# APPROXIMATE ANALYTIC SOLUTION OF THE FRACTAL FISHER'S EQUATION VIA LOCAL FRACTIONAL VARIATIONAL ITERATION METHOD

by

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*The local fractional variational iteration method is applied to a modified Fisher's equation defined on Cantor sets with the fractal conditions. The solution process is simple, and the accuracy of the approximate solution is high. The method provides an unrivaled tool for local differential equations.*

Key word: *fractal Fisher's equation, approximate analytical solutions, local fractional variational iteration method, local fractional derivative*

## Introduction

The Fisher equation is a 1-D non-linear parabolic PDE proposed by Fisher in 1937 [1]. This is a reaction-diffusion equation, which is used to test the fluctuation proliferation of beneficial quality genes in a population. It can also be called the dynamic dominance rate of a favorable gene to show the reproduction of virus mutants in infinite habitats. It is one of the simplest reaction-diffusion equations. Therefore, the study of this kind of PDE has become a related research field. The Fisher equation is now widely used in various biological and chemical processes and engineering, for example, gene propagation [1-3], combustion [4], the autocatalytic chemical reaction [5], and tissue engineering [6]. Rehman *et al.* [7] used a numerical technique called the variational iteration method to solve the Fisher's equation with great success, showing the variational iteration method is an incomparable technique for non-linear equations; the method was first proposed by Ji-Huan He in the 1990's [8, 9]. Maha [10] also employed the variational iteration method to the fractional Fisher equation and fractional Navier-Stokes equation, showing the method is also a powerful and effective tool to fractional differential equations, and now the method becomes a matured tool for various non-linear problems, for example, He [11] used the variational iteration method to study the stability of a heat conduction equation. Nadeem and He [12] coupled the variational iteration method with the Laplace transform to solve population dynamics. He, *et al.* [13] obtained an analytical solution for the MEMS oscillator. The variational iteration method is extremely effective for fractional calculus [14-20].

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The local fractional calculus [21] can be used to describe non-differentiable problems, and the local fractional variational iteration method (LFVIM) [14] is an extension of the variational iteration method [7, 8] to the local fractional differential equations.

Using the fractional complex transform [22-25], we can transform the fractional Fisher's equation into its local fractional partner on Cantor sets. The fractional complex transform can be physically explained by the two-scale fractal theory [26-28].

## Mathematics tools

In this section, we recall and review briefly basic definitions of local fractional derivatives (LFD) and local fractional integral [21]. Some basic operations of LFD on fractal space are presented as follows [21].

*Definition 1.* Assume the relation below exists:

$$\left|\psi(x)-\psi(x_0)\right|<\varepsilon^{\alpha} \tag{1}$$

with $|x-x_0|<\delta$ for $\varepsilon,\delta>0.$ Then $\psi(x)$ is local fractional continuous at $x_0$ which is denoted by $\lim_{x\to x_0}\psi(x)=\psi(x_0).$ If $\psi(x)$ is local fractional continuous on the interval $(a,b),$ it is denoted by:

$$\psi(x)\in C_{\alpha}(a,b)$$

*Definition 2.* In fractal space, let $\psi(x)\in C_{\alpha}(a,b),$ the LFD of $\psi(x)$ at the point $x=x_0$ is given by:

$$\mathrm{D}_x^{\alpha}\psi(x_0)=\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}}\psi(x)\bigg|_{x=x_0}=\psi^{(\alpha)}(x_0)=\lim_{x\to x_0}\frac{\Delta^{\alpha}[\psi(x)-\psi(x_0)]}{(x-x_0)^{\alpha}} \tag{2}$$

where $\Delta[\psi(x)-\psi(x_0)]\cong\Gamma(\alpha+1)[\psi(x)-\psi(x_0)].$

*Definition 3.* In the fractional space, the Mittag-Leffler function is given by:

$$E_{\alpha}(x^{\alpha})=\sum_{n=0}^{\infty}\frac{x^{(n\alpha)}}{\Gamma(1+n\alpha)},\quad 0<\alpha\le 1 \tag{3}$$

We recall the following properties:

$$E_{\alpha}(x^{\alpha})E_{\alpha}(y^{\alpha})=E_{\alpha}[(x+y)^{\alpha}] \tag{4}$$

*Definition 4.* A partition of the interval $[a,b]$ is denoted as $(t_j,t_{j+1}),$ $j=0,1,\cdots,\ N-1,$ $t_0=a$ and $t_N=b$ with $\Delta t_j=t_{j+1}-t_j$ and $\Delta t=\max\{\Delta t_0,\Delta t_1,\cdots\Delta t_N\}$. The local fractional integral of $\psi(x)$ in the interval $[a,b]$ is given by:

$${}_a\mathrm{I}_b^{(\alpha)}\psi(x)=\frac{1}{\Gamma(1+\alpha)}\int_a^b\psi(x)(\mathrm{d}x)^{\alpha}=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{N-1}\psi(t_j)(\Delta t_j)^{\alpha} \tag{5}$$

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b E_{\alpha}(x^{\alpha})(\mathrm{d}x)^{\alpha}=E_{\alpha}(b^{\alpha})-E_{\alpha}(a^{\alpha}) \tag{6}$$

*Definition 5.* The local fractional partial derivative operator of $\psi(x,t)$ of order $\alpha(\alpha\in(0,1])$ with respect to *t* at the point $(x,t_0)$ is defined [21]:

$$\mathrm{D}^{(\alpha)}\psi(x,t_0)=\frac{\partial^{(\alpha)}\psi(x,t_0)}{\partial t^{\alpha}}=\lim_{t\to t_0}\frac{\Delta^{\alpha}[\psi(x,t)-\psi(x,t_0)]}{(t-t_0)^{\alpha}} \tag{7}$$

where

$$\Delta^{\alpha}[\psi(x,t)-\psi(x,t_0)]\cong\Gamma(1+\alpha)[\psi(x,t)-\psi(x,t_0)] \tag{8}$$

In view of eq. (7), the local fractional partial derivative operator of $\psi(x,t)$ of order $k\alpha(\alpha\in(0,1])$ is given by [21]:

$$\mathrm{D}_t^{(k\alpha)}\psi(x,t)=\frac{\partial^{k\alpha}\psi(x,t)}{\partial t^{k\alpha}}=\overbrace{\frac{\partial^{\alpha}}{\partial t^{\alpha}}\cdots\frac{\partial^{\alpha}}{\partial t^{\alpha}}}^{k\quad times}\psi(x,t) \tag{9}$$

## Fractal model of the local fractional Fisher's equation on Cantor sets

In this section, the local fractional Fisher's equation on Cantor sets is derived with the fractional complex transform [ 22-25] *via* the LFD [21].

Considering Fisher's equation of given form [1]:

$$\frac{\partial\psi(X,T)}{\partial T}-\frac{\partial^2\psi(X,T)}{\partial X^2}-6\psi(1-\psi)=0,\quad T>0 \tag{10}$$

subject to the initial condition:

$$\psi(X,0)=\psi_0=\frac{1}{(1+\mathrm{e}^x)^2},\quad X\in\mathbb{R} \tag{11}$$

Using the fractional complex transform method [22-25] *via* local fractional derivatives, we can obtain:

$$\frac{\partial^{\alpha}\psi(x,t)}{\partial t^{\alpha}}-\frac{\partial^{2\alpha}\psi(x,t)}{\partial x^{2\alpha}}-6\psi(1-\psi)=0,\quad t>0 \tag{12}$$

## Analysis of the local fractional variational iteration method

In this section, we present the main steps of the local fractional variational iteration method [16-20]. The method gives the solution in a local fractional series form that converges to the closed-form solution if an exact solution exists.

We consider a general non-linear local fractional partial differential equation:

$$L_{\gamma}\psi(x,t)+N_{\gamma}\psi(x,t)=f(x,t),\quad t>0,\quad x\in\mathbb{R},\quad 0<\gamma\le 1 \tag{13}$$

where $L_{\gamma}$ denotes linear LFD operator, $N_{\gamma}$ – the non-linear local fractional operator, and $f(x,t)$ – the non-differentiable source term.

Local fractional variational iteration algorithm (LFVIA) can be written [16-20]:

$$\psi_{n+1}(t) = \psi_n(t) + {}_{t_0}\mathrm{I}_t^{(\gamma)}\{\eta^\gamma[L_\gamma\psi_n(s) + N_\gamma\psi_n(s) - f(s)]\} \tag{14}$$

where $\eta^\gamma$ is a fractal Lagrange multiplier, which can be identified by the fractal variational theory [28-37].

Here, according to the rule of the local fractional variational iteration method, we can construct a correction functional [7, 8]**:**

$$\psi_{n+1}(t) = \psi_n(t) + {}_{t_0}\mathrm{I}_t^{(\gamma)}\{\eta^\gamma[L_\gamma\psi_n(s) + N_\gamma\tilde{\psi}_n - f(s)]\} \tag{15}$$

where $\tilde{\psi}_n$ is considered as a restricted local fractional variation, that is, $\delta^\gamma\tilde{\psi}_n = 0$ [7, 8]. Here, $\delta^\gamma$ is the local fractional variation signal.

We consider a local fractional variational principle [28-37]:

$$\Theta(\omega) = {}_\sigma\mathrm{I}_\mu^{(\gamma)}\varphi[t, \omega(t), \omega^\gamma(t)] \tag{16}$$

where $\omega^{(\gamma)}(t)$ is taken in local fractional differential operator and $\sigma \le t \le \mu$.

Following eq. (16), we have the stationary condition, which is given by [28-37]:

$$\frac{\partial\varphi}{\partial\omega} - \frac{\mathrm{d}^\gamma}{\mathrm{d}t^\gamma}\frac{\partial\varphi}{\partial\omega^{(\gamma)}} = 0 \tag{17}$$

Equation (17) is useful for the identification of the Lagrange multiplier in the local fractional variational iteration method.

After the fractal Lagrange multipliers have been determined, the successive approximations ($\psi_{n+1}, n \ge 0$) of the solution will be readily obtained by using any selective fractal function. Therefore, we get the solution of the equation:

$$\psi = \lim_{n\to\infty}\psi_n \tag{18}$$

Here, this technology is called the local fractional variational iteration method [16-20].

## Approximate analytical solutions of fractal Fisher's equation

In this section, fractal Fisher's eq. (12) is discussed by using local fractional variational iteration method.

Start with the 0th approximation:

$$\psi(x,0) = \frac{1}{[1 + E_\gamma(x^\gamma)]^2} \tag{19}$$

We can structure a correction local fractional functional:

$$\psi_{n+1}(x,t) = \psi_n(x,t) + {}_{t_0}\mathrm{I}_t^{(\gamma)}\left\{\eta^\gamma\left[\frac{\partial^\gamma\psi_n}{\partial\tau^\gamma} - \frac{\partial^{2\gamma}\psi_n}{\partial x^{2\gamma}} - 6\psi_n(1-\psi_n)\right]\right\} \tag{20}$$

The stationary conditions of eq. (20) are presented:

$$(\eta^{\gamma})^{(\gamma)} = 0 \tag{21}$$

and

$$1+\eta^{\gamma}\big|_{\tau=t} = 0 \tag{22}$$

Therefore, it is clear that the fractal Lagrange multiplier can be determined simply:

$$\eta^{\gamma}(\tau) = -1 \tag{23}$$

From eq. (20), LFVIA for fractal Fisher's eq. (12) are structured:

$$\psi_{n+1}(x,t) = \psi_n(x,t) - {}_{t_0}\mathrm{I}_t^{(\gamma)}\left[\frac{\partial^{\gamma}\psi_n}{\partial\tau^{\gamma}} - \frac{\partial^{2\gamma}\psi_n}{\partial x^{2\gamma}} - 6\psi_n(1-\psi_n)\right] \tag{24}$$

Following eqs. (19) and (24), the formulas of non-differentiable terms are:

$$\psi_0(x,0) = \frac{1}{[1+E_{\gamma}(x^{\gamma})]^2} \tag{25}$$

$$\psi_1(x,t) = \psi_0(x,t) - {}_0\mathrm{I}_t^{(\gamma)}\left[\frac{\partial^{\gamma}\psi_0}{\partial\tau^{\gamma}} - \frac{\partial^{2\gamma}\psi_0}{\partial x^{2\gamma}} - 6\psi_0(1-\psi_0)\right] =$$

$$= \frac{1}{[1+E_{\gamma}(x^{\gamma})]^2} - {}_0\mathrm{I}_t^{(\gamma)}\left\{\frac{-10E_{\gamma}(x^{\gamma}) - 10E_{\gamma}(2x^{\gamma})}{[1+E_{\gamma}(x^{\gamma})]^4}\right\} =$$

$$= \frac{1}{[1+E_{\gamma}(x^{\gamma})]^2} + 10\frac{E_{\gamma}(x^{\gamma})}{[1+E_{\gamma}(x^{\gamma})]^3}\frac{t^{\gamma}}{\Gamma(1+\gamma)} \tag{26}$$

$$\psi_2(x,t) = \psi_1(x,t) - {}_0\mathrm{I}_t^{(\gamma)}\left[\frac{\partial^{\gamma}\psi_1}{\partial\tau^{\gamma}} - \frac{\partial^{2\gamma}\psi_1}{\partial x^{2\gamma}} - 6\psi_1(1-\psi_1)\right] =$$

$$= \frac{1}{[1+E_{\gamma}(x^{\gamma})]^2} + \frac{14E_{\gamma}(x^{\gamma}) + 7E_{\gamma}(2x^{\gamma})}{[1+E_{\gamma}(x^{\gamma})]^4}\frac{t^{\gamma}}{\Gamma(1+\gamma)} +$$

$$+ \frac{64E_{\gamma}(3x^{\gamma}) + 113E_{\gamma}(2x^{\gamma}) - 59E_{\gamma}(x^{\gamma})}{[1+E_{\gamma}(x^{\gamma})]^3}\frac{t^{2\gamma}}{\Gamma(1+2\gamma)} +$$

$$+\left\{-600\frac{E_{\gamma}(2x^{\gamma})}{[1+E_{\gamma}(x^{\gamma})]^6}\frac{\Gamma(1+2\gamma)}{[\Gamma(1+\gamma)]^2}\frac{t^{3\gamma}}{\Gamma(1+3\gamma)}\right\} \tag{27}$$

Therefore, the approximation solution is:

$$\psi(x,t)=\frac{1}{[1+E_\gamma(x^\gamma)]^2}+\frac{14E_\gamma(x^\gamma)+7E_\gamma(2x^\gamma)}{[1+E_\gamma(x^\gamma)]^4}\frac{t^\gamma}{\Gamma(1+\gamma)}+$$

$$+\frac{64E_\gamma(3x^\gamma)+113E_\gamma(2x^\gamma)-59E_\gamma(x^\gamma)}{[1+E_\gamma(x^\gamma)]^3}\frac{t^{2\gamma}}{\Gamma(1+2\gamma)}+\cdots \tag{28}$$

## Conclusion

In this paper, the Fisher's equation within the local fractional differential operator had been analyzed coupling LFCT *via* LFD and LFVIM. The non-differentiable solutions for fractal the Fisher's equation were obtained. This method is a powerful mathematical tool for solving local fractional non-linear partial differential equations.

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