DISCRETE WEIBULL-RAYLEIGH DISTRIBUTION
Properties and Parameter Estimations

by

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In this paper, a new distribution is introduced based on a continuous Weibull--Rayleigh distribution, that is a new three-parameter lifetime model called as the discrete Weibull-Rayleigh distribution. It is a distribution allowing for a bathtub-shaped hazard rate function. Its mathematical properties are discussed, and the estimation of model parameters is compared by the maximum likelihood and the least square methods. An actual data set of thermal aging is fitted to the new model, which shows the superiority of the proposed distribution.

Key words: *Weibull-Rayleigh distribution, maximum likelihood estimation, discrete Weibull-Rayleigh distribution bathtub-shaped failure rate, the least square estimation*

Introduction

In many applied sciences, such as medicine, engineering, and finance, among others, modeling and analyzing lifetime data is crucial. Several lifetime distributions have been used to model such kinds of data. The quality of the procedures used in a statistical analysis depends heavily on the selected probability models. Because of this, many scholars have extended the Weibull distribution in order to simulate more complex life data. In the past few decades, a large number of continuous life distributions have been proposed, such as Mudholkar and Kolya proposed generalized Weibull distribution [1], a three-parameter exponentiated -Weibull distribution was introduced by Mouthkar [2].

In tests, however, the lifetime measurement of a component or system is often a discrete value, sometimes, the data might be stochastically uncertain [3], and the distributed state estimation becomes important [4] for uncertain data and missing data [5]. So, some scholars put forward the discretization idea of continuous distribution and gave some different methodologies. Nakagawa and Osaki [6], Stein and Dattero [7], and Padgett and John [8] proposed three different discrete versions of the Weibull distribution. According to Nakagawa and Osaki [6], the discretization idea is: If the underlying continuous random variable X has the survival function $S_X(x) = P(X \geq x)$, then the random variable $Y = [X]$ is largest integer less or equal to X will have the probability mass function (PMF):

$$P(Y = y) = P(y \leq X < y+1) = P(X \geq y) - P(X \geq y+1) =$$

$$S_X(y) - S_X(y+1), \quad y = 0,1,2,... \tag{1}$$

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So, given any continuous distribution, it is possible to generate corresponding discrete distribution using eq. (1). The PMF of a random variable Y can be viewed as a discrete concentration of the probability density function (PDF) of X. It is well known that the geometric distribution is the discrete analogue of the exponential distribution, while the negative binomial distribution is the discrete analogue of the gamma distribution. Recently, Bebbington *et al.* [9] introduced the discrete additive Weibull, the discrete modified Weibull distribution [10], and the discrete Rayleigh distributions [11], Bourguignon *et al.* [12] proposed discrete reduced modified Weibull distribution, and Ademola [13] further discussed the properties of discrete reduced modified Weibull distribution.

The bathtub-shaped hazard rate function is widely used in many applications in reliability and lifetime analysis. Many continuous distributions with bathtub-shaped hazard rate function have been studied, such as Nadarajah [14], Duan and Liu [15]. But only a few discrete distributions have bathtub-shaped hazard rate functions.

In this paper, a discrete Weibull-Rayleigh distribution is introduced. By means of maximum likelihood and least square methods, the model parameters and some reliability quantities are estimated and compared. By the maximum likelihood and least square methods, a heat aging life data is fitted to the proposed distribution and compared with discrete reduced modified Weibull (DRMW) distribution and discrete Weibull (DW) distribution.

Weibull-Rayleigh distribution

A random variable X is said to have the Rayleigh distribution with the parameter θ, if its PDF is given by:

$$g(x,\theta) = \theta x \exp\left(-\frac{\theta x^2}{2}\right), \quad \theta > 0, \quad x \ge 0 \tag{2}$$

The corresponding cumulative distribution function (CDF) is given by:

$$G(x,\theta) = 1 - \exp\left(-\frac{\theta x^2}{2}\right), \quad \theta > 0, \quad x \ge 0\,, \tag{3}$$

where θ denotes the scale parameter.

Marcelo and Elbatal [16] introduced and studied in generality a family of univariate distributions with two additional parameters, similarly as the extend Weibull families, using the Weibull generator applied to the odds radio $[G(x)]/[1-G(x)]$. The term *generator* means that each baseline distribution G has different distributions.

A random variable, its baseline CDF is $G(x;\zeta)$ which depends on a parameter vector ζ, and its PDF is $g(x;\zeta)$. The CDF of Weibull distribution is:

$$F(x) = 1 - \exp\{-\alpha x^{\beta}\}, \quad \alpha, \beta > 0, \quad x \ge 0.$$

Based on this density, by replacing x with $[G(x;\zeta)]/[1-G(x;\zeta)]$. The CDF of Weibull-generalized distribution, say Weibull-G distribution with two extra parameters α and β, and is defined by:

$$F(x;\alpha,\beta,\zeta) = \int_0^{\frac{G(x;\zeta)}{1-G(x;\zeta)}} \alpha\beta t^{\beta-1} e^{-\alpha t^{\beta}} \, dt = 1 - \exp\left\{-\alpha\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\beta}\right\} \tag{4}$$

The corresponding family PDF becomes:

$$f(x;\alpha,\beta,\zeta)=\alpha\beta g(x;\zeta)\frac{[G(x;\zeta)]^{\beta-1}}{[1-G(x;\zeta)]^{\beta+1}}\exp\left\{-\alpha\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\beta}\right\},\quad \alpha,\beta>0 \tag{5}$$

A random variable X with PDF eq. (5) is denoted by $X\sim$ Weibull-G(α,β,ζ). The additional parameters are induced by the Weibull generator are sought as a manner to furnish a more flexible distribution.

Using the power series and generalized binomial theorem for the exponential function, we obtain the Weibull-G density function by some simple derivation:

$$f(x)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{i,j}g(x;\zeta)[G(x;\zeta)]^{\beta(i+1)+j-1}$$

where

$$\omega_{i,j}=\frac{(-1)^{i}\alpha^{i+1}\beta\,\Gamma[\beta(i+1)+j+1]}{i!\,j!\Gamma[\beta(i+1)+1]}$$

So, let us substitute the CDF $G(x;\theta)$ and PDF $g(x;\theta)$ of the Rayleigh distribution with the parameter θ into eq. (4) and eq. (5). The CDF of the Weibull Rayleigh distribution is given by:

$$F(x;\alpha,\beta,\theta)=1-\exp[-\alpha(\mathrm{e}^{\frac{\theta}{2}x^2}-1)^{\beta}],\quad \alpha,\beta>0,\quad \theta>0,\quad x\geq 0 \tag{6}$$

The PDF of the Weibull-Rayleigh distribution is:

$$f(x;\alpha,\beta,\theta)=\alpha\beta\theta x\mathrm{e}^{\frac{\theta}{2}x^2}(\mathrm{e}^{\frac{\theta}{2}x^2}-1)^{\beta-1}\exp[-\alpha(\mathrm{e}^{\frac{\theta}{2}x^2}-1)^{\beta}] \tag{7}$$

and the hazard rate function is:

$$h(x;\alpha,\beta,\theta)=\alpha\beta\theta x\mathrm{e}^{\frac{\theta x^2}{2}}\left(\mathrm{e}^{\frac{\theta}{2}x^2}-1\right)^{\beta\text{-}1} \tag{8}$$

Merovci and Elbatal [16] illustrated some of the possible shapes of the hazard function of the Weibull-Rayleigh distribution.

Discrete Weibull-Rayleigh distribution

The probability mass function and the hazard rate function

If the continuous random variable X obeys Weibull-Rayleigh distribution, then the random variable $Y=[X]$ that will have the PMF, where $[X]$ denotes the largest integer less or equal to X.

From eq. (6), and let $\mathrm{e}^{-\alpha}=p$, $\mathrm{e}^{\theta/2}=q$, we can obtain the survival function of Weibull-Rayleigh distribution:

$$S(x;p,q,\beta)=p^{(q^{x^2}-1)^{\beta}},\quad 0<p<1,\quad q>1,\quad \beta>0; x\geq 0 \tag{9}$$

So, let us substitute eq. (9) into eq. (1), getting the PMF of discrete Weibull-Rayleigh distribution, fig. 1:

$$P(Y=y)=S_X(y)-S_X(y+1)=p^{(q^{y^2}-1)^\beta}-p^{[q^{(y+1)^2}-1]^\beta}$$
$$y=0,1,2,...,n \quad 0<p<1, \quad q>1, \quad \beta>0$$

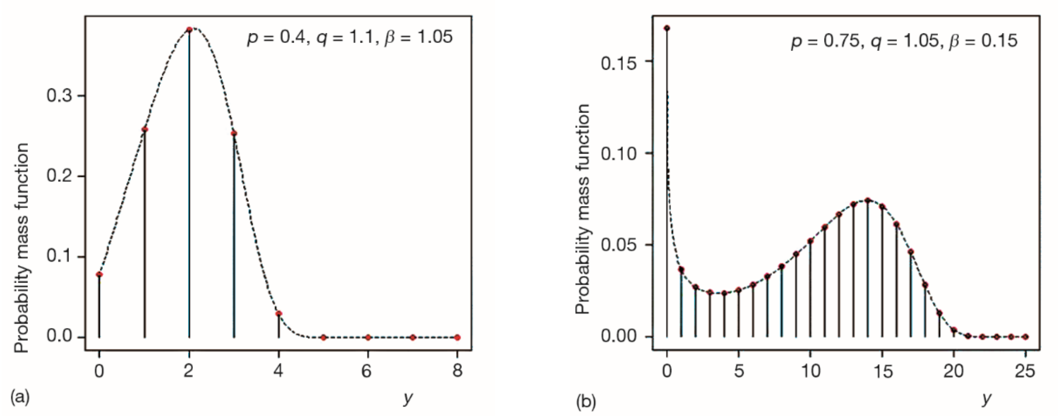

Figure 1. The probability mass function of discrete Weibull-Rayleigh distribution

Then the hazard rate function of discrete Weibull-Rayleigh distribution is:

$$h(y)=1-p^{[q^{(y+1)^2}-1]^\beta-(q^{y^2}-1)^\beta}, \quad y=0,1,2,...,n \quad 0<p<1, \quad q>1, \quad \beta>0$$

As shown in the fig. 2, the discrete Weibull-Rayleigh distribution has increasing and bathtub-shapes hazard rate function.

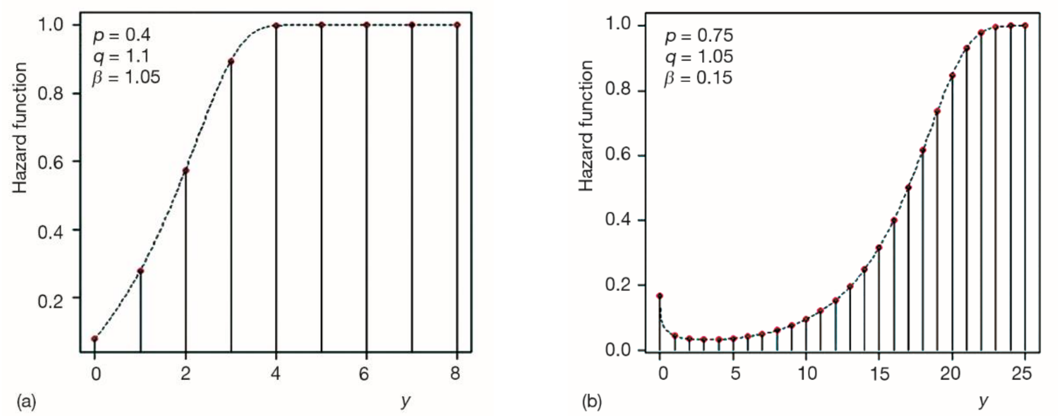

Figure 2. The hazard rate function of discrete Weibull-Rayleigh distribution

Moment of discrete Weibull-Rayleigh distribution

The r^{th} moment of discrete Weibull-Rayleigh distribution is given by:

$$\mu_r'=E(Y^r)=\sum_{y=0}^{\infty}y^r p(y)=\sum_{y=0}^{\infty}y^r\left\{p^{(q^{y^2}-1)^\beta}-p^{[q^{(y+1)^2}-1]^\beta}\right\}$$

In particular, see tab 1:

– The mean μ of a lifetime following discrete Weibull-Rayleigh distribution can be obtained:

$$\mu=\mu_1'=\sum_{y=0}^{\infty} y\{p^{(q^{y^2}-1)^\beta}-p^{[q^{(y+1)^2}-1]^\beta}\}$$

– The second moment is given by:

$$\mu_2'=\sum_{y=0}^{\infty} y^2\{p^{(q^{y^2}-1)^\beta}-p^{[q^{(y+1)^2}-1]^\beta}\}$$

– The variance V of discrete Weibull-Rayleigh distribution can be obtained:

$$V=\mu_2'-\mu^2$$

– The 3rd moment is given by:

$$\mu_3'=\sum_{y=0}^{\infty} y^3\{p^{(q^{y^2}-1)^\beta}-p^{[q^{(y+1)^2}-1]^\beta}\}$$

– The 4rd moment is given by:

$$\mu_4'=\sum_{y=0}^{\infty} y^4\{p^{(q^{y^2}-1)^\beta}-p^{[q^{(y+1)^2}-1]^\beta}\}$$

– The skewness S of discrete Weibull-Rayleigh distribution can be obtained:

$$S=\frac{\mu_3'-3\mu_2'\mu+2\mu^3}{V^{\frac{3}{2}}}$$

– The kurtosis K of discrete Weibull-Rayleigh distribution can be obtained:

$$K=\frac{\mu_4'-4\mu_3'\mu+6\mu_2'\mu^2-3\mu^4}{V^2}$$

– The index of dispersion, variance-to-mean ratio (VMR), of discrete Weibull-Rayleigh distribution can be obtained:

$$VMR=\frac{V}{\mu}$$

Table 1. Mean, variance, skewness, kurtosis of discrete Weibull-Rayleigh distribution for various values of *p*, *q*, and *β*

p	*q*	*β*	*μ*	*V*	*S*	*K*	*VMR*
0.5	1.3	2	1.031	0.151	0.290	6.519	0.146
0.5	1.3	1.2	1.082	0.377	–0.049	2.618	0.349
0.5	1.3	0.2	1.662	3.085	0.604	2.028	1.857
0.8	1.7	2	0.896	0.093	–2.601	7.770	0.104
0.8	3.7	2.5	0.069	0.064	3.400	12.558	0.931

Estimation of parameters of discrete Weibull-Rayleigh distribution

Let n items be put on test and their lifetimes are recorded as $Y_1, Y_2, ..., Y_n$. If the $Y_i (i = 1, 2, ..., n)$ is assumed to be simple random sample from discrete Weibull-Rayleigh distribution. We discuss the maximum likelihood estimation (MLE) and the least square estimation (LSE) of parameters.

Maximum likelihood estimation

The likelihood function is given by:

$$L(p,q,\beta;Y)=\prod_{i=1}^{n}p(y_i)=\prod_{i=1}^{n}\{p^{(q^{y_i^2}-1)^{\beta}}-p^{[q^{(y_i+1)^2}-1]^{\beta}}\}$$

The log-likelihood function is as:

$$l(p,q,\beta;Y)=\sum_{i=1}^{n}\ln\{p^{(q^{y_i^2}-1)^{\beta}}-p^{[q^{(y_i+1)^2}-1]^{\beta}}\}$$

In maximum likelihood method, the first derivative of likelihood function for $l(p, q, \beta; Y)$, for p, q, and β is:

$$\frac{\mathrm{d}l}{\mathrm{d}p}=\sum_{i=1}^{n}\frac{[q^{(y_i)^2}-1]^{\beta}p^{[q^{(y_i)^2}-1]^{\beta}-1}-[q^{(y_i+1)^2}-1]^{\beta}p^{[q^{(y_i+1)^2}-1]^{\beta}-1}}{p^{[q^{(y_i)^2}-1]^{\beta}}-p^{[q^{(y_i+1)^2}-1]^{\beta}}}$$

$$\frac{\mathrm{d}l}{\mathrm{d}q}=\sum_{i=1}^{n}\frac{\beta\ln p\{[q^{(y_i)^2}-1]^{\beta\text{-}1}p^{[q^{(y_i)^2}-1]^{\beta}}(y_i)^2q^{(y_i)^2\text{-}1}-[q^{(y_i+1)^2}-1]^{\beta\text{-}1}p^{[q^{(y_i+1)^2}-1]^{\beta}}(y_i+1)^2q^{(y_i+1)^2-1}\}}{p^{[q^{(y_i)^2}-1]^{\beta}}-p^{[q^{(y_i+1)^2}-1]^{\beta}}}$$

$$\frac{\mathrm{d}l}{\mathrm{d}\beta}=\sum_{i=1}^{n}\frac{\ln p\{[q^{(y_i)^2}-1]^{\beta}p^{[q^{(y_i)^2}-1]^{\beta}}\ln[q^{(y_i+1)^2}-1]-[q^{(y_i+1)^2}-1]^{\beta}p^{[q^{(y_i+1)^2}-1]^{\beta}}\ln[q^{(y_i+1)^2}-1]\}}{p^{[q^{(y_i)^2}-1]^{\beta}}-p^{[q^{(y_i+1)^2}-1]^{\beta}}}$$

By solving the previous system of three non-linear equations, we can obtain MLE of p, q, and β. It is clear that the system cannot be solved in a closed-form. It has to be solved numerically.

According to Miller [17], if $\hat{p}$, $\hat{q}$, and $\hat{\beta}$ are the MLE of p, q, and β. The asymptotic distribution of $\hat{p}$, $\hat{q}$, and $\hat{\beta}$ is normal distribution, $\sqrt{n}(\hat{p}-p,\hat{q}-q,\hat{\beta}-\beta)$ $\sim N[0, I^{-1}(p,q,\beta)]$, where $I(p,q,\beta)$ is the Fisher information matrix, and:

$$I(p,q,\beta)=\begin{pmatrix} -E(\dfrac{\partial^2 l}{\partial^2 p}) & -E(\dfrac{\partial^2 l}{\partial p\partial q}) & -E(\dfrac{\partial^2 l}{\partial p\partial\beta}) \\ -E(\dfrac{\partial^2 l}{\partial p\partial q}) & -E(\dfrac{\partial^2 l}{\partial^2 q}) & -E(\dfrac{\partial^2 l}{\partial q\partial\beta}) \\ -E(\dfrac{\partial^2 l}{\partial p\partial\beta}) & -E(\dfrac{\partial^2 l}{\partial q\partial\beta}) & -E(\dfrac{\partial^2 l}{\partial^2\beta}) \end{pmatrix}$$

We then assess the accuracy of MLE by the observed Fisher information matrix $I(p,q,\beta)$ in $(\hat{p},\hat{q},\hat{\beta})$:

$$I(\hat{p},\hat{q},\hat{\beta}) = \begin{pmatrix} -\dfrac{\partial^2 l}{\partial^2 p} & -\dfrac{\partial^2 l}{\partial p \partial q} & -\dfrac{\partial^2 l}{\partial p \partial \beta} \\ -\dfrac{\partial^2 l}{\partial p \partial q} & -\dfrac{\partial^2 l}{\partial^2 q} & -\dfrac{\partial^2 l}{\partial q \partial \beta} \\ -\dfrac{\partial^2 l}{\partial p \partial \beta} & -\dfrac{\partial^2 l}{\partial q \partial \beta} & -\dfrac{\partial^2 l}{\partial^2 \beta} \end{pmatrix}_{\hat{p},\hat{q},\hat{\beta}} \approx \begin{bmatrix} \mathrm{Var}(p) & \mathrm{Cov}(p,q) & \mathrm{Cov}(p,\beta) \\ & \mathrm{Var}(q) & \mathrm{Cov}(q,\beta) \\ & & \mathrm{Var}(\beta) \end{bmatrix}_{\hat{p},\hat{q},\hat{\beta}}$$

This would also help us compare distributions to use Akaike information criterion.

Least square estimation

In order to compare with MLE, we provide a brief discussion on the LSE of the unknown parameters:

$$S(p,q,\beta \mid y_i) = \sum_{i=1}^{n} \left[1 - p^{(q^{y_i^2} - 1)^{\beta}} - \frac{i}{n+1} \right]^2 \tag{11}$$

From eq. (11), we find the minimum value of the unknown parameters.

Thermal aging life data analysis

In this section, we illustrate the flexibility of the proposed distribution for thermal aging life data, tab. 2. The fit of the proposed distribution is compared with that of discrete reduced modified Weibull (DRMW) distribution and discrete Weibull (DW) distribution.

Table 2. Failure data of thermal aging life for enameled wires with composite insulation layers [18]

Temperature [°C]	Failure times [min]
210	3853, 3853, 4523, 5193, 5193, 5193, 5863, 5863, 5863, 5863, 5863,5863, 5863, 5863, 5863, 6871, 6871, 7015, 7375, 8023, 8023
230	1086, 1420, 1587, 1754, 1921, 2255, 2255, 2422, 2422, 2422, 2589, 2589, 2589, 2589, 3093, 3237, 3237, 3237, 4101, 4101, 4101
250	406, 268, 268, 383, 291, 268, 452, 245, 475, 268, 475, 406, 314, 291, 567, 383, 475, 475, 521, 406, 429

Since the unit of three sets of failure time data is minute, the three sets of data can be considered as discrete data. The MLE and LSE of parameters can be calculated by R software, and the *p*-value under MLE and LSE can be obtained.

From tab. 3, we can see there is little difference between MLE and LSE of parameters. The *p*-value of DWR under MLE is bigger than under LSE.

Then the maximum likelihood method is used to estimate the unknown parameters, the DWR distribution is used to fit the thermal aging life data and compares with DRMW distribution (the survival function is: $q^{\sqrt{x}(1+bc^x)}$, $0<q<1$, $b>1$, $c \geq 0$), and DW distribution (the survival function is: $q^{x^{\theta}}$, $0<q<1$, $\theta>0$). In order to better explain the fitting effect of the distribution, the following information criteria can be used.

Table 3. Comparison of MLE and LSE of the parameters under the sample

Temperature [°C]	MLE			LSE		
	Parameter		*p*-value	Parameter		*p*-value
	$\hat{p}$	0.7730		$\hat{p}$	0.8875	
210	$\hat{q}$	1.0001	0.2681	$\hat{q}$	1.0002	0.1454
	$\hat{\beta}$	1.2119		$\hat{\beta}$	1.3998	
	$\hat{p}$	0.8701		$\hat{p}$	0.8567	
230	$\hat{q}$	1.0667	0.1069	$\hat{q}$	1.0008	0.5174
	$\hat{\beta}$	0.0110		$\hat{\beta}$	1.0440	
	$\hat{p}$	0.9272		$\hat{p}$	0.9156	
250	$\hat{q}$	1.0010	0.7009	$\hat{q}$	1.0012	0.5474
	$\hat{\beta}$	0.0131		$\hat{\beta}$	0.0103	

The Akaike information criterion (AIC) due to Akaik [19], defined by:

$$AIC = 2k - 2L(\hat{p}, \hat{q}, \hat{\beta}; y_i)$$

The Bayesian information criterion (BIC) due to Schwarz [20, 21], defined by:

$$BIC = k\ln(n) - 2L(\hat{p}, \hat{q}, \hat{\beta}; y_i)$$

The consistent AIC (CAIC) due to Hurvich and Tsai [22], defined by:

$$CAIC = AIC + \frac{2k(k+1)}{n-k-1}$$

$L(\hat{p}, \hat{q}, \hat{\beta}; y_i)$ is likehood function value based on the parameter estimations.

Table 4. Comparison of DWP distribution and DW distribution under the sample (210 ℃)

Distribution	$\hat{p}$	$\hat{q}$	$\hat{\beta}$	b	$\hat{c}$	*AIC*	*BIC*	*CAIC*	*p*-value
DWR	0.7730	1.0001	1.2119			7.5460	10.8578	8.9577	0.2681
DW		0.9998	4.6564			7.9996	11.1332	9.4113	0.0508
DMW		0.9997		1.5411	1.0518	7.9994	11.1330	9.4112	0.0530

Table 5. Comparison of DWP distribution and DW distribution under the sample (230 ℃)

Distribution	$\hat{p}$	$\hat{q}$	$\hat{\beta}$	b	$\hat{c}$	*AIC*	*BIC*	*CAIC*	*p*-value
DWR	0.8701	1.0667	0.0110			7.7402	10.8737	9.1519	0.1069
DW		0.9995	4.8367			7.9990	11.1325	9.4107	0.0578
DMW		0.9996		16.6829	1.0633	7.9992	11.1328	9.4110	0.2516

Table 6. Comparison of DWP distribution and DW distribution under the sample (250 ℃)

Distribution	$\hat{p}$	$\hat{q}$	$\hat{\beta}$	b	$\hat{c}$	*AIC*	*BIC*	*CAIC*	*p*-value
DWR	0.9297	1.0010	0.0131			7.8544	10.9880	8.5602	0.7009
DW		0.9998	4.6564			7.9997	11.1331	9.4115	0.1271
DMW		0.9995		8.2072	1.8854	7.9989	11.1325	9.4107	0.0700

From tabs. 4-6, we will see that it gives the smallest value for *AIC*, *BIC*, *CAIC*, and the biggest *p*-value, so discrete Weibull-Rayleigh distribution gives the best fit for each temperature.

Conclusion

We have introduced a three-parameter discrete Weibull-Rayleigh distribution based on Weibull-Rayleigh. We have shown that the hazard rates function of this distribution exhibits increasing and bathtub-shapes, and it is more flexible than Weibull-Rayleigh distribution. For the heat aging life data set, making maximum likelihood estimate and the LSE. Furthermore, according to the K-S test and Akaike information criterion, the proposed distribution was shown to be given a better fit than discrete reduced modified Weibull distribution and discrete Weibull distribution.

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