integral Transform for Solving Some Fractional Differential Equations, *International Journal of Pure and Applied Mathematics*, *103* (2015), 4, pp. 675-682
[21] Gungor, N., Solving Convolution Type Linear Volterra Integral Equations with Kashuri Fundo Transform, *Journal of Abstract and Computational Mathematics*, *6* (2021), 2, pp. 1-7

Paper submitted: October 17, 2021
Paper revised: March 17, 2022
Paper accepted: April 29, 2022