ANALYTIC ALGORITHM FOR LOCAL FRACTIONAL CAUDREY-DODD-GIBBON-KAEADA EQUATION BASED ON THE NEW ITERATIVE METHOD

by

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In this paper, the initial value problem is discussed for the local fractional Caudrey-Dodd-Gibbon-Kaeada equation. The fractional complex transform and the new iterative method are used to solve the problem, and the approximate analytical solutions are obtained.

Key words: *Caudrey-Dodd-Gibbon-Kaeada equation, local fractional derivative, complex transform method, new iterative method*

Introduction

The differential equations with local fractional derivatives have proved to be a suitable tool for modeling many non-differentiable phenomena. Many physical problems in fractal media lead to non-linear models involving local fractional derivatives [1, 2]. Recently, several authors investigated the non-linear local fractional heat equation for the anomalous diffusion on a fractal medium [3, 4], the fractal population dynamics [5], and the fractional generalized KdV system [6].

Our interest here is to solve the following non-linear local fractional partial differential equation:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^\beta}{\partial x^\beta}\left(au^3 + bu\frac{\partial^{2\beta} u}{\partial x^{2\beta}} + c\frac{\partial^{4\beta} u}{\partial x^{4\beta}} \right) \tag{1}$$

with the condition:

$$u(x,0) = \phi\left[\frac{x^\beta}{\Gamma(1+\beta)} \right] \tag{2}$$

where $\partial^\alpha u/\partial t^\alpha$ and $\partial^\beta u/\partial x^\beta$ are the local fractional derivatives $(0<\alpha\le 1, 0<\beta\le 1)$, *a*, *b*, and *c* are constants, and $\phi(x)$ is given function.

When $\alpha=\beta=1$, eq. (1) is called the Caudrey-Dodd-Gibbon-Kaeada equation [7], it has wide applications in the description of many non-linear phenomena in physics and chem-

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istry [8-10]. This problem is often challenging to be solved analytically. In recent years, some analytical methods for solving local fractional differential equations have been proposed, for examples, the variational iteration method [11-13], the homotopy perturbation method [14, 15], the local fractional series expansion [16], and the local fractional function decomposition [17]. The new iterative method (NIM) was proposed first by Daftardar-Gejji and Jafari in [18], it is an effective procedure to find approximate analytical solutions of a wide class of non-linear local fractional differential equations [19, 20].

The main purpose of this paper is to solve the problem of eqs. (1) and (2) by using the fractional complex transform [21-24] and NIM [18-20].

Preliminaries

Local fractional derivative

In this section, we give some basic definitions and properties of the local fractional calculus theory [1-3].

Definition 1. For arbitrary $\varepsilon > 0,$ assume that the relation below exists:

$$\left| f(x) - f(x_0) \right| < \varepsilon^{\alpha} \tag{3}$$

with $|x - x_0| < \delta.$ Then $f(x)$ is called local fractional continuous at x_0 which is denoted by $\lim_{x\to x_0} f(x) = f(x_0).$ If $f(x)$ is local fractional continuous on the interval $(a,b),$ it is denoted by:

$$f(x) \in C_{\alpha}(a,b)$$

Definition 2. Let $f(x) \in C_{\alpha}(a,b).$ In fractal space, the local fractional derivative of $f(x)$ of order α at the point $x = x_0$ is given by:

$$\mathrm{D}_x^{\alpha} f(x_0) = \frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}} f(x)\bigg|_{x=x_0} = f^{(\alpha)}(x_0) = \lim_{x\to x_0} \frac{\Delta^{\alpha}[f(x) - f(x_0)]}{(x - x_0)^{\alpha}} \tag{4}$$

where $\Delta[f(x) - f(x_0)] \cong \Gamma(\alpha+1)[f(x) - f(x_0)].$

Definition 3. The local fractional partial derivative of a high order is defined in the form:

$$\frac{\partial^{k\alpha} u(x,t)}{\partial x^{k\alpha}} = \overbrace{\frac{\partial^{\alpha}}{\partial x^{\alpha}} \frac{\partial^{\alpha}}{\partial x^{\alpha}} \cdots \frac{\partial^{\alpha}}{\partial x^{\alpha}}}^{k\ \text{times}} u(x,t) \tag{5}$$

The following property holds true:

$$\frac{\partial^{\alpha} u[g(x,t)]}{\partial x^{\alpha}} = u'[g(x,t)] \frac{\partial^{\alpha} g(x,t)}{\partial x^{\alpha}} \tag{6}$$

where there exist $u'[g(x,t)]$ and $[\partial^{\alpha} g(x,t)]/\partial x^{\alpha}.$

The new iterative method

Below we illustrate the main points of NIM [18-20], by considering the following general function equation:

$$u = L(u) + \Phi(u) + \Pi \tag{7}$$

where L is a linear operator, Φ – a non-linear operator, and Π – a known function.

Assume that the solution of eq. (7) is of the form:

$$u = \sum_{i=0}^{\infty} u_i \tag{8}$$

The non-linear operator Φ can be decomposed as:

$$\Phi\left(\sum_{i=0}^{\infty} u_i\right) = \Phi(u_0) + \sum_{i=1}^{\infty}\left[\Phi\left(\sum_{j=0}^{i} u_j\right) - \Phi\left(\sum_{j=0}^{i-1} u_j\right)\right] \tag{9}$$

From eqs. (8) and (9), eq. (7) is equivalent to:

$$\sum_{i=0}^{\infty} u_i = \Pi + \sum_{i=0}^{\infty} \mathrm{T}(u_i) + \Phi(u_0) + \sum_{i=1}^{\infty}\left[\Phi\left(\sum_{j=0}^{i} u_j\right) - \Phi\left(\sum_{j=0}^{i-1} u_j\right)\right] \tag{10}$$

Define the recurrence relation:

$$\begin{aligned} &u_0 = \Pi(x) \\ &u_1 = L(u_0) + M_0 \\ &u_m = L(u_m) + M_m, \quad m = 1, 2, \cdots \end{aligned} \tag{11}$$

where

$$M_0 = \Phi(u_0) \tag{12}$$

$$M_m = \Phi\left(\sum_{i=0}^{m} u_i\right) - \Phi\left(\sum_{i=0}^{m-1} u_i\right), \quad m = 1, 2, \cdots \tag{13}$$

Then the p-term approximate solution of (7) is given by:

$$u = u_0 + u_1 + \cdots + u_{p-1} \tag{14}$$

Solution of the problem (1)-(2)

In this section, we consider the following initial value problem of non-linear local fractional Caudrey-Dodd-Gibbon-Kaeada equation:

$$\begin{gathered} \frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^{\beta}}{\partial^{\beta}}\left(au^3 + bu\frac{\partial^{2\beta} u}{\partial x^{2\beta}} + c\frac{\partial^{4\beta} u}{\partial x^{4\beta}}\right) \\ u(x,0) = \varphi\left[\frac{x^{\beta}}{\Gamma(1+\beta}\right] \end{gathered} \tag{15}$$

By using the fractional complex transform [21-24]:

$$X = \frac{x^{\beta}}{\Gamma(1+\beta)}, \quad T = \frac{t^{\alpha}}{\Gamma(1+\alpha)} \tag{16}$$

the problem (15) becomes:

$$\frac{\partial U}{\partial T} = \frac{\partial}{\partial X}\left(aU^3 + bU\frac{\partial^2 U}{\partial X^2} + \frac{\partial^4 U}{\partial X^4} \right)$$
$$U(X,0) = \Phi(X) \tag{17}$$

The understanding of eq. (16) was discussed in detail by the two-scale fractal theory for various applications [25-34]. We rewrite the problem (17):

$$U(X,T) = U(X,0) + \int_0^T \left(\sum_{k=1}^{4} N_k + c\frac{\partial^4 U}{\partial X^4} \right) \mathrm{d}T \tag{18}$$

where

$$N_1 = aU^3\frac{\partial U}{\partial X}, \quad N_2 = bU^2\frac{\partial^3 U}{\partial X^3}, \quad N_3 = bU\frac{\partial U}{\partial X}\frac{\partial^2 U}{\partial X^2}, \quad N_4 = b\frac{\partial U}{\partial X}\frac{\partial^2 U}{\partial X^2}\frac{\partial^3 U}{\partial X^3}$$

Suppose that the solution of eq. (18) takes the form:

$$U(X,T) = \sum_{k=0}^{\infty} U_k(X,T) \tag{19}$$

and the non-linear term in eq. (18) is decomposed:

$$N_1 = aU_0^2\frac{\partial U_0}{\partial X} + a\sum_{m=1}^{\infty} N_{1m} \tag{20}$$

$$N_2 = bU_0\frac{\partial^3 U_0}{\partial X^3} + b\sum_{m=1}^{\infty} N_{2m} \tag{21}$$

$$N_3 = b\frac{\partial U_0}{\partial X}\frac{\partial^2 U_0}{\partial X^2} + b\sum_{m=1}^{\infty} N_{3m} \tag{22}$$

where

$$N_{1m} = \left(\sum_{k=0}^{m} U_k\right)^2 \frac{\partial}{\partial X}\left(\sum_{k=0}^{m} U_k\right) - \left(\sum_{k=0}^{m-1} U_k\right)^2 \frac{\partial}{\partial X}\left(\sum_{k=0}^{m-1} U_k\right)$$

$$N_{2m} = \left(\sum_{k=0}^{m} U_k\right)\frac{\partial^3}{\partial X^3}\left(\sum_{k=0}^{m} U_k\right) - \left(\sum_{k=0}^{m-1} U_k\right)\frac{\partial^3}{\partial X^3}\left(\sum_{k=0}^{m-1} U_k\right)$$

$$N_{3m}=\frac{\partial}{\partial X}\left(\sum_{k=0}^{m}U_k\right)\frac{\partial^2}{\partial X^2}\left(\sum_{k=0}^{m}U_k\right)-\frac{\partial}{\partial X}\left(\sum_{k=0}^{m-1}U_k\right)\frac{\partial^2}{\partial X^2}\left(\sum_{k=0}^{m-1}U_k\right)$$

Now, we let:

$$N_0=aU_0^2\frac{\partial U_0}{\partial X}+bU_0\frac{\partial^3U_0}{\partial X^3}+b\frac{\partial U_0}{\partial X}\frac{\partial^2U_0}{\partial X^2}$$

then according to NIM, we obtain:

$$U_0(X,T)=\Phi(X)$$

$$U_1(X,T)=\int_0^T\left(N_0+c\frac{\partial^5U_0}{\partial X^5}\right)\mathrm{d}T$$

$$U_2(X,T)=\int_0^T\left(N_{11}+N_{21}+N_{31}+c\frac{\partial^5U_1}{\partial X^5}\right)\mathrm{d}T$$

$$\vdots$$

$$U_m(X,T)=\int_0^T\left(N_{1,m-1}+N_{2,m-1}+N_{3,m-1}+c\frac{\partial^5U_{m-1}}{\partial X^5}\right)\mathrm{d}T \tag{23}$$

Thus the *p-term* approximate solution of eq. (18) is given by:

$$U(X,T)=U_0(X,T)+U_1(X,T)+U_2(X,T)+\cdots+U_p(X,T)$$

From eq. (14), we can get the solution of the problem (15):

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+\cdots u_n(x,t)+\cdots$$

Application

Consider the problem (1)-(2) in the form:

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^\beta}{\partial^\beta}\left(-60u^3-30bu\frac{\partial^{2\beta}u}{\partial x^{2\beta}}-\frac{\partial^{4\beta}u}{\partial x^{4\beta}}\right)$$
$$u(x,0)=\frac{1}{4}\sec h^2\left[\frac{x^\beta}{2\Gamma(1+\beta}\right] \tag{24}$$

By the relations (16), we obtain:

$$\frac{\partial U}{\partial T}=\frac{\partial}{\partial X}\left(-60U^3-30U\frac{\partial^{2\beta}}{\partial X^2}-\frac{\partial^4U}{\partial X^4}\right)$$
$$U(X,0)=\frac{1}{4}\sec h^2\left(\frac{X}{2}\right) \tag{25}$$

Let $\exp(-2X)=E$, $\exp\left[\dfrac{-2x^{\beta}}{\Gamma(1+\beta)}\right]=F$.

Then by eq. (23), we have:

$$U_0(X,T)=\frac{4E}{(1+E)^2}$$

$$U_1(X,T)=\frac{128(E-E^2)}{(1+E)^2}T$$

$$U_2(X,T)=\frac{2048(E^3-4E^2+E)}{(1+E)^4}T^2$$

$$U_3(X,T)=\frac{65536(11E^3-11E^2-E^4+E)}{3(1+E)^5}T^2$$

$$\vdots$$

and so on.

Hence, from eq. (17), we obtain:

$$u_0(x,t)=\frac{4F}{(1+F)^2}$$

$$u_1(x,t)=\frac{128(F-F^2)}{(1+F)^3\Gamma(1+\alpha)}t^{\alpha}$$

$$u_2(x,t)=\frac{2048(F^3-4F^2+F)}{(1+F)^4\Gamma^2(1+\alpha)}t^{2\alpha}$$

$$u_3(x,t)=\frac{65536(11F^3-11F^2-F^4+F)}{3(1+F)^5\Gamma^3(1+\alpha)}t^{3\alpha}$$

$$\vdots$$

Finally, the solution of eq. (23) is given by:

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+u_4(x,t)+\cdots$$

When $\alpha=\beta=1$, if let $\exp(-2x)=G$, then we get:

$$u(x,t)=\frac{4G}{(1+G)^2}+\frac{128(G-G^2)}{(1+G)^3}t+\frac{2048(G^3-4G^2+G)}{(1+G)^4}t^2+\cdots$$

which is close to the exact solution [10]:

$$u(x,t)=\text{sech}^2(x-16t)$$

Conclusion

We have presented an analytic algorithm for non-linear local fractional Caudrey-Dodd-Gibbon-Kawada equation. The fractional complex transform method and NIM have been successfully applied to find the approximate analytical solutions of the equation. The results show that NIM is a powerful and efficient technique in finding approximate analytical solutions for non-linear local fractional differential equations.

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