TRANSITION BEHAVIORS OF SYSTEM ENERGY IN A BISTABLE VAN der POL OSCILLATOR WITH FRACTIONAL DERIVATIVE ELEMENT DRIVEN BY MULTIPLICATIVE GAUSSIAN WHITE NOISE

by

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The stochastic P-bifurcation behavior of system energy in a bi-stable Van der Pol oscillator with fractional damping under multiplicative Gaussian white noise excitation is investigated. Firstly, using the principle of minimal mean square error, the non-linear stiffness terms can be equivalent to a linear stiffness which is a function of the system amplitude, and the original system is simplified to an equivalent integer order Van der Pol system. Secondly, the system amplitude's stationary probability density function is obtained by stochastic averaging. Then, according to the singularity theory, the critical parametric conditions for the system amplitude's stochastic P-bifurcation are found. Finally, the types of the system's stationary probability density function curves of amplitude are qualitatively analyzed by choosing the corresponding parameters in each area divided by the transition set curves. The consistency between the analytical results and the numerical results obtained from Monte-Carlo simulation verifies the theoretical analysis in this paper, and the method used in this paper can directly guide the design of the fractional-order controller to adjust the response of the system.

Key words: *stochastic P-bifurcation, fractional damping, Gaussian white noise, transition set, Monte-Carlo simulation*

Introduction

Fractional calculus is a generalization of integer-order calculus, which has a history of more than 300 years. The integer-order derivative can not express the memory characteristics of the viscoelastic substances, while the definition of fractional derivative contains convolution, which can express a memory effect and show a cumulative effect over time. Therefore, the fractional derivative is a more suitable mathematical tool to describe memory characteris-

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tics [1-4] and has become a powerful mathematical tool for the study in the research fields such as anomalous diffusion, non-Newtonian fluid mechanics, viscoelastic mechanics, and soft matter physics. Comparing with the integer-order calculus, the fractional derivative can describe various reaction processes more accurately [5-7]. Thus, it is necessary and significant to study the mechanical characteristics and the fractional-order parametric influences on systems.

Recently, many scholars have studied the dynamic behavior of non-linear multi-stable systems under different noise excitations and achieved fruitful results. Liu *et al.* [8] studied the response of a strongly non-linear vibro-impact system with Coulomb friction excited by real noise and analyzed the *P*-bifurcation by a qualitative change of the friction amplitude and the restitution coefficient on the stationary probability distribution. Some researchers [9-11] studied the Van der Pol-Duffing oscillators under Levy noise, color noise, combined harmonic and random noise, respectively. The stochastic *P*-bifurcation behaviors of the noise oscillators are discussed by analyzing changes in the system's stationary probability density function (PDF). The analytical results of the bimodal stationary PDF are obtained, showing that the system parameters and noise intensity can each induce stochastic *P*-bifurcation of the systems. Hao and Wu [12-14] investigated the tristable stochastic *P*-bifurcation in a generalized Duffing-Van der Pol oscillator under additive Gaussian white noise, multiplicative colored noise, combined additive and multiplicative Gaussian white noise, respectively. They obtained an analytical expression of the system's stationary PDF of amplitude and analyzed the influences of noise intensity and system parameters on the system's stochastic P-bifurcation. Chen *et al.* [15] studied the response of a Duffing system with fractional damping under the combined white noise and harmonic excitations and showed that variation of the fractional derivative's order could arouse the system's stochastic *P*-bifurcation. Huang *et al.* [16] discussed the response and the stationary PDF of a single-degree-of-freedom strongly non-linear system under Gaussian white noise excitation. Li *et al.* [17] studied the bi-stable stochastic *P*-bifurcation behavior of a Van der Pol- Duffing system with the fractional derivative under additive and multiplicative colored noise excitations and found that changes in the linear damping coefficient, the fractional derivative' order, and the noise intensity can each lead to stochastic *P*-bifurcation in the system. Liu *et al.* [18] investigated a Duffing oscillator system with fractional damping under combined harmonic and Poisson white noise parametric excitation, and then the asymptotic Lyapunov stability with a probability of the original system is analyzed based on the largest Lyapunov exponent. Chen *et al.* [19] studied the primary resonance response of a Van der Pol system under fractional-order delayed negative feedback and forced excitation and obtained the approximate analytical solution based on the averaging method. Chen *et al.* [20] proposed a stochastic averaging technique that can be used to study the randomly excited strongly non-linear system with delayed feedback fractional-order proportional-derivative controller and obtained the stationary PDF of the system.

Due to the complexity of the fractional derivative, the parametric vibration characteristics of the fractional system can only be analyzed qualitatively, while the critical conditions of the parametric influences can not be obtained. In practice, the critical conditions of the parametric influences play a vital role in the analysis and design of fractional-order systems. Additionally, the stochastic *P*-bifurcation of bi-stability for the generalized Van der Pol system with fractional damping has not been reported in the open literature. In this paper, taking a generalized Van der Pol system with fractional damping excited by multiplicative Gaussian white noise excitation as the example, non-linear vibration of this kind of fractional-order system is studied through the fractional derivative. The transition set curves and critical parameter conditions for the system's stochastic *P*-bifurcation are obtained by the singularity method. The types of the system's stationary PDF curves in each area of the parameter plane are analyzed. We also compare the numerical results from the Monte-Carlo simulation with analytical solutions obtained by stochastic averaging. The comparison shows that the numerical results are in good agreement with the analytical solutions, verifying our theoretical analysis.

Derivation of the equivalent system

The initial condition of the Riemann-Liouville derivative has no physical meaning, while the initial condition of the system described by the Caputo derivative has not only clear physical meaning but also forms the same initial condition with the integer-order differential equation. Therefore, in this paper, we adopt the Caputo fractional derivative:

$$ {}_a^C\mathrm{D}^p[x(t)] = \frac{1}{\Gamma(m-p)} \int_a^t \frac{x^{(m)}(u)}{(t-u)^{1+p-m}} \mathrm{d}u \tag{1} $$

where $m-1 < p \le m, m \in N,\ t \in [a,b],\ x^{(m)}(t)$ – the m-order derivative of $x(t)$, and $\Gamma(m)$ – the Gamma function.

For a given physical system, the initial moment of oscillators is $t = 0,$ and the Caputo derivative is usually expressed:

$$ {}_0^C\mathrm{D}^p[x(t)] = \frac{1}{\Gamma(m-p)} \int_0^t \frac{x^{(m)}(u)}{(t-u)^{1+p-m}} \mathrm{d}u \tag{2} $$

where $m-1 < p \le m, m \in N.$

In this paper, we study the generalized Van der Pol system with the fractional damping excited by multiplicative Gaussian white noise excitation:

$$ \ddot{x}(t) - [-\varepsilon + \alpha_1 x^2(t) - \alpha_2 x^4(t)]\dot{x}(t) + {}_0^C\mathrm{D}^p[x(t)] + w^2 x(t) = x(t)\xi(t) \tag{3} $$

where ε represents the linear damping coefficient, α_1 and α_2 – the non-linear damping coefficients of the system, and w – the system's natural frequency. The ${}_0^C\mathrm{D}^p[x(t)]$ is the $p(0 \le p \le 1)$) order Caputo derivative of $x(t)$, which is defined by eq. (2). The $\xi(t)$ is the Gaussian white noise excitation, which satisfies:

$$ E[\xi(t)] = 0, \quad E[\xi(t)\xi(t-\tau)] = 2D\delta(\tau) \tag{4} $$

where D denotes the intensity of Gaussian white noises $\xi(t)$ and $\delta(\tau)$ – the Dirac function.

The fractional derivative has the contributions of the damping force and restoring force [21]. Hence, we introduce the equivalent system:

$$ \ddot{x}(t) - [-\varepsilon + \alpha_1 x^2(t) - \alpha_2 x^4(t)]\dot{x}(t) + [C(p)\dot{x}(t) + K(p)x(t)] = x(t)\xi(t) \tag{5} $$

where $C(p)$ and $K(p)$ are the coefficients of the equivalent damping and equivalent restoring forces of the fractional derivative ${}_0^C\mathrm{D}^p[x(t)],$ respectively.

Applying the equivalent method mentioned in [22], we get the ultimate forms of $C(p)$ and $K(p)$:

$$C(p) = -w^{p-1}\sin\frac{p\pi}{2}, \quad K(p) = w^p\cos\frac{p\pi}{2} \tag{6}$$

Therefore, the equivalent Van der Pol oscillator associated with system (5) can be written:

$$\ddot{x}(t) - \gamma\dot{x}(t) + w_0^2 x(t) = x(t)\xi(t) \tag{7}$$

where

$$\begin{gathered} \gamma = -\varepsilon + \alpha_1 x^2 - \alpha_2 x^4 - w^{p-1}\sin\frac{p\pi}{2} \\ w_0^2 = w^2 + w^p\cos\frac{p\pi}{2 - w\tau} \end{gathered} \tag{8}$$

Stationary PDF of the system amplitude

Assuming the system (7) has the solution of the periodic form, we introduce the following transformation [23]:

$$\begin{gathered} X = x(t) = a(t)\cos\Phi(t) \\ Y = \dot{x}(t) = -a(t)w_0\sin\Phi(t) \\ \Phi(t) = w_0 t + \theta(t) \end{gathered} \tag{9}$$

where w_0 is the natural frequency of the above equivalent system (8) and $a(t)$ and $\theta(t)$ – the amplitude and phase processes of the system's response, respectively, and they are both random processes.

Substituting eq. (9) into eq. (7), we obtain:

$$\begin{gathered} \frac{\mathrm{d}a}{\mathrm{d}t} = F_{11}(a,\theta) + G_{11}(a,\theta)\xi(t) \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} = F_{21}(a,\theta) + G_{21}(a,\theta)\xi(t) \end{gathered} \tag{10}$$

in which:

$$\begin{gathered} F_{11}(a,\theta) = a\sin^2\Phi(-\varepsilon + \alpha_1 a^2\cos^2\Phi - \alpha_2 a^4\cos^4\Phi) \\ F_{21}(a,\theta) = \sin\Phi\cos\Phi(-\varepsilon + \alpha_1 a^2\cos^2\Phi - \alpha_2 a^4\cos^4\Phi) \\ G_{11} = -\frac{\mathrm{a}\sin\Phi\cos\Phi}{w_0} \\ G_{21} = -\frac{\cos^2\Phi}{w_0} \end{gathered} \tag{11}$$

Equation (11) can be treated as the Stratonovich stochastic differential equation, and by adding the relevant Wong-Zakai correction term, we transform it into the corresponding Itô stochastic differential equation:

$$\begin{aligned} \mathrm{d}a &= [F_{11}(a,\theta) + F_{12}(a,\theta)]\mathrm{d}t + \sqrt{2D}\,\mathrm{G}_{11}(a,\theta)\mathrm{d}B(t) \\ \mathrm{d}\theta &= [F_{21}(a,\theta) + F_{22}(a,\theta)]\mathrm{d}t + \sqrt{2D}G_{21}(a,\theta)\mathrm{d}B(t) \end{aligned} \tag{12}$$

where $B(t)$ is the normalized Wiener process and:

$$\begin{aligned} F_{12}(a,\theta) &= D\frac{\partial G_{11}}{\partial a}G_{11} + D\frac{\partial G_{11}}{\partial \theta}G_{21} \\ F_{22}(a,\theta) &= D\frac{\partial G_{21}}{\partial a}G_{11} + D\frac{\partial G_{21}}{\partial \theta}G_{21} \end{aligned} \tag{13}$$

By stochastic averaging [24] of eq. (12) over Φ, we obtain the following averaged Ito equation:

$$\begin{aligned} \mathrm{d}a &= m_1(a)\mathrm{d}t + \sigma_{11}(a)\mathrm{d}B(t) \\ \mathrm{d}\theta &= m_2(a)\mathrm{d}t + \sigma_{21}(a)\mathrm{d}B(t) \end{aligned} \tag{14}$$

where

$$\begin{gathered} m_1(a) = -\frac{1}{2}\left(w^{p-1}\sin\frac{p\pi}{2} + \varepsilon\right)a + \frac{1}{8}\alpha_1 a^3 - \frac{1}{16}\alpha_2 a^5 + \frac{3D_2 a}{8w_0^2} \\ \sigma_{11}{}^2(a) = \frac{D_2 a^2}{4w_0^2} \\ m_2(a) = 0 \\ \sigma_{21}{}^2(a) = \frac{3D_2}{4w_0^2} \end{gathered} \tag{15}$$

Equations (14) and (15) show that $\mathrm{d}a$ does not depend on θ, the averaged Ito equation of $a(t)$ is independent of $\theta(t)$ and that the random process $a(t)$ is a 1-D diffusion process.

Thus, the correspondingly reduced Fokker-Planck-Kolmogorov (FPK) equation of $a(t)$ can be written:

$$0 = -\frac{\partial}{\partial a}[m_1(a)p(a)] + \frac{1}{2}\frac{\partial^2}{\partial a^2}\{[\sigma_{11}{}^2(a)]p(a)\} \tag{16}$$

The boundary conditions are:

$$\begin{aligned} &p(a) = c, \quad c \in (-\infty, +\infty) \quad \text{as } a = 0 \\ &p(a) \to 0, \quad \frac{\partial p}{\partial a} \to 0 \quad \text{as } a \to \infty \end{aligned} \tag{17}$$

Based on the boundary conditions given in eq. (17), the amplitude's stationary PDF can be obtained:

$$p(a) = \frac{C}{\sigma_{11}{}^2(a)}\exp\left\{\int_0^a \frac{2m_1(u)}{[\sigma_{11}{}^2(u)]}\mathrm{d}u\right\} \tag{18}$$

where C is the normalizing constant that satisfies:

$$C=\left[\int_0^\infty\left(\frac{1}{\sigma_{11}{}^2(a)}\exp\left\{\int_0^a\frac{2m_1(u)}{[\sigma_{11}{}^2(u)]}\mathrm{d}u\right\}\right)\mathrm{d}a\right]^{-1} \tag{19}$$

Substituting eq. (15) into eq. (18), we get the explicit expression of stationary PDF of the system amplitude a:

$$p(a)=\frac{4Cw_0^2}{D}a^{-\frac{4[\varepsilon+w^{p-1}\sin(p\pi/2)]w_0^2-D}{D}}\exp\left[-\frac{a^2w_0^2(-48\alpha_1+12\alpha_2a^2)}{96D_2}\right] \tag{20}$$

Further, we can obtain the stationary PDF of system energy as following [25]:

$$p(E)=\frac{cw_0^2}{3D}E^{-\frac{2}{3}\frac{(\varepsilon w_0^2+\beta w^{p-1}\sin\frac{p\pi}{2}w_0^2+2D)}{D}}\exp\left[-\frac{E(-2w_0^2\alpha_1+E\alpha_2)}{6Dw_0^2}\right] \tag{21}$$

Stochastic *P*-bifurcation of the system energy

Stochastic *P*-bifurcation means that the changes in a number of the stationary PDF curve's peaks. To obtain the critical parametric conditions for stochastic *P*-bifurcation, we analyze the influences of parameters on the system's stochastic *P*-bifurcation by using the singularity theory in this section.

For the sake of convenience, $p(E)$ is expressed by:

$$p(a)=R(E,D,\varepsilon,w,p,\alpha_1,\alpha_2)\exp[Q(E,D,\varepsilon,w,p,\alpha_1,\alpha_2)] \tag{22}$$

in which:

$$\begin{aligned}R_1(\mathrm{E},D,\varepsilon,w,p,\alpha_1,\alpha_2,\alpha_3,\alpha_4)&=\frac{cw_0^2}{3D}E^{-\frac{2}{3}\frac{(\varepsilon w_0^2+\beta w^{p-1}\sin\frac{p\pi}{2}w_0^2+2D)}{D}}\\ Q_1(\mathrm{E},D,\varepsilon,w,p,\alpha_1,\alpha_2,\alpha_3,\alpha_4)&=-\frac{E(-2w_0^2\alpha_1+E\alpha_2)}{6Dw_0^2}\end{aligned} \tag{23}$$

Based on the singularity theory [26], the stationary PDF of the system amplitude needs to satisfy:

$$\frac{\partial p(E)}{\partial E}=0,\quad \frac{\partial^2p(E)}{\partial E^2}=0 \tag{24}$$

Substituting eq. (22) into eq. (24), we obtain:

$$H=\left\{R_1'+R_1Q_1'=0,\ R_1''+2R_1'Q_1'+R_1Q_1''+R_1Q_1'^2=0\right\} \tag{25}$$

where H is the critical condition for the changes in a number of the PDF curve's peaks.

The influence of the fractional-order, p, and the noise intensity, D, on the system energy

In this part, the influences of p and D on the system are investigated. The parameters $\alpha_1 = 1.51$, $\alpha_2 = 2.85$, $w = 1$, and $\varepsilon = -0.8$ are chosen as examples for illustration. According to eqs. (23) and (25), we obtain the transition set for stochastic P-bifurcation of the system energy with the unfolding parameters p and D shown in fig. 1.

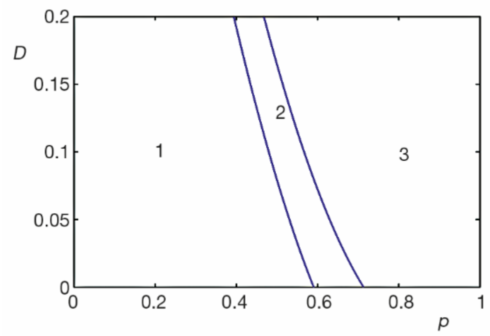

Figure 1. Transition set curves (taking *p* and *D* as unfolding parameters)

Based on the singularity theory, the topological structures of the stationary PDF curves of different points (p, D) in the same area are qualitatively identical. By taking a point (p, D) in each area, we can obtain all varieties of the system's stationary PDF curves that are qualitatively different. The unfolding parameter p–D plane is divided into three sub-areas by the transition set curve. For the sake of convenience, each area in fig. 1 is marked with a number.

We first analyze the stationary PDF of system energy $p(E)$ for a point (p, D) in each of the three sub-areas of fig. 1, and then compare the analytical solutions with the numerical results obtained by Monte-Carlo simulation from the original system (3) using the numerical method for fractional derivative [20]. The corresponding results are shown in fig. 2.

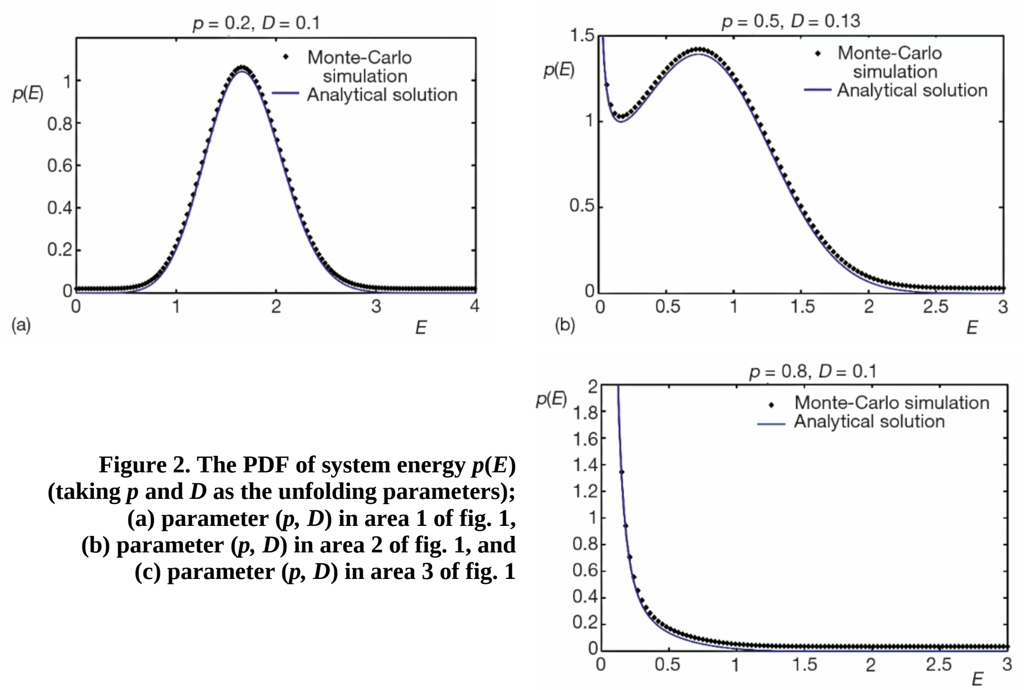

Figure 2. The PDF of system energy *p*(*E*) (taking *p* and *D* as the unfolding parameters); (a) parameter (*p, D*) in area 1 of fig. 1, (b) parameter (*p, D*) in area 2 of fig. 1, and (c) parameter (*p, D*) in area 3 of fig. 1

From fig. 1 we can see that the parameter area where the PDF occurs bimodal is surrounded by two curves. When the parameter (p, D) is taken as $p = 0.2, D = 0.1$ in area 1, fig. 2(a), the PDF $p(E)$ has a stable limit cycle. When the parameter (p, D) is taken as

$p=0.5, D=0.13$ in area 2, fig. 2(b), the PDF $p(E)$ still has a stable limit cycle, but the probability is not zero near the origin. The system has a response whose amplitude is approximately zero, just like in the deterministic system at the moment. In addition, there are both the limit cycle and equilibrium in the system simultaneously. When the parameter (p, D) is taken as $p=0.8, D=0.1$ in area 3, the PDF $p(E)$ appears in the form of the Dirac function, and the system's steady-state response amplitude is constant at 0, similar to the stable equilibrium in the deterministic system at this time. The randomness of the system is suppressed.

Apparently, the stationary PDF $p(E)$ in any two adjacent areas in fig. 1 are very qualitatively different. Regardless of the exact values of the unfolding parameters, if they cross any line in these figures, the system will demonstrate stochastic *P*-bifurcation behavior. Therefore, the transition set curves are just the critical parametric conditions of the system's stochastic *P*-bifurcation. The analytic results shown in fig. 2 are well consistent with those numerical results obtained by Monte-Carlo simulation from the original system (3), further verifying the theoretical analysis and showing that it is feasible to use the methods in this paper to analyze the stochastic *P*-bifurcation behavior of fractional-order systems.

Compared with the integral-order controllers [27-29], the fractional-order controllers have better dynamic performances and robustness [20]. In the past several years, various fractional-order controllers have been developed [30-34]. In the above analysis, we obtained the areas where the stochastic *P*-bifurcation occurs in system (3), which can make the system switch between monostable and bistable states by selecting the corresponding unfolding parameters. This could provide theoretical guidance for the analysis and design of the fractional-order controllers, and our results can extend to fractal oscillators with two-scale fractal derivatives [35-41].

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