A SHORT REVIEW ON APPROXIMATE ANALYTICAL METHODS FOR NON-LINEAR PROBLEMS

by

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In this paper, some approximate analytical methods are reviewed, including the variational iteration method and the homotopy perturbation method. Based on the ideas of the two methods, a new method for solving a class of system of Fredholm integral equations of the second kind is proposed. Some examples are given to show the simple solution process and the accuracy of the solution for each method. The emphasis of this review paper focuses on Ji-Huan He's unapproachable ideas and the mathematics beauty, so the references are not exhaustive.

Key words: *variational iteration method, homotopy perturbation method, Fredholm integral equation of the second kind*

Introduction

The PDE arise in many physical fields like the condensed matter physics, fluid mechanics, plasma physics and optics, *etc.* The investigation of analytical solutions plays an important role in the study of physical systems, and it is one of the central themes in mathematics and physics as well. In the past decades, many advanced methods were developed to obtain analytical solutions of PDE, among which the homotopy perturbation method (HPM) [1] and the variational iteration method (VIM) [2] are the most used tool in engineering.

The Taylor series method is simple and accessible to all engineers, but it is low convergence hinders its wide applications [3]. The exp-function method can lead to the exact solutions, but its complex calculation makes those inaccessible who are not familiar with some mathematics software [4-6]. The variational-based methods [7-14] can obtain a globally valid solution. However, it is extremely difficult to establish a needed variational principle for a complex non-linear problem. Among all analytical methods, the HPM [1] and the VIM [2] are mathematically beautiful and physically meaningful.

In this paper, we first review the VIM and the HPM. Some examples are given to show the simple solution process and high accuracy of the two methods. Inspired by the idea of Ji-Huan He's unapproachable ideas and the mathematics beauty of the two methods, the author proposes a new method for solving a class of system of Fredholm integral equations of the second kind, the solution process leads to solving a system of algebraic equations, which seems rather difficult to be solved by hand, even by software, such as MATHEMATICA, MAPLE, and MATLAB, this paper suggests an effective method for this purpose, that is Wu's method [15].

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The variational iteration method

The VIM was first proposed in the later 1990's [2], and was further developed by Ji-Huan He and his students [16-19], and it is now thoroughly used by mathematicians to handle a wide variety of non-linear equations, the publications arose gradually and 2000 saw more than 300 articles on the VIM according to Clarivate Analytics' Web of Science, fig. 1.

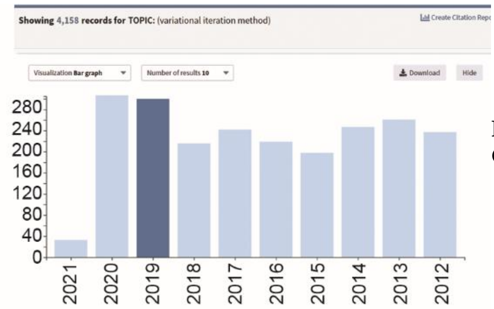

Figure 1. Publications on the VIM according to Clarivate's Web of Science (March 2, 2021)

It has been proved that this method is effective and reliable for various approximate analytical purposes. The method gives rapidly convergent successive approximations of the exact solution if such a solution exists.

Consider the following differential equation

$$Lu + Nu = g(t) \tag{1}$$

where L and N are linear and non-linear operators, respectively, and $g(t)$ – the inhomogeneous source term.

The VIM presents a correction functional for eq. (1) in the following form:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\xi)[Lu_n(\xi) + N\bar{u}_n(\xi) - g(\xi)]\mathrm{d}\xi \tag{2}$$

where λ is a general Lagrange multiplier and $\bar{u}_n(\xi)$ – the restricted variation.

It is obvious that the main steps of the He's VIM require first the determination of the Lagrange multiplier, $\lambda(\xi)$. Integration by parts is usually used for the determination of the Lagrange multiplier, $\lambda(\xi)$. In other words, we can use:

$$\int \lambda(\xi)u_n'(\xi)\mathrm{d}\xi = \lambda(\xi)u_n(\xi) - \int \lambda'(\xi)u_n(\xi)\mathrm{d}\xi \tag{3}$$

$$\int \lambda(\xi)u_n''(\xi)\mathrm{d}\xi = \lambda(\xi)u_n'(\xi) - \lambda'(\xi)u_n(\xi) + \int \lambda''(\xi)u_n(\xi)\mathrm{d}\xi \tag{4}$$

and so on.

The original VIM uses the variational theory [1] to optimally identify the Lagrange multiplier involved in the variational iteration algorithm. Now the problem is completely changed, the Laplace transform can make the identification extremely simple [17]. If we have determined the Lagrange multiplier, $\lambda(\xi)$, the successive approximations u_{n+1}, $n \geq 0$ of the solution u will be readily obtained upon using any selective function u_0. Consequently, the solution:

$$u = \lim_{n\to\infty} u_n$$

Remark 1

The determination of the Lagrange multiplier plays an important role in the determination of the solution of the problem. In what follows, we summarize some iteration formulae that show ODE, its corresponding Lagrange multipliers, and its correction functional, respectively [18]:

$$\text{(i)}\begin{cases} u' + f[u(\xi), u'(\xi)] = 0, \ \lambda = -1 \\ u_{n+1} = u_n - \int_0^x [u_n' + f(u_n, u_n')]\,\mathrm{d}\xi \end{cases}$$

$$\text{(ii)}\begin{cases} u'' + f[u(\xi), u'(\xi), u''(\xi)] = 0, \ \lambda = (\xi - x) \\ u_{n+1} = u_n + \int_0^x (\xi - x)[u_n'' + f(u_n, u_n', u_n'')]\mathrm{d}\xi \end{cases}$$

$$\text{(iii)}\begin{cases} u''' + f[u(\xi), u'(\xi), u''(\xi), u'''(\xi)] = 0, \ \lambda = -\frac{1}{2!}(\xi - x)^2 \\ u_{n+1} = u_n - \int_0^x \frac{1}{2!}(\xi - x)^2[u_n''' + f(u_n, \cdots, u_n''')]\mathrm{d}\xi \end{cases}$$

$$\text{(iv)}\begin{cases} u^{(\mathrm{iv})} + f[u(\xi), u'(\xi), u''(\xi), u'''(\xi), u^{(\mathrm{iv})}(\xi)] = 0, \ \lambda = \frac{1}{3!}(\xi - x)^3 \\ u_{n+1} = u_n + \int_0^x \frac{1}{3!}(\xi - x)^3[u_n''' + f(u_n, u_n', \cdots, u_n^{(iv)})]\mathrm{d}\xi \end{cases}$$

$$\cdots$$

$$\text{(n)}\begin{cases} u^{(n)} + f[u(\xi), u'(\xi), \cdots, u^{(\mathrm{n})}(\xi)] = 0, \ \lambda = (-1)^n \frac{1}{(n-1)!}(\xi - x)^{(n-1)} \\ u_{n+1} = u_n + (-1)^n \int_0^x \frac{1}{(n-1)!}(\xi - x)^{(n-1)}[u_n''' + f(u_n, \cdots, u_n^{(n)})]\mathrm{d}\xi \end{cases}$$

Example 1

Consider the following inhomogeneous PDE [20]:

$$u_x + u_y = 2x + 2y, \qquad u(x,0) = x^2, \qquad u(0,y) = y^2 \tag{5}$$

The correction functional for eq. (5) is:

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^x \lambda(\xi)\left[\frac{\partial u_n(\xi,y)}{\partial \xi} + \frac{\partial \overline{u_n}(\xi,y)}{\partial y} - 2\xi - 2y\right]\mathrm{d}\xi \tag{6}$$

Using eqs. (3) and (4), the stationary conditions:

$$1+\lambda|_{\xi=x}=0 \tag{7}$$

$$\lambda'|_{\xi=x}=0 \tag{8}$$

Equations (7) and (8) give the result:

$$\lambda=-1$$

substituting this value of the Lagrange multiplier $\lambda = -1$ into the functional (6) gives the iteration formula:

$$u_{n+1}(x,y)=u_n(x,y)-\int_0^x\left[\frac{\partial u_n(\xi,y)}{\partial \xi}+\frac{\partial \overline{u_n}(\xi,y)}{\partial y}-2\xi-2y\right]\mathrm{d}\xi, \quad n\geq 0 \tag{9}$$

as stated before, we can select $u_0(x, y) = u(0, y) = y^2$ from the given conditions. Using this selection into eq. (9) we obtain the following successive approximations:

$$u_0(x,y)=y^2$$

$$u_1(x,y)=y^2-\int_0^x\left[\frac{\partial u_0(\xi,y)}{\partial \xi}+\frac{\partial u_0(\xi,y)}{\partial y}-2\xi-2y\right]\mathrm{d}\xi=x^2+y^2$$

$$u_2(x,y)=x^2+y^2-\int_0^x\left[\frac{\partial u_1(\xi,y)}{\partial \xi}+\frac{\partial u_1(\xi,y)}{\partial y}-2\xi-2y\right]\mathrm{d}\xi=x^2+y^2$$

$$u_3(x,y)=x^2+y^2-\int_0^x\left[\frac{\partial u_2(\xi,y)}{\partial \xi}+\frac{\partial u_2(\xi,y)}{\partial y}-2\xi-2y\right]\mathrm{d}\xi=x^2+y^2$$

$$\vdots$$

$$u_n(x,y)=x^2+y^2$$

the VIM admits the use of:

$$u(x,y)=\lim_{n\to\infty}u_n(x,y) \tag{10}$$

that gives the exact solution by:

$$u(x,y)=x^2+y^2 \tag{11}$$

Example 2

Consider the following Volterra integral equation [21]:

$$u(x)=1-\frac{1}{2}x^2+\int_0^x u(t)\mathrm{d}t \tag{12}$$

Using Leibnitz rule to differentiate both sides of eq. (12) once gives:

$$u'(x)+x-u(x)=0 \tag{13}$$

Substituting x = 0 into eq. (12) gives the initial condition $u(0)$ = 1, the correction functional for eq. (13) is:

$$u_{n+1}(x)=u_n(x)+\int_0^x \lambda(\xi)[u_n'(\xi)+\xi-\overline{u_n}(\xi)]\mathrm{d}\xi \tag{14}$$

using the formula (i) given below in the **Remark1** leads to:

$$\lambda = -1 \tag{15}$$

Substituting this value of the Lagrange multiplier λ = –1 into the functional (14) gives the iteration formula:

$$u_{n+1}(x)=u_n(x)-\int_0^x [u_n'(\xi)+\xi-u_n(\xi)]\mathrm{d}\xi \tag{16}$$

As stated before, we can use the initial conditions to select $u_0(x) = u(0) = 1$. Using this selection into eq. (14) gives the following successive approximations:

$$u_0(x)=1$$

$$u_1(x)=1-\int_0^x [u_0'(\xi)+\xi-u_0(\xi)]\mathrm{d}\xi = 1+x-\frac{1}{2}x^2$$

$$u_2(x)=1+x-\frac{x^2}{2}-\int_0^x [u_1'(\xi)+\xi-u_1(\xi)]\mathrm{d}\xi = 1+x-\frac{x^3}{6}$$

$$u_3(x)=1+x-\frac{x^2}{6}-\int_0^x [u_2'(\xi)+\xi-u_2(\xi)]\mathrm{d}\xi = 1+x-\frac{x^4}{24}$$

$$u_4(x)=1+x-\frac{x^4}{24}-\int_0^x [u_3'(\xi)+\xi-u_3(\xi)]\mathrm{d}\xi = 1+x-\frac{x^5}{120}$$

$$\vdots$$

The VIM admits the use of:

$$u(x)=\lim_{n\to\infty} u_n(x) \tag{17}$$

that gives the exact solution by:

$$u(x)=1+x \tag{18}$$

The homotopy perturbation method

The HPM was introduced and developed by Ji-Huan He in the late 1990's [1] and was further developed by Ji-Huan He and his student [22-37]. The method couples a homotopy technique of topology and a perturbation technique. A homotopy with an embedding parameter $p \in [0, 1]$ is constructed, and the impeding parameter p is considered a small parameter. The coupling of the perturbation method and the homotopy method has eliminated the limitations of the traditional perturbation technique, and now the method has become a completely matured tool for various problems, and there have been many modifications. Figure 2 shows publications on the HPM according to Clarivate's Web of Science per year.

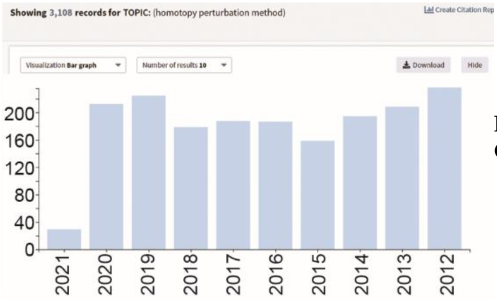

Figure 2. Publications on the HPM according to Clarivate's Web of Science (March 2, 2021)

The modification with an auxiliary parameter [27, 28] is highly suitable for a non-linear vibration system, and the modification with two expanding parameters [29] or three expansion [30] is for complex problems with more than one non-linear term. The Elzaki transform makes the solution process much simpler [31]. He's multiple scales method [32] and the reducing rank method [33, 34] are two effective modifications of the HPM for non-linear oscillators. Li-He's modification is very much promising, it is to improve the original differential equation to a higher-order partner, then the HPM is used for the resultant equation. This modification is called as Li-He's modified homotopy perturbation [35, 36] or the higher-order HPM [37]. The HPM is now extended to fractal or fractional differential equations [38-42] and can deal with various periodic and instable properties of non-linear systems [43-45].

In the next, we illustrate the HPM to handle Fredholm integral equations of the first kind of the form:

$$f(x) = \int_a^b K(x,t)u(t)\mathrm{d}t \tag{19}$$

We now define the operator:

$$L(u) = f(x) - \int_a^b K(x,t)u(t)\mathrm{d}t = 0 \tag{20}$$

and construct a convex homotopy of the form:

$$H(u,p) = (1-p)u(x) + pL(u)(x) = 0 \tag{21}$$

The embedding parameter p monotonically increases from 0 to 1. The HPM admits the use of the expansion:

$$u = \sum_{n=0}^{\infty} p^n u_n \tag{22}$$

and consequently:

$$u(x)=\lim_{p\to 1}\sum_{n=0}^{\infty}p^n u_n(x) \tag{23}$$

The series (23) converges to the exact solution if such a solution exists by setting $p = 1$.

Substituting eq. (22) into eq. (21), we obtain the recurrence relations:

$$u_0(x)=0 \tag{24}$$

$$u_1(x)=f(x) \tag{25}$$

$$u_{n+1}(x)=u_n(x)-\int_a^b K(x,t)u_n(t)\mathrm{d}t,\quad n\geq 1 \tag{26}$$

Remark 2

If the kernel is separable, *i.e.* $K(x, t) = g(x)h(t)$, then the following condition:

$$\left|1-\int_a^b K(t,t)\mathrm{d}t\right|<1 \tag{27}$$

must be justified for convergence.

Example 3

Consider the following Fredholm integral equation [23]:

$$\frac{1}{2}\mathrm{e}^{3x}=\int_0^{1/2}\mathrm{e}^{3x-3t}u(t)\mathrm{d}t \tag{28}$$

Noticing that:

$$\left|1-\int_0^{1/2}K(t,t)\mathrm{d}t\right|=0<1 \tag{29}$$

and using the recurrence relation (24)-(26), we find:

$$u_0(x)=0$$

$$u_1(x)=\frac{1}{2}\mathrm{e}^{3x}$$

$$u_{n+1}(x)=u_n(x)-\int_0^{1/2}\mathrm{e}^{3x-3t}u_n(t)\mathrm{d}t,\quad n\geq 1$$

This in turn gives:

$$u_0(x)=0$$

$$u_1(x)=\frac{1}{2}\mathrm{e}^{3x}$$

$$u_2(x)=u_1(x)-\int_0^{1/2}\mathrm{e}^{3x-3t}u_1(t)\mathrm{d}t=\frac{1}{4}\mathrm{e}^{3x}$$

$$u_3(x)=u_2(x)-\int_0^{1/2}\mathrm{e}^{3x-3t}u_2(t)\mathrm{d}t=\frac{1}{8}\mathrm{e}^{3x}$$

$$u_4(x)=u_3(x)-\int_0^{1/2}\mathrm{e}^{3x-3t}u_3(t)\mathrm{d}t=\frac{1}{16}\mathrm{e}^{3x}$$

and so on. Consequently, the approximate solution is given by:

$$u(x)=\mathrm{e}^{3x}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right) \tag{30}$$

That converges to the exact solution:

$$u(x)=\mathrm{e}^{3x} \tag{31}$$

A new analytical method for solving a class of system of Fredholm integral equations

In this section, we will study systems of non-linear Fredholm integral equations of the second kind given by:

$$u(x)=f_1(x)+\int_a^b\{K_1(x,t)F_1[u(t)]+\overline{K_1}(x,t)\overline{F_1}[v(t)]\}\mathrm{d}t$$

$$v(x)=f_2(x)+\int_a^b\{K_2(x,t)F_2[u(t)]+\overline{K_2}(x,t)\overline{F_2}[v(t)]\}\mathrm{d}t \tag{32}$$

The unknown functions $u(x)$ and $v(x)$, that will be determined later, occur inside and outside the integral sign. The kernels $K_1(x, t)$, $K_2(x, t)$, $\overline{K_1}(x,t), \overline{K_2}(x,t)$, and the function $f_1(x)$ and $f_2(x)$ are given real-valued polynomial functions, $F_1[u(t)], \overline{F_1}[v(t)], F_2[u(t)], \overline{F_2}[v(t)]$, are linear or non-linear functions of $u(x)$ and $v(x)$. The $K_1(x, t)$, $K_2(x, t)$, $\overline{K_1}(x,t), \overline{K_2}(x,t)$ are degenerate or separable kernels of the form:

$$K_1(x,t)=\sum_{k=1}^{n}g_k(x)h_k(t),\quad \overline{K_1}(x,t)=\sum_{k=1}^{n}\overline{g_k}(x)\overline{h_k}(t) \tag{33}$$

$$K_2(x,t)=\sum_{k=1}^{n}r_k(x)s_k(t),\quad \overline{K_2}(x,t)=\sum_{k=1}^{n}\overline{r_k}(x)\overline{s_k}(t) \tag{34}$$

Step1. Substitute eqs. (33) and (34) into the system (32) to obtain:

$$u(x)=f_1(x)+\sum_{k=1}^{n}g_k(x)\int_a^b h_k(t)F_1[u(t)]\mathrm{d}t+\sum_{k=1}^{n}\overline{g_k}(x)\int_a^b\overline{h_k}(t)\overline{F_1}[v(t)]\mathrm{d}t$$

$$v(x)=f_2(x)+\sum_{k=1}^{n}r_k(x)\int_a^b s_k(t)F_2[u(t)]\mathrm{d}t+\sum_{k=1}^{n}\overline{r_k}(x)\int_a^b \overline{s_k}(t)\overline{F_2}[v(t)]\mathrm{d}t \tag{35}$$

Step2. Each integral at the right side depends only on variable t with constant limits of integration for t. This means that each integral is equivalent to a constant. Based on this, eq. (35) becomes:

$$u(x)=f_1(x)+\alpha_1 g_1(x)+\cdots+\alpha_n g_n(x)+\beta_1\overline{g_1}(x)+\cdots+\beta_n\overline{g_n}(x)$$
$$v(x)=f_2(x)+\gamma_1 r_1(x)+\cdots+\gamma_n r_n(x)+\delta_1\overline{r_1}(x)+\cdots+\delta_n\overline{r_n}(x) \tag{36}$$

where

$$\alpha_i=\int_a^b h_i(t)F_1[u(t)]\mathrm{d}t,\quad 1\le i\le n \tag{37}$$

$$\beta_i=\int_a^b \overline{h_i}(t)\overline{F_1}[v(t)]\mathrm{d}t,\quad 1\le i\le n \tag{38}$$

$$\gamma_i=\int_a^b s_i(t)F_2[u(t)]\mathrm{d}t,\quad 1\le i\le n \tag{39}$$

$$\delta_i=\int_a^b \overline{s_i}(t)\overline{F_2}[v(t)]\mathrm{d}t,\quad 1\le i\le n \tag{40}$$

Step 3. Substituting eq. (36) into eq. (32) gives a system of algebraic equations about α_i, β_i, γ_i, δ_i ($1\le i\le n$), the system of algebraic equations is difficult to solve, we use Wu's method [15] to solve it.

Example 4

Let us consider the following non-linear system [46]:

$$f_1(x)=x-\frac{5}{18}+\frac{1}{3}\int_0^1[f_1(t)+f_2(t)]\mathrm{d}t$$

$$f_2(x)=x^2-\frac{2}{9}+\frac{1}{3}\int_0^1[f_1^2(t)+f_2(t)]\mathrm{d}t \tag{41}$$

This system can be rewritten as:

$$f_1(x)=x-\frac{5}{18}+\frac{1}{3}(c_1+c_2)$$

$$f_2(x)=x^2-\frac{2}{9}+\frac{1}{3}(c_3+c_2) \tag{42}$$

where

$$c_1 = \int_0^1 f_1(t)\mathrm{d}t, \quad c_2 = \int_0^1 f_2(t)\mathrm{d}t, \quad c_3 = \int_0^1 f_1^2(t)\mathrm{d}t \tag{43}$$

Substitute eq. (42) into eq. (41), we obtain:

$$\frac{1}{3}(-c_1 - c_2) + \frac{1}{3}\left(\frac{1}{3} + \frac{c_1}{3} + \frac{2c_2}{3} + \frac{c_3}{3}\right) = 0$$

$$\frac{1}{972}[-79 - 36c_1^2 + 168c_2 - 36c_2^2 - 24c_1(2 + 3c_2) + 216c_3] = 0 \tag{44}$$

Using the algebraic form of Wu's method to solve eq. (44), we have:

$$\begin{matrix} -1 + 6c_1 + 6c_3 = 0 \\ 3c_2 - 2 + 3c_3 = 0 \end{matrix} \quad \text{and} \quad \begin{matrix} 53 + 6c_1 - 6c_3 = 0 \\ 3c_2 - 56 + 3c_3 = 0 \end{matrix} \tag{45}$$

which lead to:

$$\begin{matrix} c_1 + c_2 = \dfrac{5}{6} \\ c_2 + c_3 = \dfrac{2}{3} \end{matrix} \quad \text{and} \quad \begin{matrix} c_1 + c_2 = \dfrac{59}{6} \\ c_2 + c_3 = \dfrac{56}{3} \end{matrix} \tag{46}$$

Substituting eq. (46) into eq. (42), we obtain:

$$f_1(x) = x, \quad f_2(x) = x^2 \tag{47}$$

and

$$f_1(x) = x + 3, \quad f_2(x) = x^2 + 6 \tag{48}$$

Remark 3

By using the method proposed in [46], only the solution (47) is obtained, well the solution (48) is not obtained.

Conclusions

In this paper, the VIM and the HPM are reviewed firstly, inspired by the idea of the two methods. The author proposes a new method for solving a class of system of Fredholm integral equations of the second kind, the solution process can be come down to solving a system of algebraic equations, which rather difficult to be solved, the algebraic form of Wu's method is used to solve this problem, and very well results are obtained.

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