VARIATIONAL ITERATION METHOD FOR TWO FRACTIONAL SYSTEMS WITH BOUNDARY CONDITIONS

by

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Under investigation in this paper are two local fractional partial differential systems, one is the homogeneous linear partial differential system with initial values, and the other is the inhomogeneous non-linear partial differential system with initial and boundary values. To solve these two local fractional systems, we employ the local fractional variational iteration method and obtain exact solutions. It is shown that the method provides an effective mathematical tool for solving linear and non-linear local fractional partial differential systems with initial and boundary values.

Key words: *homogeneous linear local fractional partial differential system, inhomogeneous non-linear local fractional partial differential system, local fractional variational iteration method, exact solution*

Introduction

Fractional calculus or fractal calculus is very important, which has many applications in different fields [1]. He [2, 3] gave a good review on mathematical foundation of the fractional calculus, and gave its geometrical explanation. Physical laws in a fractal space have similar properties as those in a smooth space, for example, a variational principle can be established in a fractal space [4] or in a smooth space [5]. A non-linear dynamical system, *e.g.*, a Duffing oscillator [6], can be converted into a fractal vibration system if the vibration medium is considered as a fractal space [7, 8].

Since the concept of local fractional derivative was presented by Kolwankar and Gangal [9] in 1996, it was widely studied, and much achievement was obtained [10-20]. One of the graceful properties of the local fractional calculus is that it can be used to describe the non-differential problems in science and engineering. The celebrated variational iteration method (VIM) [21-28] has received a wide of applications since being proposed by He in 1990 [21]. It is worth mentioning that the VIM was successfully extended to the local fractional differential equations, and the modified method is called the local fractional VIM [13].

With the development of fractional calculus, solving fractional differential equations have been attached much attention, it is because solutions of fractional differential equations have theoretical and practical values. In this paper, we shall employ the local fractional VIM [13] to solve two local fractional partial differential systems, one is the homogeneous linear

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partial differential system with initial values, and the other one is the inhomogeneous non-linear partial differential system with initial and boundary values.

For such purpose, it is necessary to recall in this section the local fractional calculus and its some properties [10]. The local fractional partial derivative of order $\alpha(0<\alpha\le 1)$ at the point $x=x_0$ is defined:

$$u_x^{(\alpha)}(x_0,t)=\left.\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}\right|_{x=x_0}=\lim_{x\to x_0}\frac{\Delta^\alpha[u(x,t)-u(x_0,t)]}{(x-x_0)^\alpha} \tag{1}$$

where $\Delta^\alpha[u(x,t)-u(x_0,t)]\cong\Gamma(1+\alpha)[u(x,t)-u(x_0,t)]$.

The local fractional derivative has the following properties and basic operations [10]:

$$[\lambda f(x,t)+\mu g(x,t)]_x^{(\alpha)}=\lambda f_x^{(\alpha)}(x,t)+\mu g_x^{(\alpha)}(x,t)$$

$$[f(x,t)g(x,t)]_x^{(\alpha)}=f_x^{(\alpha)}(x,t)g(x,t)+f(x,t)g_x^{(\alpha)}(x,t)$$

$$\left[\frac{f(x,t)}{g(x,t)}\right]_x^{(\alpha)}=\frac{f_x^{(\alpha)}(x,t)}{g(x,t)}-\frac{f(x,t)g_x^{(\alpha)}(x,t)}{g^2(x,t)}$$

$$C_x^{(\alpha)}=0,\qquad u_x^{(2\alpha)}(x,t)=\left[u_x^{(\alpha)}(x,t)\right]_x^{(\alpha)},\qquad \left[\frac{x^{k\alpha}}{\Gamma(1+k\alpha)}h(t)\right]_x^{(\alpha)}=\frac{x^{(k-1)\alpha}}{\Gamma[1+(k-1)\alpha]}h(t)$$

$$[\sinh_\alpha(x^\alpha)h(t)]_x^{(\alpha)}=\cosh_\alpha(x^\alpha)h(t),\qquad [\cosh_\alpha(x^\alpha)h(t)]_x^{(\alpha)}=\sinh_\alpha(x^\alpha)h(t)$$

where λ, μ, and C are constants, $h(t)$ is an arbitrary function of t, while:

$$\sinh_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{x^{(2k+1)\alpha}}{\Gamma[1+(2k+1)\alpha]},\qquad \cosh_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{x^{2k\alpha}}{\Gamma(1+2k\alpha)}$$

The local fractional integral of $u(x,t)$ of order $\alpha(0<\alpha\le 1)$ in the integral $[a,b]$ is defined:

$${}_a\mathrm{I}_b^\alpha u(x,t)=\frac{1}{\Gamma(1+\alpha)}\int_a^b u(x,t)(\mathrm{d}t)^\alpha=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta x_k\to 0}\sum_{k=0}^{N-1}u(x_k,t)(\Delta x_k)^\alpha \tag{2}$$

where $\Delta x_k=x_{k+1}-x_k$ with $x_0=a<x_1<\cdots<x_{N-1}<x_N=b$.

The local fractional integral has the following properties and basic operations:

$${}_a\mathrm{I}_b^\alpha[\lambda f(x,t)+\mu g(x,t)]=\lambda\,{}_a\mathrm{I}_b^\alpha f(x,t)+\mu\,{}_a\mathrm{I}_b^\alpha g(x,t)$$

$${}_a\mathrm{I}_b^\alpha\{[f_x^{(\alpha)}(x,t)]g(x,t)\}=[f(x,t)g(x,t)]\big|_a^b-{}_a\mathrm{I}_b^\alpha\{f(x,t)[g_x^{(\alpha)}(x,t)]\}$$

$${}_0\mathrm{I}_x^\alpha C=\frac{Cx^\alpha}{\Gamma(1+\alpha)},\qquad {}_0\mathrm{I}_x^\alpha\left[\frac{x^{k\alpha}}{\Gamma(1+k\alpha)}h(t)\right]=\frac{x^{(k+1)\alpha}}{\Gamma[1+(k+1)\alpha]}h(t)$$

$$ {}_0\mathrm{I}_x^{\alpha}[\sinh_\alpha(x^\alpha)h(t)] = [\cosh_\alpha(x^\alpha)-1]h(t), \qquad {}_0\mathrm{I}_x^{\alpha}[\cosh_\alpha(x^\alpha)h(t)] = \sinh_\alpha(x^\alpha)h(t) $$

Local fractional VIM for local fractional systems

In this section, we recall the basic idea of the local fractional VIM [2, 13] for local fractional partial differential systems. For convenience, we suppose that the considered local fractional system can be written:

$$ L_\alpha(u) + N_\alpha(u) = f(x,t) \tag{3} $$

where L_α and N_α are linear and non-linear local fractional partial differential operators, respectively, while $f(x,t)$ is the source term.

The local fractional correction functional for eq. (3) reads:

$$ u_{n+1} = u_n + \frac{1}{\Gamma(1+\alpha)}\int_0^t \lambda_\alpha(t,\xi)[L_\alpha u_n(x,\xi) + N_\alpha \tilde{u}_n(x,\xi) - f(x,\xi)](\mathrm{d}\xi)^\alpha \tag{4} $$

where λ_α is a fractional Lagrange multiplier which can be identified optimally *via* the variational theory and integration by parts, $\tilde{u}_n$ is considered as a restricted local fractional variation, that is $\delta_\alpha \tilde{u}_n = 0.$ With the help of the identified Lagrange multiplier λ_α and the selected function u_0, the successive approximations $u_{n+1}(n \ge 0)$ can be determined. Then the solution u is immediately obtained by:

$$ u = \lim_{n\to\infty} u_n(x,t) \tag{5} $$

Solution of the homogeneous linear local fractional system

In this section, we apply the local fractional VIM to the following homogeneous linear local fractional system:

$$ \frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^\alpha v}{\partial x^\alpha} + u + v = 0 \tag{6} $$

$$ \frac{\partial^\alpha v}{\partial t^\alpha} - \frac{\partial^\alpha u}{\partial x^\alpha} + u + v = 0 \tag{7} $$

subject to initial conditions:

$$ u(x,0) = \sinh_\alpha x^\alpha, \qquad v(x,0) = \cosh_\alpha x^\alpha \tag{8} $$

Firstly, we let the local fractional correction functionals read:

$$ u_{n+1} = u_n + \frac{1}{\Gamma(1+\alpha)}\int_0^t \lambda_{1,\alpha}(t,\xi)\left[\frac{\partial^\alpha u_n(x,\xi)}{\partial \xi^\alpha} - \frac{\partial^\alpha \tilde{v}_n(x,\xi)}{\partial x^\alpha} + \tilde{u}_n(x,\xi) + \tilde{v}_n(x,\xi)\right](\mathrm{d}\xi)^\alpha \tag{9} $$

$$ v_{n+1} = v_n + \frac{1}{\Gamma(1+\alpha)}\int_0^t \lambda_{2,\alpha}(t,\xi)\left[\frac{\partial^\alpha v_n(x,\xi)}{\partial \xi^\alpha} - \frac{\partial^\alpha \tilde{u}_n(x,\xi)}{\partial x^\alpha} + \tilde{u}_n(x,\xi) + \tilde{v}_n(x,\xi)\right](\mathrm{d}\xi)^\alpha \tag{10} $$

and then set

$$\delta_\alpha u_{n+1} = \delta_\alpha u_n + \delta_\alpha \frac{1}{\Gamma(1+\alpha)} \int_0^t \lambda_{1,\alpha}(t,\xi) \left[\frac{\partial^\alpha u_n(x,\xi)}{\partial \xi^\alpha} - \frac{\partial^\alpha \tilde{v}_n(x,\xi)}{\partial x^\alpha} + \tilde{u}_n(x,\xi) + \tilde{v}_n(x,\xi) \right] (\mathrm{d}\xi)^\alpha = 0$$

$$\delta_\alpha v_{n+1} = \delta_\alpha v_n + \delta_\alpha \frac{1}{\Gamma(1+\alpha)} \int_0^t \lambda_{2,\alpha}(t,\xi) \left[\frac{\partial^\alpha v_n(x,\xi)}{\partial \xi^\alpha} - \frac{\partial^\alpha \tilde{u}_n(x,\xi)}{\partial x^\alpha} + \tilde{u}_n(x,\xi) + \tilde{v}_n(x,\xi) \right] (\mathrm{d}\xi)^\alpha = 0$$

namely:

$$\delta_\alpha u_{n+1} = [1 + \lambda_{1,\alpha}(t,\xi)\big|_{\xi=t}]\delta_\alpha u_n - \frac{1}{\Gamma(1+\alpha)} \int_0^t \frac{\partial^\alpha \lambda_{1,\alpha}(x,\xi)}{\partial \xi^\alpha} \delta_\alpha u_n(x,\xi)(\mathrm{d}\xi)^\alpha = 0$$

$$\delta_\alpha v_{n+1} = [1 + \lambda_{2,\alpha}(t,\xi)\big|_{\xi=t}]\delta_\alpha v_n - \frac{1}{\Gamma(1+\alpha)} \int_0^t \frac{\partial^\alpha \lambda_{2,\alpha}(x,\xi)}{\partial \xi^\alpha} \delta_\alpha v_n(x,\xi)(\mathrm{d}\xi)^\alpha = 0$$

Here $\delta_\alpha \tilde{u}_n = 0$ and $\delta_\alpha \tilde{v}_n = 0$ had been used. We than have:

$$1 + \lambda_{1,\alpha}(t,\xi)\big|_{\xi=t} = 0, \quad \left. \frac{\partial^\alpha \lambda_{1,\alpha}(t,\xi)}{\partial \xi^\alpha} \right|_{\xi=t} = 0, \quad 1 + \lambda_{2,\alpha}(t,\xi)\big|_{\xi=t} = 0, \quad \left. \frac{\partial^\alpha \lambda_{2,\alpha}(t,\xi)}{\partial \xi^\alpha} \right|_{\xi=t} = 0$$

which gives:

$$\lambda_1(t,\xi) = -1, \quad \lambda_2(t,\xi) = -1 \tag{11}$$

Substituting eq. (11) into the functionals (9) and (10) yields

$$u_{n+1} = u_n - \frac{1}{\Gamma(1+\alpha)} \int_0^t \left[\frac{\partial^\alpha u_n(x,\xi)}{\partial \xi^\alpha} - \frac{\partial^\alpha v_n(x,\xi)}{\partial x^\alpha} + u_n(x,\xi) + v_n(x,\xi) \right] (\mathrm{d}\xi)^\alpha$$

$$v_{n+1} = v_n - \frac{1}{\Gamma(1+\alpha)} \int_0^t \left[\frac{\partial^\alpha v_n(x,\xi)}{\partial \xi^\alpha} - \frac{\partial^\alpha u_n(x,\xi)}{\partial x^\alpha} + u_n(x,\xi) + v_n(x,\xi) \right] (\mathrm{d}\xi)^\alpha$$

In view of the initial conditions in eq. (8), we next select $u_0(x,t) = \sinh_\alpha x^\alpha$, $v_0(x,t) = \cosh_\alpha x^\alpha$ and then obtain the successive approximations:

$$u_1 = u_0 - \frac{1}{\Gamma(1+\alpha)} \int_0^t \cosh_\alpha x^\alpha (\mathrm{d}\xi)^\alpha = \sinh_\alpha x^\alpha - \frac{t^\alpha}{\Gamma(1+\alpha)} \cosh_\alpha x^\alpha$$

$$v_1 = v_0 - \frac{1}{\Gamma(1+\alpha)} \int_0^t \sinh_\alpha x^\alpha (\mathrm{d}\xi)^\alpha = \cosh_\alpha x^\alpha - \frac{t^\alpha}{\Gamma(1+\alpha)} \sinh_\alpha x^\alpha$$

$$u_2 = u_1 - \frac{1}{\Gamma(1+\alpha)} \int_0^t \left[-\frac{\xi^\alpha}{\Gamma(1+\alpha)} \sinh_\alpha x^\alpha \right] (\mathrm{d}\xi)^\alpha = \sinh_\alpha x^\alpha - \frac{t^\alpha}{\Gamma(1+\alpha)} \cosh_\alpha x^\alpha +$$

$$+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\sinh_\alpha x^\alpha$$

$$v_2 = v_1 - \frac{1}{\Gamma(1+\alpha)}\int_0^t \left[-\frac{\xi^\alpha}{\Gamma(1+\alpha)}\cosh_\alpha x^\alpha\right](\mathrm{d}\xi)^\alpha = \cosh_\alpha x^\alpha - \frac{t^\alpha}{\Gamma(1+\alpha)}\sinh_\alpha x^\alpha +$$

$$+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\cosh_\alpha x^\alpha$$

$$u_3 = u_2 - \frac{1}{\Gamma(1+\alpha)}\int_0^t \frac{\xi^{2\alpha}}{\Gamma(1+2\alpha)}(\mathrm{d}\xi)^\alpha = \sinh_\alpha x^\alpha\left[1+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right] -$$

$$-\cosh_\alpha x^\alpha\left[\frac{t^\alpha}{\Gamma(1+\alpha)}+\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right]$$

$$v_3 = v_2 - \frac{1}{\Gamma(1+\alpha)}\int_0^t \frac{\xi^{2\alpha}}{\Gamma(1+2\alpha)}\sinh_\alpha x^\alpha(\mathrm{d}\xi)^\alpha = \cosh_\alpha x^\alpha\left[1+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right] -$$

$$-\sinh_\alpha x^\alpha\left[\frac{t^\alpha}{\Gamma(1+\alpha)}+\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right]$$

$$\vdots$$

and so on.

Finally, we obtain a pair of exact solutions of eqs. (6) and (7):

$$u = \lim_{n\to\infty} u_n(x) = \sinh_\alpha (x^\alpha - t^\alpha) \tag{12}$$

$$v = \lim_{n\to\infty} v_n(x) = \cosh_\alpha (x^\alpha - t^\alpha) \tag{13}$$

When $\alpha = 1$, solutions (12) and (13) become $u = \sinh(x-t)$ and $u = \cosh(x-t)$ which are the known solutions of the homogeneous linear system [29]:

$$\frac{\partial u}{\partial t} - \frac{\partial v}{\partial x} + u + v = 0 \tag{14}$$

$$\frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} + u + v = 0 \tag{15}$$

with the initial conditions:

$$u(x,0) = \sinh x,\ \ v(x,0) = \cosh x \tag{16}$$

Solution of the inhomogeneous non-linear local fractional system

In this section, we apply the local fractional VIM to the following inhomogeneous non-linear local fractional system:

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}=\frac{\partial^{2\alpha}u}{\partial x^{2\alpha}}+u+u^2-x^\alpha t^\alpha-x^{2\alpha}t^{2\alpha}, \quad (0<x<\pi,\ t>0) \tag{17}$$

subject to boundary conditions:

$$u(0,t)=0, \ \ u(\pi,t)=\pi^\alpha t^\alpha \tag{18}$$

and the initial conditions:

$$u(x,0)=0, \ \ \frac{\partial^\alpha u(x,0)}{\partial t^\alpha}=\Gamma(1+\alpha)x^\alpha \tag{19}$$

Firstly, we suppose that the correction functional for eq. (17) reads:

$$u_{n+1}=u_n+\frac{1}{\Gamma(1+\alpha)}\int_0^t \lambda_\alpha(t,\xi)\cdot$$

$$\cdot\left[\frac{\partial^{2\alpha}u_n(x,\xi)}{\partial\xi^{2\alpha}}-\frac{\partial^{2\alpha}\tilde{u}_n(x,\xi)}{\partial x^{2\alpha}}+\tilde{u}_n(x,\xi)+\tilde{u}_n^2(x,\xi)-x^\alpha\xi^\alpha-x^{2\alpha}\xi^{2\alpha}\right](\mathrm{d}\xi)^\alpha \tag{20}$$

and set:

$$\delta_\alpha u_{n+1}=\delta_\alpha u_n+\delta_\alpha\frac{1}{\Gamma(1+\alpha)}\int_0^t \lambda_\alpha(t,\xi)\cdot$$

$$\cdot\left[\frac{\partial^{2\alpha}u_n(x,\xi)}{\partial\xi^{2\alpha}}-\frac{\partial^{2\alpha}\tilde{u}_n(x,\xi)}{\partial x^{2\alpha}}+\tilde{u}_n(x,\xi)+\tilde{u}_n^2(x,\xi)-x^\alpha\xi^\alpha-x^\alpha\xi^{2\alpha}\right](\mathrm{d}\xi)^\alpha=0$$

Then using $\delta_\alpha\tilde{u}_n=0$, we have:

$$\delta_\alpha u_{n+1}=\delta_\alpha u_n\left[1-\left.\frac{\partial^\alpha\lambda_\alpha(t,\xi)}{\partial\xi^\alpha}\right|_{\xi=t}\right]+\lambda_\alpha(t,\xi)\,\delta_\alpha\left.\frac{\partial^\alpha u(x,\xi)}{\partial\xi^\alpha}\right|_{\xi=t}+$$

$$+\delta_\alpha\frac{1}{\Gamma(1+\alpha)}\int_0^t\frac{\partial^{2\alpha}\lambda_\alpha(x,\xi)}{\partial\xi^{2\alpha}}\frac{\partial^\alpha u(x,\xi)}{\partial\xi^\alpha}(\mathrm{d}\xi)^\alpha=0$$

namely:

$$1-\left.\frac{\partial^\alpha\lambda_\alpha(t,\xi)}{\partial\xi^\alpha}\right|_{\xi=t}=0 \tag{21}$$

$$\lambda_\alpha(t,\xi)\big|_{\xi=t}=0 \tag{22}$$

$$\left.\frac{\partial^{2\alpha}\lambda(x,\xi)}{\partial\xi^{2\alpha}}\right|_{\xi=t}=0 \tag{23}$$

Solving eqs. (21)-(23), we obtain:

$$\lambda(x,\xi)=\frac{(\xi-t)^\alpha}{\Gamma(1+\alpha)}$$

Thus, the functional (20) reads:

$$u_{n+1}=u_n+\frac{1}{\Gamma(1+\alpha)}\int_0^t\frac{(\xi-t)^\alpha}{\Gamma(1+\alpha)}\left[\frac{\partial^{2\alpha}u_n(x,\xi)}{\partial\xi^{2\alpha}}-\frac{\partial^{2\alpha}u_n(x,\xi)}{\partial x^{2\alpha}}+u_n(x,\xi)+\right.$$

$$\left.+u_n^2(x,\xi)-x^\alpha\xi^\alpha-x^{2\alpha}\xi^{2\alpha}\right](\mathrm{d}\xi)^\alpha$$

Secondly, we select $u_0=x^\alpha t^\alpha$ and obtain the following successive approximations:

$$u_1=u_0+\frac{1}{\Gamma(1+\alpha)}\int_0^t\frac{(\xi-t)^\alpha}{\Gamma(1+\alpha)}\left[u_0(x,\xi)+u_0^2(x,\xi)-x^\alpha\xi^\alpha-x^{2\alpha}\xi^{2\alpha}\right](\mathrm{d}\xi)^\alpha=x^\alpha t^\alpha$$

$$u_2=u_1+\frac{1}{\Gamma(1+\alpha)}\int_0^t\frac{(\xi-t)^\alpha}{\Gamma(1+\alpha)}\left[u_1(x,\xi)+u_1^2(x,\xi)-x^\alpha\xi^\alpha-x^{2\alpha}\xi^{2\alpha}\right](\mathrm{d}\xi)^\alpha=x^\alpha t^\alpha$$

$$u_3=u_2+\frac{1}{\Gamma(1+\alpha)}\int_0^t\frac{(\xi-t)^\alpha}{\Gamma(1+\alpha)}\left[u_2(x,\xi)+u_2^2(x,\xi)-x^\alpha\xi^\alpha-x^{2\alpha}\xi^{2\alpha}\right](\mathrm{d}\xi)^\alpha=x^\alpha t^\alpha$$

$$\vdots$$

and so on.

Finally, we reach an exact solution of eq. (17):

$$u=\lim_{n\to\infty}u_n(x)=x^\alpha t^\alpha \tag{24}$$

In particular, if we let $\alpha=1$ then solution (24) becomes $u=xt$, which is the exact solution of the known inhomogeneous non-linear system [30]:

$$\frac{\partial^2u(x,t)}{\partial t^2}=\frac{\partial^2u(x,t)}{\partial x^2}+u+u^2-xt-x^2t^2 \tag{25}$$

subject to boundary conditions:

$$u(0,t)=0,\quad u(\pi,t)=\pi t \tag{26}$$

and the initial conditions:

$$u(x,0)=0,\quad \frac{\partial u(x,0)}{\partial t}=x \tag{27}$$

Discussion and conclusion

In the open literature, there are many analytical methods for fractional differential equations, for example, the exp-function method [31], the direct algebraic method [32, 33], the variational approach [34-36], Fourier spectral method [37] and the reproducing kernel method [38], and the frequency analysis method [39], this paper shows the VIM is as effective as the homotopy perturbation method for fractional calculus. The examples show the solution process is simple, and the results are of high accuracy. This paper concludes that the VIM is a powerful tool for fractional calculus.

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