STOCHASTIC TRANSITION BEHAVIORS IN A TRI-STABLE VAN der POL OSCILLATOR WITH FRACTIONAL DELAYED ELEMENT SUBJECT TO GAUSSIAN WHITE NOISE

by

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Original scientific paper
https://doi.org/10.2298/TSCI2203713L

The stochastic P-bifurcation behavior of tri-stability in a generalized Van der Pol system with fractional derivative under additive Gaussian white noise excitation is investigated. Firstly, based on the minimal mean square error principle, the fractional derivative is found to be equivalent to a linear combination of damping and restoring forces, and the original system is simplified into an equivalent integer order system. Secondly, the stationary probability density function of the system amplitude is obtained by stochastic averaging, and according to the singularity theory, the critical parameters for stochastic P-bifurcation of the system are found. Finally, the nature of stationary probability density function curves of the system amplitude is qualitatively analyzed by choosing the corresponding parameters in each region divided by the transition set curves. The consistency between the analytical solutions and Monte-Carlo simulation results verifies the theoretical results in this paper.

Key words: *stochastic P-bifurcation, fractional derivative, gaussian white noises, transition set curves, Monte-Carlo simulation*

Introduction

Due to the limitation of the definition of integer-order derivatives, the classical integer operators cannot express memory properties and do not have sufficient parameters to handle the different shapes of the hysteresis loops describing the behaviors of viscoelastic materials and structures. While fractional derivatives contain convolution, which can describe a memory effect and express a cumulative effect over time; hence they are more suitable to describe memory characteristics [1-4] and have become a powerful mathematical tool to study fields such as anomalous diffusion, non-Newtonian fluid mechanics, viscoelastic mechanics, and soft matter physics. Compared with integer-order calculus, the fractional derivative can

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describe various reaction processes more accurately with fewer parameters, [5-11]. Thus, it is necessary and significant to investigate the fractional differential equations on the typical mechanical properties and the influences of fractional order parameters on the system.

Recently, many scholars have studied the dynamic behavior of non-linear multi-stable systems under different noise excitations and achieved fruitful results. Li and Huang [12] investigated the mean first-passage time of a delayed tumor cell growth system driven by colored cross-correlated noises and then thoroughly discussed the effects of different kinds of delays and noise parameters on the mean first passage time. Wang *et al.* [13] established the governing equations for the non-linear transverse vibration of an axially moving viscoelastic beam with finite deformation using the Hamiltonian principle and produced nanoscale crimped fibers using stuffer box crimping and bubble electrospinning. Li *et al.* [14] solved a paradox in an electrochemical sensor by a fractal modification of the surface coverage model and elucidated a simple solution process to the fractal model. He *et al.* [15] pointed out the so-called enhanced variational iteration method for a non-linear equation arising in electrospinning and the vibration-electrospinning process is the standard variational iteration method and an effective algorithm using the variational iteration algorithm-II is suggested for Bratu-like equation arising in electrospinning. Anjum and He [16] suggested an easier approach by the Laplace transform to determining the Lagrange multiplier for the variational iteration method, making the method accessible to researchers facing various non-linear problems, and adopted a non-linear oscillator as an example to elucidate the identification process and the solution process. Yu *et al.* [17] improved the homotopy perturbation method by constructing a homotopy equation with one or more auxiliary parameters embedding in the linear term with a clear advantage in accelerating and controlling the approximation convergence speed. Wu and He [18] elucidated that the homotopy perturbation method is valid for non-linear oscillators with negative linear terms, and conditions for the periodic solutions can be easily obtained. Wang and An [19] adopted He's fractional derivative which is defined through a variational iteration algorithm to describe a non-linear vibration in microphysics, and used He's amplitude-frequency formulation to solve the fractional Duffing equation. Wang and Wang [20] modified the reduced differential transform method for obtaining the approximate analytical solutions of the fractional heat transfer equations. Wang and Liu [21] suggested a modification of the reduced differential transform method and a new iterative Elzaki transform method and then applied them to obtain the analytical solutions of the time-fractional Navier-Stokes equations. In addition, some researchers investigated the Van der Pol-Duffing oscillators under the Lévy noise, colored noise, combined harmonic and random excitations, respectively. Moreover, the stochastic P-bifurcation behaviors of the noise oscillators were discussed by analyzing changes in the stationary PDF of the systems [22-24]. Hao and Wu [25-27] investigated the stochastic *P*-bifurcation of tri-stability in a generalized Duffing-Van der Pol oscillator system excited by additive Gaussian white noise, multiplicative colored noise, combined additive, and multiplicative Gaussian white noises, respectively, obtained an analytical expression of the stationary probability density function (PDF) of system amplitude, and analyzed the influences of noise intensity and system parameters on stochastic *P*-bifurcation of the system. Chen and Zhu [28] studied the response of a Duffing system with fractional damping under combined white noise and harmonic excitations and showed that variation in the order of fractional derivative could cause stochastic *P*-bifurcation of the system. Li *et al.* [29] investigated the stochastic *P*-bifurcation behavior of a bistable Van der Pol-Duffing oscillator with fractional derivative excited by additive and multiplicative Gaussian colored noise excitations and found that changes in the linear damping coefficient, the order of fractional derivative, and the noise intensity can each lead to stochastic *P*-bifurcation of the system. Liu *et al.* [30] investigated the stochastic stability of a Duffing oscillator with fractional derivative damping under combined harmonic and Poisson white noise parametric excitations and analyzed the asymptotic Lyapunov stability with probability one of the original system by using the largest Lyapunov exponent.

For the dynamics of time-delay systems, Liu *et al.* [31] studied a two-degree-of-freedom non-linear vibration of a quarter vehicle suspension system by using a feedback control method considered the fractional-order derivative damping and obtained the asymptotic stability conditions of the non-linear system by using the Routh-Hurwitz criterion. Leung *et al.* [32] investigated two Duffing-Van der Pol oscillators that have fractional damping and time delay, found periodic solutions using the residue harmonic balance method, and then accurately captured the limit cycle frequency. Chen *et al.* [33] studied the primary resonance response of a Van der Pol system under fractional-order delayed negative feedback and forced excitation and obtained an approximate analytical solution based on the averaging method. Leung *et al.* [34] investigated a Van der Pol-Duffing oscillator with both fractional derivative and time delay according to the harmonic residue method, then examined the periodic bifurcations using the order of fractional derivative, time delay, and feedback gain as the continuation parameters. Chen *et al.* [35] proposed a stochastic averaging technique, which can be used to study randomly excited strongly non-linear systems with a delayed feedback fractional-order proportional-derivative controller, and obtained stationary PDF of the system. Wen *et al.* [36] studied the deterministic and autonomous Duffing systems with fractional time-delay coupled feedback and found that fractional time-delay coupled feedback plays the roles of both velocity time-delay feedback and displacement time-delay feedback. Jiang *et al.* [37] considered a classical Van der Pol oscillator with general time-delay feedback and found that there are the Bogdanov-Takens bifurcation, triple-zero, and Hopf-zero singularities in the system by analyzing the distribution of the associated characteristic roots.

Because of the complexity of fractional derivatives, analyzing them is difficult, and the influences of system parameters on vibration characteristics are mostly studied numerically, which are usually limited to qualitative analysis. It is difficult to find the critical condition of parametric influence, which affects the analysis and design of such systems, in part because the bistable stochastic *P*-bifurcation of the fractional delayed feedback system has not been reported. Accordingly, we take the non-linear vibration of a generalized Van der Pol oscillator excited by both additive and multiplicative Gaussian white noise excitations simultaneously as an example and obtain the critical parametric conditions for stochastic *P*-bifurcation using the singularity method. Furthermore, we compare the Monte-Carlo simulation results with the analytical solutions obtained by stochastic averaging. Their consistency verifies the theoretical analysis in this paper.

Derivation of the equivalent system

The Riemann-Liouville derivative, Caputo derivative, and two-scale fractal derivative [38-40] are most commonly used. The initial condition corresponding to Riemann-Liouville derivative has no physical meaning. However, the initial condition of the system described by Caputo derivative has both clear physical meaning and forms the same as in the integer-order differential equation. In this paper, we adopt the Caputo derivative.

For a given physical system, because the moment when the oscillator begins to vibrate is always $t = 0$, and the Caputo derivative is often used in the following form:

$$ {}_0^C\mathrm{D}^p[x(t)] = \frac{1}{\Gamma(m-p)}\int_0^t \frac{x^{(m)}(u)}{(t-u)^{1+p-m}}\mathrm{d}u \tag{1} $$

where $m-1 < p \leq m, m \in N$.

In this paper, we study the generalized Van der Pol oscillator system with fractional order time-delay coupled feedback driven by Gaussian white noise excitations:

$$ \ddot{x}(t) - [-\varepsilon + \alpha_1 x^2(t) - \alpha_2 x^4(t) + \alpha_3 x^6(t) - \alpha_4 x^8(t)]\dot{x}(t) + w^2 x(t) + {}_0^C D^p[x(t-\tau)] = \xi(t) \tag{2} $$

where ε represents the linear damping coefficient, α_1, α_2, α_3, and α_4 – the non-linear damping coefficients of the system, w – the natural frequency, and τ – the time-delay introduced in the system. The ${}_0^C\mathrm{D}^p[x(t-\tau)]$ is the p ($0 \leq p \leq 1$) order Caputo derivative of $x(t-\tau)$ with respect to *t*, which is defined by eq. (4). The $\xi(t)$ is Gaussian white noises, which satisfy:

$$ E[\xi(t)] = 0, \quad E[\xi(t)\xi(t-s)] = 2D\delta(s) \tag{3} $$

where D denotes the intensity of Gaussian white noises $\xi(t)$, respectively, and $\delta(s)$ is the Dirac function.

The fractional derivative has the contributions of the damping force and restoring force, [41-44]. Hence, we introduce the equivalent system:

$$ \ddot{x}(t) - [-\varepsilon + \alpha_1 x^2(t) - \alpha_2 x^4(t) + \alpha_3 x^6(t) - \alpha_4 x^8(t) + C(p,\tau)]\dot{x}(t) + [K(p,\tau) + w^2]x(t) = \xi(t) \tag{4} $$

where $C(p,\tau)$ and $K(p,\tau)$ are coefficients of the equivalent damping and restoring forces of fractional derivative ${}_0^C\mathrm{D}^p[x(t-\tau)]$, respectively.

Applying the equivalent methods mentioned in, [28, 43, 44] the concrete forms of $C(p,\tau)$ and $K(p,\tau)$ are:

$$ C(p,\tau) = -w^{p-1}\sin\frac{p\pi}{2-w\tau}, \quad K(p,\tau) = w^p\cos\frac{p\pi}{2-w\tau} \tag{5} $$

Therefore, the equivalent Van der Pol oscillator associated with the system (5) can be rewritten as:

$$ \ddot{x}(t) - \gamma\dot{x}(t) + w_0^2 x(t) = \xi(t) \tag{6} $$

where

$$ \gamma = -\varepsilon + \alpha_1 x^2 - \alpha_2 x^4 + \alpha_3 x^6 - \alpha_4 x^8 - w^{p-1}\sin\frac{p\pi}{2-w\tau} $$

$$ w_0^2 = w^2 + w^p\cos\frac{p\pi}{2-w\tau} \tag{7} $$

The stationary PDF of system amplitude

In our first example, we examine the system in eq. (6), with linear and non-linear damping coefficients $\varepsilon = 0.2$, $\alpha_1 = 1.51$, $\alpha_2 = 2.85$, $\alpha_3 = 1.693$, $\alpha_4 = 0.312$, natural frequency $w = 1$, and time-delay $\tau = 0.5$. For convenience in discussing parametric influence, the bifurcation diagram of system amplitude with the variation of fractional order *p* is shown in fig. 1 when $D_1 = D_2 = 0$.

As can be seen from fig. 1, there are two attractors when p changes in [0.19, 0.264): equilibrium and the large limit cycle. There are three attractors when p changes in [0.264, 0.267): equilibrium, the small limit cycle, and the large limit cycle. There are also two attractors when p changes in [0.267, 0.278): equilibrium and the small limit cycle.

For the system with linear and non-linear damping coefficients $\varepsilon = 0.2$, $\alpha_1 = 1.51$, $\alpha_2 = 2.85$, $\alpha_3 = -1.693$, $\alpha_4 = 0.312$, natural frequency $w = 1$, and the fractional-order $p = 0.2$, the bifurcation diagram of system amplitude with the variation of time-delay τ is shown in fig. 2 when $D_1 = D_2 = 0$.

As can be seen from fig. 2, it shows that there are two attractors where $0.377 \le p < 0.395$: the small limit cycle and equilibrium. There exists three attractors where $0.395 \le p < 0.40$: equilibrium, the small and the large limit cycles. There are also two attractors where $0.40 \le p < 0.516$: equilibrium and the large limit cycle.

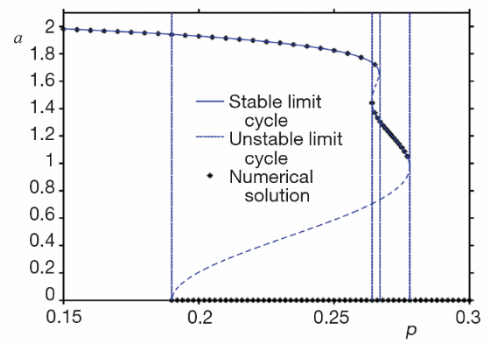

Figure 1. Bifurcation diagram of the deterministic system (with variation in *p*)

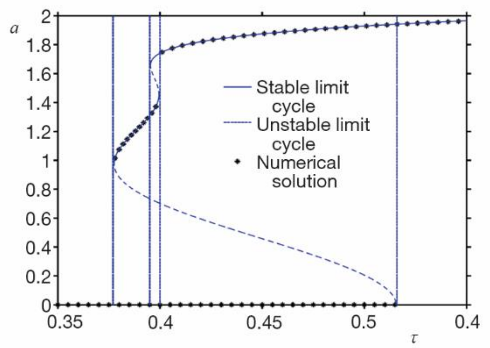

Figure 2. Bifurcation diagram of the deterministic system (with variation in τ)

Assuming that the solution of system (6) has the periodic form, and we introduce the following transformation [45]:

$$
\begin{aligned}
X &= x(t) = a(t)\cos\Phi(t) \\
Y &= \dot{x}(t) = -a(t)w_0 \sin\Phi(t) \\
\Phi(t) &= w_0 t + \theta(t)
\end{aligned}
\tag{8}
$$

where w_0 is the natural frequency of the equivalent system (6), $a(t)$ and $\theta(t)$ – the amplitude and phase processes of system response, respectively, and they are both random processes.

Substituting eq. (8) into eq. (6), we can obtain:

$$
\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= F_{11}(a,\theta) + \mathrm{G}_{11}(a,\theta)\xi(t) \\
\frac{\mathrm{d}\theta}{\mathrm{d}t} &= F_{21}(a,\theta) + G_{21}(a,\theta)\xi(t)
\end{aligned}
\tag{9}
$$

where

$$F_{11}(a,\theta)=a\sin^2\Phi\left[-\varepsilon+\alpha_1 a^2\cos^2\Phi-\alpha_2 a^4\cos^4\Phi+\alpha_3 a^6\cos^6\Phi-w^{p-1}\sin\left(\frac{p\pi}{2}-w\tau\right)\right]$$

$$F_{21}(a,\theta)=\sin\Phi\cos\Phi\left[-\varepsilon+\alpha_1 a^2\cos^2\Phi-\alpha_2 a^4\cos^4\Phi+\alpha_3 a^6\cos^6\Phi-w^{p-1}\sin\left(\frac{p\pi}{2}-w\tau\right)\right]$$

$$G_{11}=-\frac{\sin\Phi}{w_0}$$

$$G_{21}=-\frac{\cos\Phi}{aw_0} \tag{10}$$

Equation (9) can be treated as the Stratonovich differential equation, and by adding the relevant correction term, we can transform it into the corresponding Ito stochastic differential equation:

$$\mathrm{d}a=[F_{11}(a,\theta)+F_{12}(a,\theta)]\mathrm{d}t+\sqrt{2D}\,\mathrm{G}_{11}(a,\theta)\mathrm{d}B(t)$$

$$\mathrm{d}\theta=[F_{21}(a,\theta)+F_{22}(a,\theta)]\mathrm{d}t+\sqrt{2D}G_{21}(a,\theta)\mathrm{d}B(t) \tag{11}$$

where $B(t)$ is the normalized Wiener processes, in addition:

$$F_{12}(a,\theta)=D\frac{\partial G_{11}}{\partial a}G_{11}+D\frac{\partial G_{11}}{\partial\theta}G_{21}$$

$$F_{22}(a,\theta)=D\frac{\partial G_{21}}{\partial a}G_{11}+D\frac{\partial G_{21}}{\partial\theta}G_{21} \tag{12}$$

By stochastic averaging of averaging eq. (11) over Φ, [46] we can obtain the following averaged Ito equations:

$$\mathrm{d}a=m_1(a)\mathrm{d}t+\sigma_{11}(a)\mathrm{d}B(t)$$

$$\mathrm{d}\theta=m_2(a)\mathrm{d}t+\sigma_{21}(a)\mathrm{d}B(t) \tag{13}$$

where $B(t)$ is a unit Wiener process and:

$$m_1(a)=-\frac{1}{2}\left(w^{p-1}\sin\frac{p\pi}{2-w\tau}+\varepsilon\right)a+\frac{1}{8}\alpha_1 a^3-\frac{1}{16}\alpha_2 a^5+\frac{5}{128}\alpha_3 a^7+\frac{D_1}{2aw_0^2}+\frac{3D_2 a}{8w_0^2}$$

$$\sigma_{11}{}^2(a)=\frac{D}{w_0^2}$$

$$m_2(a)=0$$

$$\sigma_{21}{}^2(a)=\frac{D}{a^2w_0^2} \tag{14}$$

Equations (13) and (14) show that the averaged Ito equation of $a(t)$ is independent of $\theta(t)$, and the process $a(t)$ is actually a 1-D diffusion process. Then the reduced Fokker--Planck-Kolmogorov (FPK) equation of $a(t)$ can be written:

$$0=-\frac{\partial}{\partial a}[m_1(a)p(a)]+\frac{1}{2}\frac{\partial^2}{\partial a^2}\{[\sigma_{11}{}^2(a)]p(a)\} \tag{15}$$

The boundary conditions satisfy:

$$p(a)=c,\quad c\in(-\infty,+\infty),\quad \text{as}\, a=0$$

$$p(a)\to 0,\quad \frac{\partial p(a)}{\partial a\to 0},\quad \text{as}\, a\to\infty \tag{16}$$

Based on the boundary conditions (16), the stationary PDF of system amplitude a can be obtained:

$$p(a)=\frac{C}{\sigma_{11}{}^2(a)}\exp\left[\int_0^a\frac{2m_1(u)}{(\sigma_{11}{}^2(u)}\mathrm{d}u\right] \tag{17}$$

where C is the normalization constant that satisfies:

$$C=\left(\int_0^\infty\left\{\frac{1}{\sigma_{11}{}^2(a)}\exp\left[\int_0^a\frac{2m_1(u)}{(\sigma_{11}{}^2(u)}\mathrm{d}u\right]\right\}\mathrm{d}a\right)^{-1} \tag{18}$$

Substituting eq. (14) into eq. (17), the explicit expression of stationary PDF of system amplitude, a, can be described:

$$p(a)=\frac{Caw_0^2}{D}\exp\left(-\frac{a^2w_0^2\Delta}{7680D}\right) \tag{19}$$

where

$$\Delta=3840\varepsilon+3840w^{p-1}\sin\frac{p\pi}{2-w\tau}-480\alpha_1a^2+160\alpha_2a^4-75\alpha_3a^6+42\alpha_4a^8 \tag{20}$$

Stochastic *P*-bifurcation of system amplitude

Stochastic *P*-bifurcation means that the changes in the number of peaks in the PDF curve. To obtain the critical parametric conditions for stochastic *P*-bifurcation, we analyze the parametric influences on stochastic *P*-bifurcation of the system by using the singularity theory in this section.

For convenience, $p(a)$ can be expressed:

$$p(a)=CR(a,D_1,\varepsilon,w,p,\alpha_1,\alpha_2,\alpha_3,\alpha_4)\exp[Q(a,D_1,\varepsilon,w,p,\alpha_1,\alpha_2,\alpha_3,\alpha_4)] \tag{21}$$

where

$$R(a, D_1, \varepsilon, w, p, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \frac{aw_0^2}{D}$$

$$Q(a, D_1, \varepsilon, w, p, \alpha_1, \alpha_2, \alpha_3, \alpha_4) =$$

$$= -\frac{a^2 w_0^2}{7680D}\left(3840\varepsilon + 3840w^{p-1}\sin\frac{p\pi}{2-w\tau} - 480\alpha_1 a^2 + 160\alpha_2 a^4 - 75\alpha_3 a^6 + 42\alpha_4 a^8\right) \quad (22)$$

Based on the singularity theory, [47] the stationary PDF of system amplitude needs to meet the following two conditions:

$$\frac{\partial p(a)}{\partial a} = 0, \quad \frac{\partial^2 p(a)}{\partial a^2} = 0 \quad (23)$$

Substituting eq. (21) into eq. (23), we can obtain the following condition [26, 29]:

$$H = \left\{R' + RQ' = 0, \quad R'' + 2R'Q' + RQ'' + RQ'^2 = 0\right\} \quad (24)$$

where H is the condition for the changes in the number of peaks in the PDF curve.

Since the relationship of the 3-D surface is not easy to describe and display, here, we only give the 2-D section of the transition set to represent the influences of the fractional-order p, the time delay τ, and the noise intensities D below.

Substituting eq. (22) into eq. (24), we can get the critical parametric conditions for stochastic P-bifurcation of the system with respect to fractional order p and noise intensity D:

$$D = \frac{1}{4}\alpha_1 w^2{}_0 a^4 - \frac{1}{4}\alpha_2 w^2{}_0 a^6 + \frac{15}{64}\alpha_3 w^2{}_0 a^8 - \frac{7}{32}\alpha_4 w^2{}_0 a^{10} \quad (25)$$

where amplitude a satisfies:

$$128\varepsilon - 64\alpha_1 a^2 + 48\alpha_2 a^4 - 40\alpha_3 a^6 + 35\alpha_4 a^8 + 128w^{p-1}\sin\frac{p\pi}{2-w\tau} = 0 \quad (26)$$

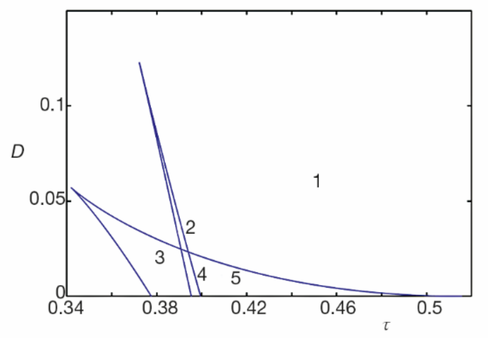

Figure 3. Transition set curves of the system (5) (taking *p* and *D* as the unfolding parameters)

The influence of the fractional-order, τ, and the noise intensity, D, on the system

With parameters ε = 0,2 α_1 = 1.51, α_2 = 2.85, α_3 = 1.693, α_4 = 0.312, w = 1, p = 0.2, and calculate the corresponding transition sets according to eqs. (28) and (29), we obtain the critical conditions for stochastic P-bifurcation of the system with unfolding parameters p and D shown in fig. 3.

Based on singularity theory, types of the stationary PDF curves of system amplitude at different points (τ, D) in the same region are qualitatively identical. By taking one point (τ, D) in each region, we can obtain all varieties of stationary PDF curves that are qualitatively different. The unfolding parametric plane $\tau - D$ is divided into five sub-regions by the transition set curve. For the sake of convenience, each region in fig. 3 is marked with a number.

Without loss of generality, we analyze the stationary PDF of amplitude $p(a)$ only for one point (τ, D) in each of the five sub-regions in fig. 3, and then compare the analytical solutions with the Monte Carlo simulation results from the original system (5) using the numerical method for fractional derivative, [36, 48] and show the corresponding results in fig. 4.

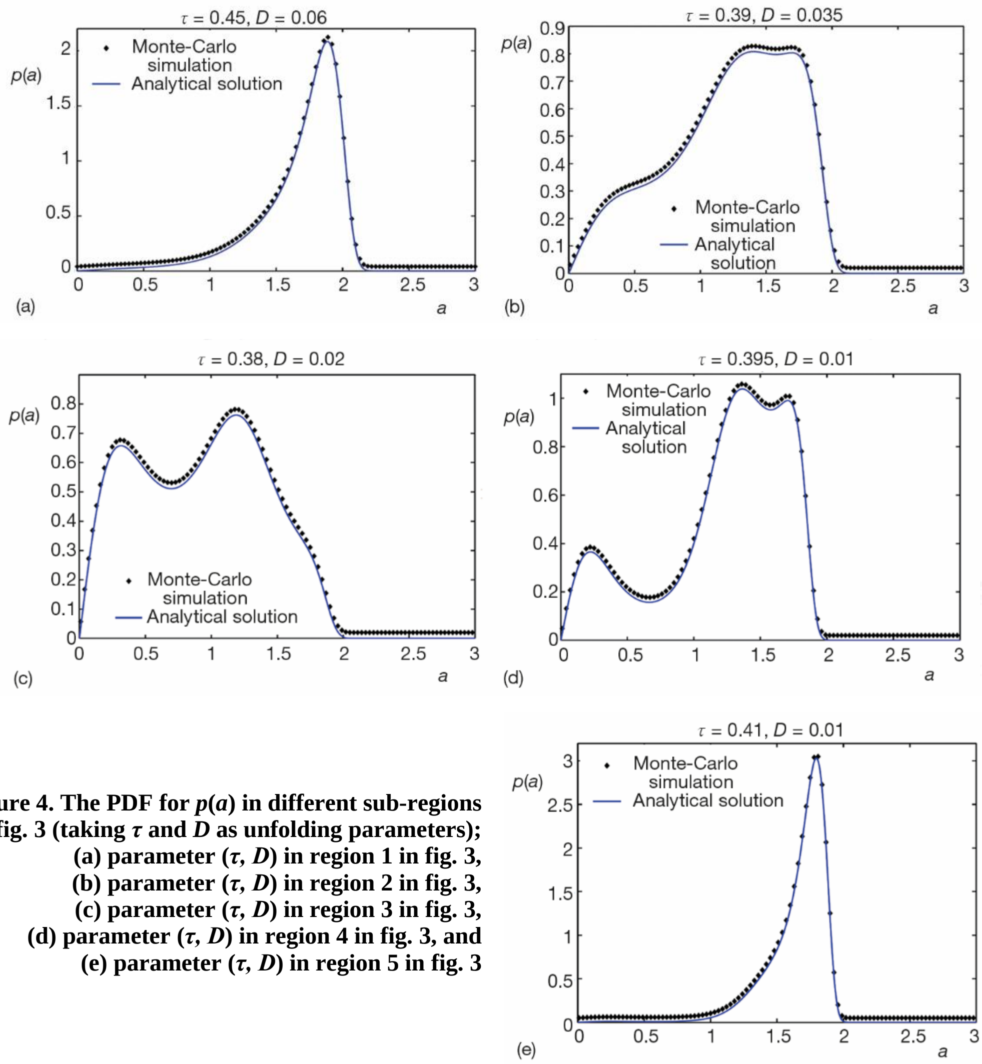

Figure 4. The PDF for $p(a)$ in different sub-regions in fig. 3 (taking τ and D as unfolding parameters); (a) parameter (τ, D) in region 1 in fig. 3, (b) parameter (τ, D) in region 2 in fig. 3, (c) parameter (τ, D) in region 3 in fig. 3, (d) parameter (τ, D) in region 4 in fig. 3, and (e) parameter (τ, D) in region 5 in fig. 3

As can be seen from fig. 3, the parameter region where the PDF curve appears multimodal is surrounded by two approximately triangular regions. Particularly, region 4, where the two triangular regions are coincident, forms a tri-modal region of the stationary PDF curve for the system. When the parameter (τ, D) is taken in region 1, the PDF curve has a distinct peak far away from the origin, as shown in fig. 4(a). In region 2, the PDF curve has two distinct peaks farther away from the origin, there are both small and large limit cycles in the system, as shown in fig. 4(b). In region 3, the PDF curve still has a distinct peak farther away from the origin, but the probability is obviously not zero near the origin, there are both the equilibrium and large limit cycle in the system simultaneously, as shown in fig. 4(c). In region 4, the PDF curve has three peaks, it shows that the equilibrium coexists with the small and large limit cycles in the system which is tri-stable, as shown in fig. 4(d). In region 5, the amplitude corresponding to the peak deviating from the origin of the PDF curve is smaller than the corresponding amplitude in fig. 4(c), there exists both the equilibrium and small limit cycle in the system simultaneously, as shown in fig. 4(e).

According to all of the mentioned, the results show that the stationary PDF for $p(a)$ in any two adjacent regions in fig. 3 are qualitatively different. No matter the exact values of the unfolding parameters that cross any curve in these figures, the system will occur stochastic P-bifurcation behavior. Thus, the transition set curves obtained are just the critical parametric conditions for stochastic P-bifurcation of the system. The analytic solutions shown in figure 4 are well consistent with the Monte-Carlo simulation results from the original system (5), thus further verifying the theoretical analysis in this paper.

Compared with integer-order controllers [49-51], the fractional-order controllers have better dynamic performances and robustness, and recently, various fractional-order controllers have been developed [52-57]. And we obtained the critical conditions when the system (5) will exhibit stochastic P-bifurcation through the above analysis, which can make the system switch between mono-stable and multi-stable states by selecting the corresponding unfolding parameters, this can provide theoretical guidance for the design of fractional-order controllers.

Conclusion

In this paper, we studied the bistable stochastic P-bifurcation of a generalized Van der Pol system with fractional time-delay feedback excited by additive and multiplicative Gaussian white noise excitations simultaneously and discussed the influences of parameters p, τ, and D on the system. Based on the minimum mean square error principle, the original system was transformed into an equivalent integer-order system, and we obtained the stationary PDF of system amplitude using the stochastic averaging method. Then, the critical parametric conditions for stochastic P-bifurcation of the system were obtained based on singularity theory, according to which we can maintain the system response at a small amplitude near the equilibrium or monostability by selecting the corresponding unfolding parameters, which can provide the theoretical guidance for the design of such systems and avoid the damage and instability caused by the non-linear jump or large amplitude vibration of the system. The consistency between the Monte-Carlo simulation results and the analytical solutions can also verify the theoretical analysis. It shows that the fractional-order, p, time-delay, τ, and noise intensities, D, can each arouse stochastic P-bifurcation of the system, and the number of peaks in the stationary PDF curves of system amplitude can be controlled from three to one by selecting the corresponding unfolding parameters. It also illustrates that the method used in this paper is feasible to analyze the stochastic P-bifurcation behaviors of non-linear oscillators with fractional derivatives.

Acknowledgment

This work in this paper was supported by the National Basic Research Program of China (Grant No. 2014CB046805), the National Natural Science Foundation of China (Grant

No. 11372211, 11672349, 11502162, 61503122 and 12002120), the Foundation of Henan Department of Science and Technology (Grant No. 162300410087), the Key Research and Development and Promotion of Special (Science and Technology) Project of Henan Province under Grant 202102210142, the Foundation of Henan Education Committee under Grant 18A120008 and the Aeronautical Science Foundation of China (Grant No. ASFC-201915048001). The authors would like to thank the reviewers and editors for their conscientious reading and comments which were extremely helpful and useful for improving the article.

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Paper submitted: March 8, 2020
Paper revised: October 1, 2021
Paper accepted: October 1, 2021