TAYLOR SERIES SOLUTION FOR THE NON-LINEAR EMDEN-FOWLER EQUATIONS

by

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Original scientific paper
https://doi.org/10.2298/TSCI2203693L

The well-known Emden-Fowler equation is widely used to model many problems arising in thermal science, physics, and astrophysics. Although there are some analytical solutions available, the high requirement for mathematical knowledge has hindered researchers from direct applications. This paper suggests a straightforward method with a simple solution process and highly accurate results. Two examples are given to verify the accuracy and reliability of the proposed method.

Key words: *Emden-Fowler equations, Taylor series method, astrophysics, non-linear differential equation*

Introduction

Non-linear differential equations appear in thermal science, physics, and engineering. Without them, progress in science and technology is impossible. The differential models for engineering applications should be simple; the simpler, the better [1]. A simple mathematical treatment has revealed the mechanism of a long-lost technology called Fangzhu, an ancient technology to collect water from the air [2]. A simple mathematical formulation can insight into the frequency-amplitude property of a complex vibration system [3-5] or instability property [6-8], a simple bond stress-slip relationship can be used for prediction of the mechanical property of a 3-D printed concrete [9]. The variational theory [10-15] is a useful mathematical tool for analysis of a differential equation. Mathematics itself has been developing, and some traditional problems are more effectively solved by new mathematics concepts, for example, the local fractional calculus [16-21].

The well-known Emden-Fowler equation (EFE) is a second-order non-linear ODE and is widely used to model many facts in thermal science, physical sciences, and astrophysics [22, 23].

The general form of EFE is given as:

$$y'' + \frac{a}{x}y' + bf(x)g(y) = s(x), \quad a \ge 0 \tag{1}$$

with the following initial conditions (IC):

$$y(0) = m, \quad y'(0) = n$$

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where *a, b, m,* and *n* are constants, *f*(*x*) and *s*(*x*) represent functions of *x* and *g*(*y*) is the function of *y*.

Different forms of the *g*(*y*) and *s*(*x*) lead to different phenomena in mathematical physics, and there has been a substantial amount of work done on the solution of the EFE [22-24]. In this paper, the Taylor series method [25, 26] is adopted to solve eq. (1).

Taylor series solution for the EFE

Taylor series plays an essential role in approximate calculation and is accessible to all students and engineers [25, 26]. To better illustrate our approach, we rewrite eq. (1) in the form:

$$xy'' + ay' + bxf(x)g(y) = xs(x) \tag{2}$$

Differentiating the eq. (1) with respect to *x* yields:

$$y'' + xy''' + bf(x)g(y) + bx[f'(x)g(y) + f(x)g'(y)y'] = s(x) + xs'(x) \tag{3}$$

Let $x = 0$ in eq. (3), we obtain:

$$y'' + bf(0)g(m) = s(0) \tag{4}$$

which leads to:

$$y''(0) = s(0) - bf(0)g(m) \tag{5}$$

In the same manner, we have the following expression by differentiating eq. (3) with respect to *x* and letting *x* = 0:

$$y'''(0) = s'(0) - bf'(0)g(n) - bf(0)g'(n)n \tag{6}$$

Similarly, we can get $y^{(4)}(0),\quad y^{(5)}(0),\quad y^{(6)}(0)$ ... and so on.

Then the approximate analytical solution of the *y*(*x*) in the form of Taylor series can be obtained:

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \frac{y'''(0)}{6}x^3 + \frac{y^{(4)}(0)}{24}x^4 + \frac{y^{(5)}(0)}{120}x^5 + \frac{y^{(6)}(0)}{720}x^6 + ... \tag{7}$$

In order to obtain higher accuracy, it is only necessary to increase the order of the Taylor series. Next, we use two examples to verify the correctness and reliability of the proposed method.

Example 1. Considering the EFE takes form:

$$y'' + \frac{8}{x}y' + xy = x^5 - x^4 + 44x^2 - 30x \tag{8}$$

There is the following IC:

$$y(0) = 0, \qquad y'(0) = 0$$

The exact solution of *Example 1* is [24]:

$$y(x) = -x^3 + x^4 \tag{9}$$

Equation (8) can be converted into:

$$xy'' + 8y' + x^2 y = x^6 - x^5 + 44x^3 - 30x^2 \tag{10}$$

Applying our proposed method, we can get:

$$y''(0) = 0, \quad y'''(0) = -6, \quad y^{(4)}(0) = 24, \quad y^{(5)}(0) = 0, \quad y^{(6)}(0) = 0, \quad y^{(7)}(0) = 0$$

The solution of eq. (8) with 7th Taylor series can be written:

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \frac{y'''(0)}{6}x^3 + \frac{y^{(4)}(0)}{24}x^4 + \frac{y^{(5)}(0)}{120}x^5 +$$

$$+ \frac{y^{(6)}(0)}{720}x^6 + \frac{y^{(7)}(0)}{5040}x^7 = -x^3 + x^4 \tag{11}$$

This is the exact solution.

Example 2. In this case, we try to solve the following EFE:

$$y'' + \frac{5}{x}y' + e^y + 2e^{\frac{y}{2}} = 0 \tag{12}$$

with the IC:

$$y(0) = 0, \quad y'(0) = 0$$

The exact solution of eq. (12) is given [24]:

$$y(x) = -2\ln\left(1 + \frac{1}{8}x^2\right) \tag{13}$$

We rewrite eq. (12):

$$xy'' + 5y' + xe^y + 2xe^{\frac{y}{2}} = 0$$

Using the method we proposed, there is:

$$y''(0) = -\frac{1}{2}, \quad y'''(0) = 0, \quad y^{(4)}(0) = \frac{3}{8}, \quad y^{(5)}(0) = 0, \quad y^{(6)}(0) = -\frac{15}{16}$$

Then we obtain the solution of eq. (12) with 6th order Taylor series:

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \frac{y'''(0)}{6}x^3 + \frac{y^{(4)}(0)}{24}x^4 +$$

$$+ \frac{y^{(5)}(0)}{120}x^5 + \frac{y^{(6)}(0)}{720}x^6 = -\frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{768}x^6 \tag{14}$$

Comparing the approximate solution and exact solution in fig. 1 reveals that our presented method is efficient and reliable.

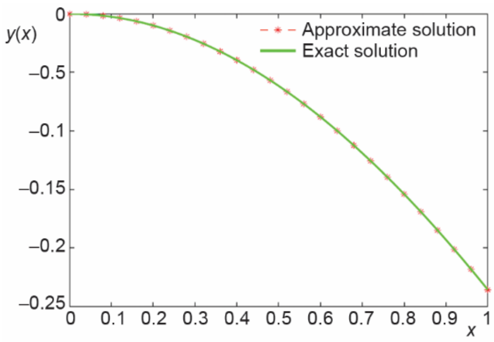

Figure 1. Comparison between the approximate solution and exact solution for $0 < x \leq 1$

Discussion and conclusion

In this paper, a simple but effective approximate solution of the EFE is presented. The whole solution process is extremely easy. The proposed method is expected to be helpful to the solution of EFE arising in astrophysics and space science.

Acknowledgment

This work is supported by the Key Programs of Universities in Henan Province of China (22A140006), the Fundamental Research Funds for the Universities of Henan Province (NSFRF210324), Program of Henan Polytechnic University (B2018-40).

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Paper submitted: June 8, 2020
Paper revised: October 2, 2021
Paper accepted: October 2, 2021