FRACTIONAL RESIDUAL METHOD COUPLED WITH ADOMIAN DECOMPOSITION METHOD FOR SOLVING LOCAL FRACTIONAL DIFFERENTIAL EQUATIONS

by

Yong-Ju YANG [*] ***and Shun-Qin WANG***

School of Mathematics and Statistics, Nanyang Normal University, Nanyang, China

Original scientific paper
https://doi.org/10.2298/TSCI2203667Y

In order to solve the local fractional differential equations, we couple the fractional residual method with the Adomian decomposition method via the local fractional calculus operator. Several examples are given to illustrate the solution process and the reliability of the method.

Key words*: local fractional derivative, fractional residual method, Adomian decomposition method*

Introduction

It is an obvious fact that fractional differential equations can describe non-differentiable problems much better than classic differential equations [1-8]. Now the fractional calculus has been developing rapidly, and it has been widely applied to physics, mathematics, engineering, and many other fields [1-5]. However, there are still many problems which are needed to be solved urgently. The main barrier is the definition of the fractional derivative. There are too many definitions, and new definitions always appear in literature, however, most fractional derivatives are almost incompatible, and the physical understanding of the fractional models is unclear. The aforementioned issues are worthy of further study in the future.

The local fractional derivative [9, 10] is defined in Cantor fractal space and can model many practical problems. Fractal theory is also a powerful tool to the analysis of biologic and material phenomena [11, 12]. How to solve such problems has become a hot topic in mathematics, and some effective methods have appeared in literature, *e.g.*, the variational iteration method [13-15], the fractional residual method [16], the homotopy perturbation method [17-20], the local fractional Fourier series method [21-23], The Yang Laplace transform-DJ iteration method [24], He-Laplace method [25, 26], the fractional complex transform (two-scale transform) [27, 28], the coupled method of the variational iteration and reduced differential transform method [29], the differential transform approach [30], the asymptotic perturbation method [31], the coupled method of the Sumudu transform and the variational iteration method [32], the direct algebraic method [33, 34], the exp-function method [35], the variational approach [36-39], the Fourier spectral method [40] and the reproducing kernel method [41].

In this paper, we attempt to solve the local differential equation by coupling the fractional residual method [16] with the Adomian decomposition method [42] *via* the local differential equations.

* Corresponding author, e-mail: tomjohn1007@126.com

Basic definitions of local fractional calculus

In this section, we introduce some mathematical preliminaries of the local fractional calculus theory in fractal space for our subsequent development [6].

Definition 1. In fractal space, let $u(t)\in C_\alpha(a,b)$, the local fractional derivative of $u(t)$ of order α at $t=t_0$ is given by [6, 9, 10]:

$$\mathrm{D}_t^{(\alpha)}u(t_0)=u^{(\alpha)}(t_0)=\frac{\mathrm{d}^\alpha u(t)}{\mathrm{d}t^\alpha}\Big|_{t=t_0}=\lim_{t\to t_0}\frac{\Delta^\alpha[u(t)-u(t_0)]}{(t-t_0)^\alpha} \tag{1}$$

where

$$\Delta^\alpha[u(t)-u(t_0)]\cong\Gamma(1+\alpha)\Delta[u(t)-u(t_0)]$$

Definition 2. [6] let $u(t)\in C_\alpha(a,b)$, the local fractional integral of (t) of order α in the interval $[a, b]$ is defined by [6, 9, 10]:

$${}_a\mathrm{I}_b^{(\alpha)}u(t)=\frac{1}{\Gamma(1+\alpha)}\int_a^b u(t)(\mathrm{d}t)^\alpha=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{j=N-1}u(t_j)(\Delta t_j)^\alpha \tag{2}$$

where $\Delta t_j=t_{j+1}-t_j$, $\Delta t=\max\{\Delta t_1,\Delta t_2,\Delta t_j,...\}$ and (t_j,t_{j+1}), $j=0,...,N-1$, $t_0=a$, $t_N=b$, is a partition of the interval $[a, b]$.

The fractional residual method coupled with the Adomian decomposition method

In this section, we shall present the iterative process of the fractional residual method [16] coupled with the Adomian decomposition method [42] to search for the exact solution of some local fractional differential equations.

Firstly, we discuss the following local fractional differential equation on the fractal set, and its form is given:

$$Lu(x,t)-Ru(x,t)-Pu(x,t)=g(x,t) \tag{3}$$

where L, P are all linear operators, R is a non-linear operator and $g(x, t)$ is an inhomogeneous term.

In order to solve eq. (3), we apply the local fractional reverse operator $L^{-1}(\bullet)$ on both sides of eq. (3), then we obtain:

$$u(x,t)=\tilde{u}(x,t)+L^{-1}[Ru(x,t)+Pu(x,t)+g(x,t)] \tag{4}$$

where $\tilde{u}(x,t)$ is derived from the initial condition.

Supposing $u_0(x,t)$ is a function to be determined, we can rewrite eq. (4):

$$u(x,t)=\tilde{u}(x,t)+L^{-1}u_0+L^{-1}[-u_0+Ru(x,t)+Pu(x,t)+g(x,t)] \tag{5}$$

If we let:

$$L^{-1}[-u_0+Ru(x,t)+Pu(x,t)+g(x,t)]=0 \tag{6}$$

Solving eq. (6), we can determine $u_0(x,t)$, and then substituting $u_0(x,t)$ into eq. (5), we can get the exact solution of eq. (4):

$$u(x,t) = \tilde{u}(x,t) + L^{-1}u_0 \tag{7}$$

According to eq. (6), we can construct the following equation:

$$-u_0 + R[\tilde{u}(x,t) + L^{-1}u_0] + P[\tilde{u}(x,t) + L^{-1}u_0] + g(x,t) = 0 \tag{8}$$

By analyzing the process of this method, we can deduce that it is critical to get the exact solution of eq. (3) to choose $u_0(x,t) = 0$.

Let the solution $u_0(x,t)$ of eq. (8) has the following series form:

$$u_0(x,t) = \sum_{n=0}^{\infty} v_n(x,t), \quad n = 1, 2, ... \tag{9}$$

Substituting eq. (9) into eq. (8), we can get:

$$\sum_{n=0}^{\infty} v_n(x,t) = R\left\{\tilde{u}(x,t) + L^{-1}\left[\sum_{n=0}^{\infty} v_n(x,t)\right]\right\} + P\left\{\tilde{u}(x,t) + L^{-1}\left[\sum_{n=0}^{\infty} v_n(x,t)\right]\right\} + g(x,t) \tag{10}$$

We let $A_n(u)$ be the Adomian polynomials, which are:

$$A_n(u) = \frac{1}{\Gamma(1+n\alpha)} \frac{\mathrm{d}^{n\alpha}}{\mathrm{d}\lambda^{n\alpha}} \left(R\left\{\tilde{u}(x,t) + L^{-1}\left[\sum_{i=0}^{\infty} \lambda^{i\alpha} v_i(x,t)\right]\right\}\right)_{\lambda=0} \tag{11}$$

And we suppose $g(x,t)+P[\tilde{u}(x,t)]$ can be decomposed as the following series:

$$g(x,t)+P[\tilde{u}(x,t)]=\sum_{n=0}^{\infty} g_n \tag{12}$$

According to eqs. (11) and (12) and the Adomian decomposition method, we can construct the following new recursion scheme:

$$\begin{aligned} v_0 &= g_0 \\ v_1 &= g_1 + P(v_0) + \mathrm{A}_0 \\ v_{m+1} &= g_n + P(v_n) + \mathrm{A}_n \end{aligned} \tag{13}$$

Obviously, similarly to the classic Adomian decomposition method, we can easily verify that:

$$u_0 = \sum_{n=0}^{\infty} v_n \tag{14}$$

converges.

Then, by virtue of eqs. (7) and (14), the exact solution of eq. (4) can be easily derived.

Illustrative examples

To demonstrate the effectiveness of the method, several local fractional PDE are presented.

Example 1. Consider the following local fractional Schrodinger conduction equation:

$$i^{\alpha}\frac{\partial^{\alpha}u}{\partial x^{\alpha}}(x,t)+\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}(x,t)+|u(x,t)|^{2}[u(x,t)]+E_{\alpha}\left[2i(t-x)^{\alpha}\right]=0 \tag{15}$$

subject to the initial condition:

$$u(0,t)=E_{\alpha}(2i^{\alpha}t^{\alpha}) \tag{16}$$

According to eq. (5), eq. (15) can be transformed into the following equation:

$$u_0(x,t)=i^{\alpha}E_{\alpha}[2i(t-x)^{\alpha}]+$$

$$+i^{\alpha}\left(\frac{\partial^{2\alpha}[\tilde{u}(x,t)+L^{-1}u_0]}{\partial t^{2\alpha}}(x,t)+\left|L^{-1}[\tilde{u}(x,t)+L^{-1}u_0](x,t)\right|^{2}\{L^{-1}[\tilde{u}(x,t)+L^{-1}u_0](x,t)\}\right) \tag{17}$$

Letting $u_0(x,t)=\Sigma_{n=0}^{\infty}v_n(x,t)$ and substituting it into eq. (17), we can derive:

$$\sum_{n=0}^{\infty}v_n(x,t)=\frac{-3x^{\alpha}i^{\alpha}}{\Gamma(1+\alpha)}E_{\alpha}(2i^{\alpha}t^{\alpha})+i^{\alpha}E_{\alpha}[2i(t-x)^{\alpha}]+i^{\alpha}\left[\sum_{n=0}^{\infty}\frac{\partial^{2\alpha}L^{-1}v_n}{\partial t^{2\alpha}}(x,t)+\sum_{n=0}^{\infty}L^{-1}A_n(u)\right]=$$

$$=\frac{-3x^{\alpha}i^{\alpha}}{\Gamma(1+\alpha)}E_{\alpha}(2i^{\alpha}t^{\alpha})+i^{\alpha}E_{\alpha}[2i(t-x)^{\alpha}]+i^{\alpha}\left[\sum_{n=0}^{\infty}\frac{\partial^{2\alpha}L^{-1}v_n}{\partial t^{2\alpha}}(x,t)+\sum_{n=0}^{\infty}L^{-1}A_n(u)\right] \tag{18}$$

where $A_n(u)$ is the Adomian polynomials for the non-linear term $|u(t,x)|^2u(t,x)$ and $A_n(u)$ is given:

$$A_n(u)=\frac{1}{\Gamma(1+n\alpha)}\frac{\mathrm{d}^{n\alpha}}{\mathrm{d}\lambda^{n\alpha}}\left\{\left|\left[\sum_{n=0}^{\infty}\lambda^{i\alpha}L^{-1}v_i(x,t)\right]\right|^{2}\left[\sum_{n=0}^{\infty}\lambda^{i\alpha}L^{-1}v_i(x,t)\right]\right\}_{\lambda=0} \tag{19}$$

Obviously, according to eq. (13), the first few Adomian polynomials are respectively given by:

$$A_0(x,t)=v_0\bar{v}_0v_0=E_\alpha(2it)^\alpha$$

$$A_1(x,t)=v_0\bar{v}_0v_1+v_0v_0\bar{v}_1+v_0\bar{v}_0v_1=\frac{(-2ix)^\alpha}{\Gamma(1+\alpha)}E_\alpha(2it)^\alpha$$

$$A_2(x,t)=v_0v_0\bar{v}_2+v_0\bar{v}_0v_2+v_0\bar{v}_1v_1+v_0\bar{v}_0v_2+v_1v_1\bar{v}_0+v_0\bar{v}_1v_1=\frac{(-2ix)^{2\alpha}}{\Gamma(1+2\alpha)}E_\alpha(2it)^\alpha \tag{20}$$

$$......$$

$$A_n(x,t)=...=\frac{(-2ix)^{n\alpha}}{\Gamma(1+n\alpha)}E_\alpha(2it)^\alpha$$

In the light of eq. (12), we let:

$$G=\sum_{n=0}^{\infty}G_n=\frac{-3x^\alpha i^\alpha}{\Gamma(1+\alpha)}E_\alpha(2i^\alpha t^\alpha)+i^\alpha E_\alpha[2i(t-x)^\alpha] \tag{21}$$

where

$$G_0=i^\alpha E_\alpha(2it)^\alpha,\quad G_1=\frac{-5(ix)^\alpha}{\Gamma(1+\alpha)}E_\alpha(2it)^\alpha$$
$$G_2=2^2\frac{(ix)^{2\alpha}}{\Gamma(1+\alpha)}E_\alpha(2it)^\alpha,...,\quad G_n=(-2)^m\frac{(ix)^{m\alpha}}{\Gamma(1+m\alpha)}E_\alpha(2it)^\alpha \tag{22}$$

Using eqs. (12), (13), (20), and (22), gives:

$$v_0=G_0=i^\alpha E_\alpha(2it)^\alpha$$

$$v_1=\frac{-5(ix)^\alpha}{\Gamma(1+\alpha)}E_\alpha(2it)^\alpha-i^\alpha\left[\frac{\partial^{2\alpha}L^{-1}v_0}{\partial t^{2\alpha}}(x,t)+L^{-1}A_0\right]\mathrm{d}\tau=-(2i)^\alpha\frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(2it)^\alpha$$

$$v_2=-2^2\frac{(ix)^{2\alpha}}{\Gamma(1+2\alpha)}E_\alpha(2it)^\alpha-i^\alpha\left[\frac{\partial^{2\alpha}L^{-1}v_1}{\partial t^{2\alpha}}(x,t)+L^{-1}A_1\right]\mathrm{d}\tau=(2i)^{2\alpha}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}E_\alpha(2it)^\alpha$$

$$v_3=8\frac{(ix)^{3\alpha}}{\Gamma(1+3\alpha)}E_\alpha(2it)^\alpha-i^\alpha\left[\frac{\partial^{2\alpha}L^{-1}v_2}{\partial t^{2\alpha}}(x,t)+L^{-1}A_2\right]\mathrm{d}\tau=-(2i)^{3\alpha}\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}E_\alpha(2it)^\alpha$$

$$v_m=(-2)^m\frac{(ix)^{m\alpha}}{\Gamma(1+m\alpha)}E_\alpha(2it)^\alpha+i^\alpha\left[\frac{\partial^{2\alpha}v_{m-1}}{\partial t^{2\alpha}}(x,t)+A_m\right]\mathrm{d}\tau=(-2i)^{m\alpha}\frac{x^{m\alpha}}{\Gamma(1+m\alpha)}E_\alpha(2it)^\alpha$$

(23)

Proceeding in this manner, the rest of the components $u_n(x,t)$ can be completely determined and then, making using of eq. (7), the exact solution of eq. (15) is:

$$u(x,t)=\sum_{m=0}^{\infty}(-2i)^{m\alpha}\frac{x^{m\alpha}}{\Gamma(1+m\alpha)}E_\alpha(2it)^\alpha=E_\alpha[2i(t-x)]^\alpha \tag{24}$$

Example 2. Consider the following non-linear gas dynamic equation:

$$\frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}}+u(x,t)\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}-u(x,t)[1-u(x,t)]=0 \tag{25}$$

subject to the boundary and initial conditions:

$$\begin{aligned} u(0,t)&=\sin_\alpha(t^\alpha) \\ \frac{\partial^{\alpha}u(0,t)}{\partial x^{\alpha}}&=\sin_\alpha(t^\alpha) \end{aligned} \tag{26}$$

Applying the inverse operator $L^{-1}(\bullet)={}_0\mathrm{I}_x{}^{(2\alpha)}(\bullet)$ on both sides of eq. (25), and making use of eq. (5), we obtain:

$$u(x,t)=\sin_\alpha(t^\alpha)\left[1+\frac{x^\alpha}{\Gamma(1+\alpha)}\right]+{}_0I_x{}^{(2\alpha)}u_0-$$
$$-{}_0\mathrm{I}_x{}^{(2\alpha)}\left\{u_0+u(x,t)\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}-u(x,t)[1-u(x,t)]\right\} \tag{27}$$

According to eqs. (6) and (7), we let:

$$u(x,t)=\sin_\alpha(t^\alpha)\left[1+\frac{x^\alpha}{\Gamma(1+\alpha)}\right]+{}_0\mathrm{I}_x{}^{(2\alpha)}u_0 \tag{28}$$

and

$${}_0\mathrm{I}_t{}^{(2\alpha)}\left\{u_0+u(x,t)\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}-u(x,t)[1-u(x,t)]\right\}=0 \tag{29}$$

Then, we suppose:

$$u_0+u(x,t)\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}-u(x,t)[1-u(x,t)]=0 \tag{30}$$

and

$$u_0(x,t)=\sum_{n=0}^{\infty}v_n(x,t) \quad ,n=1,2,... \tag{31}$$

where $u_n(t)$ are all constants to be determined.

Substituting the eqs. (28) and (31) into the eq. (30), we can obtain:

$$\sum_{n=0}^{\infty}v_n(x,t)-\sin_\alpha(t^\alpha)\left[1+\frac{x^\alpha}{\Gamma(1+\alpha)}\right]-\sum_{n=1}^{\infty}{}_0\mathrm{I}_t{}^{(2\alpha)}v_n(x,t)+$$
$$+\left\{\sin_\alpha(t^\alpha)\left[1+\frac{x^\alpha}{\Gamma(1+\alpha)}\right]+\sum_{n=1}^{\infty}{}_0\mathrm{I}_x{}^{(2\alpha)}v_n(x,t)\right\}\sin_\alpha t^\alpha\left[1+\frac{x^\alpha}{\Gamma(1+\alpha)}\right]=0 \tag{32}$$

In the light of eq. (12), we let:

$$G=\sum_{n=0}^{1}G_n=\sin_\alpha(t^\alpha)\left[1+\frac{x^\alpha}{\Gamma(1+\alpha)}\right] \tag{33}$$

where

$$G_0=\sin_\alpha t^\alpha,\quad G_1=\sin_\alpha(t^\alpha)\frac{x^\alpha}{\Gamma(1+\alpha)} \tag{34}$$

Using eqs. (11), (13), (32), and (34), gives:

$$\begin{gathered}
v_0=G_0=\sin_\alpha(t^\alpha)\\
v_1=\sin_\alpha(t^\alpha)\frac{x^\alpha}{\Gamma(1+\alpha)}+{}_0I_x^{(2\alpha)}v_0(x,t)=\sin_\alpha(t^\alpha)\left[\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\right]\\
v_2={}_0I_x^{(2\alpha)}v_1(x,t)=\sin_\alpha(t^\alpha)\left[\frac{x^{3\alpha}}{\Gamma(1+\alpha)}+\frac{x^{4\alpha}}{\Gamma(1+2\alpha)}\right]\\
v_m={}_0I_x^{(2\alpha)}v_{m-1}(x,t)=\sin_\alpha(t^\alpha)\left\{\frac{x^{(m+1)\alpha}}{\Gamma[1+(m+1)\alpha]}+\frac{x^{(m+2)\alpha}}{\Gamma[1+(m+2)\alpha]}\right\}\\
\ldots\ldots
\end{gathered} \tag{35}$$

Substituting the eq. (35) into eq. (31), and then substituting the result into eq. (30), we can obtain the following exact solution of the eq. (25):

$$u(x,t)=E_\alpha(x^\alpha)\sin_\alpha(t^\alpha) \tag{36}$$

Conclusions

In this article, we have suggested the fractional residual method coupled with the Adomian decomposition method for solving the local fractional equation. The test examples show that the method is simple and feasible, and it is effective for solving some local differential equation problems.

Acknowledgment

This work is supported by the teacher education curriculum reform project in Henan Province (2021-JSJYZD-027), the key scientific research of Institutions of higher learning in Henan Province Project plan (21A110018: Study on dynamic characteristics of a new Degas-peris-Procesi fractal equation) and the Natural Science Foundation of Henan Province (2223004205 07: Local fractional calculus and its application to Cantor type stepped surfaces).

References

[1] Kilbas, A. A., *et al.*, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, The Netherlands, 2006
[2] He, J. H., When Mathematics Meets Thermal Science, the Simpler is the Better, *Thermal Science, 25* (2021), 3, pp. 2039-2042
[3] He, J. H., Seeing with a Single Scale is Always Unbelieving: From Magic to Two-Scale Fractal, *Thermal Science, 25* (2021), 2, pp. 1217-1219

[4] He, J. H., A Tutorial Review on Fractal Space Time and Fractional Calculus, *Int. J. Theor. Phys., 53* (2014), June, pp. 3698-718

[5] He, J. H., Fractal Calculus and Its Geometrical Explanation, *Result in Physics, 10* (2018), Sept., pp. 272-276

[6] Yang, X. J., *et al., Local Fractional Integral Transforms and their Applications*, Academic Press, Hong Kong, 2015

[7] He, J.-H., *et al.*, Periodic Property and Instability of a Rotating Pendulum System, *Axioms, 10* (2021), 3, 10030191

[8] Feng, G. Q., He's Frequency Formula to Fractal Undamped Duffing Equation, *Journal of Low Frequency Noise Vibration and Active Control, 40* (2021), 4, pp. 1671-1676

[9] Yang, X. J. *Local Fractional Functional Analysis and Its Applications*, Asian Academic publisher Limited, Hong Kong, China, 2011

[10] Yang, X. J., *Advanced Local Fractional Calculus and Its Applications*, World Science Publisher, New York, USA, 2012

[11] Li, X. X., He, J. H., Along the Evolution Process: Kleiber's 3/4 Law Makes Way for Rubner's Surface Law: A Fractal Approach, *Fractals, 27* (2019), 2, 1950015

[12] Tian, D, *et al.*, Hall-Petch Effect and Inverse Hall-Petch Effect: A Fractal Unification, *Fractals*, *26* (2018), 6, 1850083

[13] Baleanu, D., *et al.*, A Modified Fractional Variational Iteration Method for Solving Non-linear Gas Dynamic and Coupled KdV Equations Involving Local Fractional Operator, *Thermal Science, 22* (2018), Suppl. 1, pp. S165-S175

[14] Anjum, N., He, J. H., Analysis of Non-linear Vibration of Nano/Microelectromechanical System Switch Induced by Electromagnetic Force Under Zero Initial Conditions, *Alexandria Engineering Journal, 59* (2020), 6, pp. 4343-4352

[15] He, J. H., *et al*., Approximate Periodic Solutions to Microelectromechanical System Oscillator Subject to Magnetostatic Excitation, *Mathematical Methods in Applied Sciences*, On-line first, http://doi.org/10.1002/mma.7018, 2020

[16] Yang, Y. J., The Fractional Residual Method for Solving the Local Fractional Differential Equations, *Thermal Science, 24* (2020), 4, pp. 2535-2542

[17] Yang, Y. J., Wang, S. Q., An Improved Homotopy Perturbation Method for Solving Local Fractional Nonlinear Oscillators, *Journal of Low Frequency Noise Vibration and Active Control, 38* (2019), 3-4, pp. 918-927

[18] He, J.-H., *et al.*, Homotopy Perturbation Method for the Fractal Toda Oscillator, *Fractal Fract., 5* (2021), 5030093

[19] Li, X. X., He, C. H., Homotopy Perturbation Method Coupled with the Enhanced Perturbation Method, *Journal of Low Frequency Noise Vibration and Active Control, 38* (2019), 3-4, pp. 1399-1403

[20] He, J. H., El-Dib, Y. O., Periodic Property of the Time-Fractional Kundu-Mukherjee-Naskar Equation, *Results in Physics, 19* (2020), Dec., 103345

[21] Yang. Y. J. Wang, S. Q., Local Fractional Fourier Series Method for Solving Nonlinear Equations with Local Fractional Operators, *Mathematical Problems in Engineering, 2015* (2015), ID104481905

[22] Yang, Y. J., *et al.*, Local fractional Fourier method for solving modified diffusion equations with local fractional derivative, *Journal of Non-linear Sciences and Applications, 9* (2016), 12, pp. 6153-6160

[23] Zhang, Y., Solving Initial-Boundary Value Problems for Local Fractional Differential Equation by Local Fractional Fourier Series Method, *Abstract and Applied Analysis, 2014* (2014), ID 912464

[24] Yang, Y. J., *et al.*, The Yang Laplace Transform – DJ Iteration Method for Solving the Local Fractional Differential Equation, *J. Nonlinear Sci. Appl., 10* (2017), 6, pp. 3023-3029

[25] Habib, S., *et al.*, Study of Non-linear Hirota-Satsuma Coupled KdV and Coupled mKdV System with Time Fractional Derivative, *Fractals, 29* (2021), 5, 2150108

[26] He, J. H., *et al.*, Non-linear Instability of Two Streaming-Superposed Magnetic Reiner-Rivlin Fluids by He-Laplace Method, *Journal of Electroanalytical Chemistry, 895* (2021), Aug., 115388

[27] Anjum, N., *et al.*, Two-Scale Fractal Theory for the Population Dynamics, *Fractals, 29* (2021), 7, 21501826

[28] He, J. H., *et al.*, A Fractal Modification of Chen-Lee-Liu Equation and its Fractal Variational Principle, *International Journal of Modern Physics B, 35* (2021), 21, 2150214

[29] Yang, Y. J., A New Method Solving Local Fractional Differential Equations in Heat Transfer, *Thermal Science, 23* (2019), 3, pp. 1663-1669

[30] Yang, X. J., *et al.*, A New Numerical Technique for Solving the Local Frac-Tional Diffusion Equation: Two-Dimensional Extended Differential Transform Approach, *Applied Mathematics and Computation 274* (2016), Feb., pp. 143-151
[31] Yang, X. J., Srivastava, H. M., An Asymptotic Perturbation Solution for a Linear Oscillator of Free Damped Vibrations in Fractal Medium Described by Local Fractional Derivatives, *Communications in Non-linear Science and Numerical Simulation, 29* (2015), 1-3, pp. 499-504
[32] Gao, F., *et al.*, A Coupling Method Involving the Sumudu Transform and the Variational Iteration Method for a Class of Local Fractional Diffusion Equations, *Non-linear Sci. Appl., 9* (2016), 11, pp. 5830-5835
[33] Tian, Y., Liu, J., Direct Algebraic Method for Solving Fractional Fokas Equation, *Thermal Science, 25* (2021), 3, pp. 2235-2244
[34] Tian, Y., Wan, J. X., Exact Solutions of Space-Time Fractional 2+1 Dimensional Breaking Soliton Equation, *Thermal Science, 25* (2021), 2, pp. 1229-1235
[35] Tian, Y., Liu, J., A Modified Exp-Function Method for Fractional Partial Differential Equations, *Thermal Science*, *25* (2021), 2, pp. 1237-1241
[36] Wang, K. J., On New Abundant Exact Traveling Wave Solutions to the Local Fractional Gardner Equation Defined on Cantor Sets, *Mathematical Methods in the Applied Sciences*, *45* (2021), 4, pp. 1904-1915
[37] Wang, K. J., Generalized Variational Principle and Periodic Wave Solution to the Modified Equal Width-Burgers Equation in Non-linear Dispersion Media, *Physics Letters A*, *419* (2021), Dec., 127723
[38] Wang, K. J., Zhang, P. L., Investigation of the Periodic Solution of the Time-Space Fractional Sasa-Satsuma Equation Arising in the Monomode Optical Fibers, *EPL*, *137* (2021), 6, 62001
[39] He, J. H., *et al.*, Variational Approach to Fractal Solitary Waves, *Fractals, 29* (2021), 7, 2150199
[40] Han, C., *et al.*, Numerical Solutions of Space Fractional Variable-Coefficient KdV-Modified KdV Equation by Fourier Spectral Method, *Fractals*, *29* (2021), 8, 2150246
[41] Dan, D. D., *et al.*, Using Piecewise Reproducing Kernel Method and Legendre Polynomial for Solving a Class of the Time Variable Fractional Order Advection-Reaction-Diffusion Equation, *Thermal Science, 25* (2021), 2B, pp. 1261-1268
[42] Duan, J. S., Rach, R., A New Modification of the Adomian Decomposition Method for Solving Boundary Value Problems for Higher Order Nonlinear Differential Equations, *Applied Mathematics and Computation, 218* (2011), 8, pp. 4090-4118

Paper submitted: October 1, 2020
Paper revised: October 1, 2021
Paper accepted: October 1, 2021