# APPROXIMATE ANALYTICAL SOLUTION OF TIME-FRACTIONAL NON-LINEAR HEAT EQUATION VIA FRACTIONAL POWER SERIES METHOD

by

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*A time-fractional heat equation arising in a quiescent medium is established, and its approximate analytical solution is obtained by the fractional power series method. The results show that the method performs extremely well in terms of efficiency and simplicity.*

Key words: *fractional heat equations, Caputo fractional derivative, fractional power series*

## Introduction

In this work, we study the following time-fractional non-linear heat equation:

$$\mathrm{D}_t^{\alpha} u(x,t) = a\frac{\partial}{\partial x}\left(\mathrm{e}^{\lambda u}\frac{\partial u}{\partial x}\right),\ \ 0<\alpha\le 1 \tag{1}$$

with the initial condition:

$$u(x,0) = f(x) \tag{2}$$

where $\lambda$ and $a$ are constants. The fractional power series method (FPSM) [1] is used to study eq. (1), which governs an unsteady heat transfer in a quiescent medium in the case where the thermal diffusivity is exponentially dependent on temperature [2-4].

In eq. (1), $\mathrm{D}_t^{\alpha} u$ denotes the Caputo fractional derivative of $u$ defined as [5-7]:

$$\mathrm{D}_t^{\alpha} u(x,t) = J_t^{1-\alpha}\left[\frac{\partial u(x,t)}{\partial t}\right] \tag{3}$$

where $J_t^{1-\alpha}$ is the Riemann-Liouville fractional integral operator given by [5-7]:

$$J_t^{1-\alpha} u(x,t) = \frac{1}{\Gamma(\mu)}\int_0^t (t-s)^{-\alpha} u(x,s)\mathrm{d}s \tag{4}$$

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The following properties hold true [5, 6]:

$$J_t^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)} t^{\gamma+\alpha}, \quad \alpha \ge 0, \quad \gamma > -1 \tag{5}$$

$$J_t^{\alpha} \mathrm{D}_t^{\alpha} u(x,t) = u(x,t) - u(x,0), \quad 0 < \alpha < 1 \tag{6}$$

It is generally difficult to solve non-linear fractional partial differential equations. In the last two decades, some analytical methods for solving non-linear differential equations can be extended to solve fractional differential equations, for example, the variational method [8-13], the variational iteration method [14-18], the homotopy perturbation method [19, 20], the direct algebraic method [21, 22], the exp-function method [23], the Fourier spectral method [24], the reproducing kernel method [25]. The FPSM was proposed by El-Ajou *et al.* [1], and recently, many researchers [26-31] have obtained the series solution of fractional differential equations by using FPSM.

## Fractional power series

The fractional power series and its applicability for various kinds of partial differential equations are given in [1, 26-31]. It is an extension of the well-known Taylor series method [32] to fractional calculus. For the convenience of the reader, we will recall some basic concepts.

*Definition 1*. A power series representation of the form:

$$\sum_{n=0}^{\infty} c_n (t-t_0)^{n\alpha} = c_0 + c_1(t-t_0)^{\alpha} + c_2(t-t_0)^{2\alpha} + \cdots \tag{7}$$

where $0 \le m-1 < \alpha \le m, m \in N^+$ and $t \ge t_0$ is called a fractional power series (FPS) about $t_0$, where $t$ is a variable and $c_n$ are the coefficients of the series.

*Theorem 1*. We have the following two cases for the $\Sigma_{n=0}^{\infty} c_n t^{n\alpha}, \quad t \ge 0$:

– If $\Sigma_{n=0}^{\infty} c_n t^{n\alpha}$ converges when $t = b > 0$, then it converges whenever $0 \le t < b$.
– If $\Sigma_{n=0}^{\infty} c_n t^{n\alpha}$ diverges when $t = d > 0$, then it diverges whenever $t > d$.

*Theorem 2*. For the series $\Sigma_{n=0}^{\infty} c_n t^{n\alpha}, t \ge 0$, there are only three possibilities:

– The series converges only when $t = 0$,
– The series converges for each $t \ge 0$,
– There is a positive real number $R$ such that the series converges whenever $0 \le t < R$ and diverges whenever $t > R$.

*Theorem 3*. The series $\Sigma_{n=0}^{\infty} c_n t^{n\alpha} \; -\infty < t < \infty$ has a radius of convergence $R$ if and only if the series $\Sigma_{n=0}^{\infty} c_n t^{n\alpha}, t \ge 0$ has a radius of convergence $R^{1/\alpha}$.

The following property plays an important role in the next section:

*Theorem 4*. Suppose that the FPS has a radius of $\Sigma_{n=0}^{\infty} c_n t^{n\alpha}$ convergence $R > 0$. If $f(t)$ is a function defined by on $0 \le t < R$, $f(t) = \Sigma_{n=0}^{\infty} c_n t^{n\alpha}$ then for $m-1 < \alpha \le m$ and $0 \le t < R$, we have:

$$\mathrm{D}^{\alpha} f(t) = \sum_{n=0}^{\infty} c_n \frac{\Gamma(n\alpha+1)}{\Gamma[(n-1)\alpha+1]} t^{(n-1)\alpha} \tag{8}$$

## Solution of the problem

In this section, we solve the initial problem:

$$\begin{gathered} \mathrm{D}_t^\alpha u(x,t) = a\frac{\partial}{\partial x}\left(\mathrm{e}^{\lambda u}\frac{\partial u}{\partial x}\right) \\ u(x,0) = f(x) \end{gathered} \tag{9}$$

We will derive the algorithms of the FPSM for solving the problem.
Firstly, let:

$$w(x,t) = \mathrm{e}^{\lambda u(x,t)} \tag{10}$$

Then the initial problem becomes:

$$\begin{gathered} D_t^\alpha \ln[w(x,t)] = a\frac{\partial^2 w}{\partial x^2} \\ w(x,0) = \exp\left[\lambda f(x)\right] \end{gathered} \tag{11}$$

Now, suppose that the function $\ln[w(x, t)]$ takes the form:

$$\ln[w(x,t)] = \sum_{k=0}^{\infty} a_k(x) t^{k\alpha} \tag{12}$$

Then the solution of eq. (11) is decomposed:

$$w(x,t) = \sum_{k=0}^{\infty} H_k(x) t^{k\alpha} \tag{13}$$

where

$$\begin{gathered} H_0(x) = \exp[a_0(x)] \\ H_1(x) = a_1(x)\exp[a_0(x)] \\ \vdots \\ H_n(x) = \frac{1}{n!}\frac{\mathrm{d}}{\mathrm{d}T^n}\left\{\exp\left[\sum_{k=0}^{\infty} a_k(x) T^k\right]\right\}_{T=0} \end{gathered} \tag{14}$$

and so on.

Thus, we obtain:

$$\mathrm{D}_t^\alpha(\ln w) = \sum_{k=1}^{\infty} a_k(x)\frac{\Gamma(k\alpha+1)}{\Gamma[(k-1)\alpha+1]} t^{(k-1)\alpha}$$

and

$$\frac{\partial^2 w}{\partial x^2} = \sum_{k=0}^{\infty}\frac{\mathrm{d}^2}{\mathrm{d}x^2}[H_k(x)] t^{k\alpha}$$

So, by (11), we can conclude that:

$$\sum_{k=1}^{\infty} a_k \frac{\Gamma(k\alpha+1)}{\Gamma[(k-1)\alpha+1]} t^{(k-1)\alpha} = a\sum_{k=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x^2}[H_k(x)]t^{k\alpha} \tag{15}$$

Hence:

$$a_{k+1}(x) = \frac{a\Gamma(k\alpha+1)}{\Gamma[(k+1)\alpha+1]} \frac{\mathrm{d}^2}{\mathrm{d}x^2}[H_k(x)], \quad (k=0,1,2,\cdots n) \tag{16}$$

By the initial condition, we have:

$$a_0(x) = \lambda f(x) \tag{17}$$

$$H_0(x) = \exp[\lambda f(x)] \tag{18}$$

Therefore, from eqs. (14) and (16), we can obtain $a_k(x)$ and $H_k(x)$, successively. For example, we have:

$$a_1(x) = \frac{a}{\Gamma(1+\alpha)} \frac{\mathrm{d}^2}{\mathrm{d}x^2} \exp[\lambda f(x)]$$

$$H_1(x) = a_1(x)\exp[\lambda f(x)]$$

$$a_2(x) = \frac{a\Gamma(\alpha+1)}{\Gamma(1+2\alpha)} \frac{\mathrm{d}^2}{\mathrm{d}x^2}[H_1(x)]$$

$$H_2(x) = \frac{1}{2}[a_1^2(x) + 2a_2(x)]\exp[\lambda f(x)]$$

$$a_3(x) = \frac{a\Gamma(2\alpha+1)}{\Gamma(1+3\alpha)} \frac{\mathrm{d}^2}{\mathrm{d}x^2}[H_2(x)]$$

and so on.

Finally, we get the solution of the initial value problem (9):

$$u(x,t) = \frac{1}{\lambda}\sum_{k=0}^{\infty} a_k(x)t^{k\alpha}$$

## Application

Now, we study the following problem:

$$\begin{gathered} \mathrm{D}_t^{\alpha} u(x,t) = a\frac{\partial}{\partial x}\left(\mathrm{e}^{2u}\frac{\partial u}{\partial x}\right) \\ u(x,0) = \ln(x+1) \end{gathered} \tag{19}$$

Following the algorithm presented in the previous section, we have:

$$a_0(x) = 2\ln(x+1)$$

$$a_1(x)=\frac{2}{\Gamma(1+\alpha)}$$

$$a_2(x)=\frac{4}{\Gamma(1+2\alpha)}$$

$$a_3(x)=\frac{4}{\Gamma(1+3\alpha)}\frac{\Gamma(2\alpha+1)+2\Gamma^2(\alpha+1)}{\Gamma^2(\alpha+1)}$$

and so on.

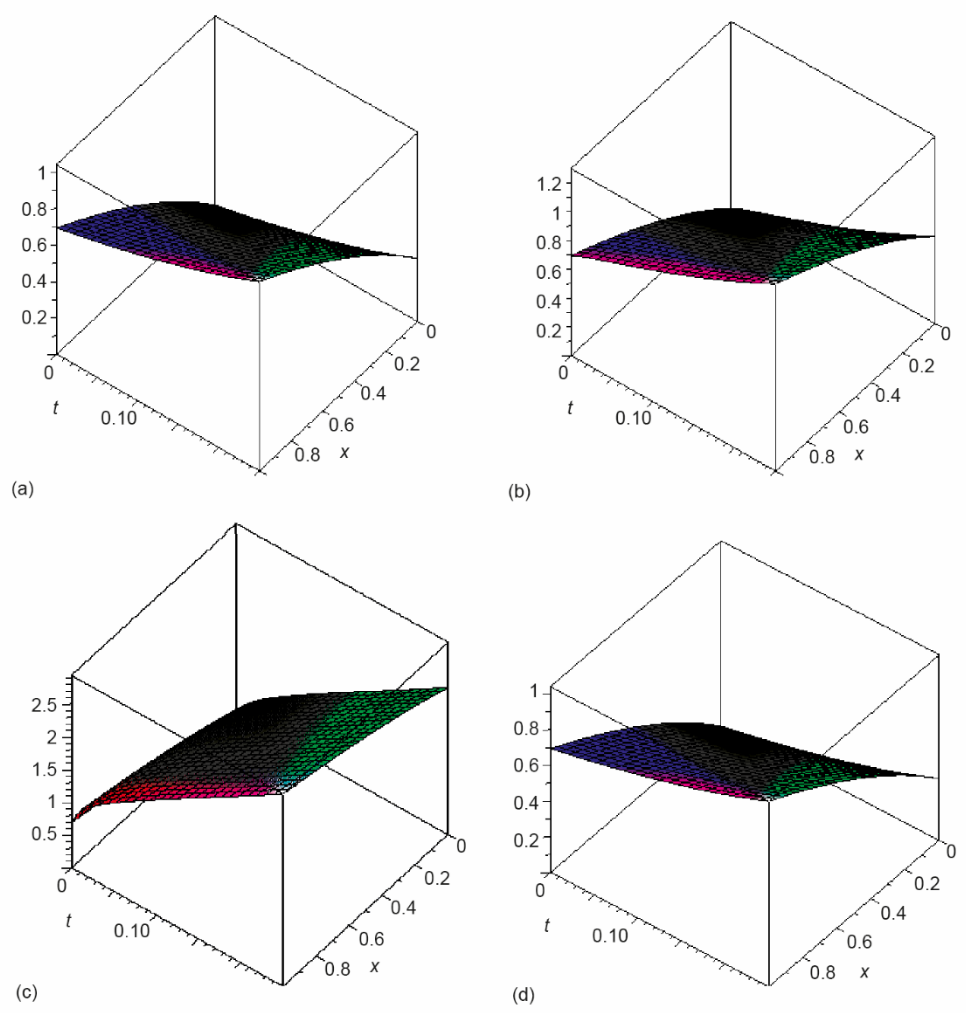


**Figure 1. For the problem (19), FPSM result for *u*(*x*, *t*) is, respectively, (a) *α* = 1, (b) *α* = 0.8, (c) *α* = 0.5, and (d) exact solution**

Thus, we obtain:

$$u(x,t)=\ln(1+x)+\frac{1}{\Gamma(1+\alpha)}t^{\alpha}+\frac{2}{\Gamma(1+2\alpha)}t^{2\alpha}+\cdots \tag{20}$$

When $\alpha=1$, we have:

$$u(x,t)=\ln(1+x)+t+t^2+\frac{4}{3}t^3+\cdots=\ln(1+x)-\frac{1}{2}\ln(1-2t) \tag{21}$$

which is the exact solution.

The exact solution (21) and FPSM solution (20) for different particular cases of $\alpha$ are presented graphically in fig. 1.

## Conclusion

In this paper, we presented the application of FPSM to a time-fractional non-linear heat equation. The results show that FPSM is a powerful tool for solving non-linear differential equations having wide applications in science and engineering.

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