TWO ANALYTICAL METHODS FOR TIME FRACTIONAL CAUDREY-DODD-GIBBON-SAWADA-KOTERA EQUATION

by

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This paper focuses on solving the time fractional Caudrey-Dodd-Gibbon-Sawada-Kotera equation (FCDGSKE). We propose two analytical methods based on the fractional complex transform, the variational iteration method and the homotopy perturbation method. The approximated solutions to the initial value problems associated with FCDGSKE are provided without linearization and complicated calculation. Numerical results show the main merits of the analytical approaches.

Key words: *time FCDGSKE, fractional complex transform, homotopy perturbation method, variational iteration method*

Introduction

Consider the FCDGSKE [1]:

$$u_t^\alpha + u_{xxxxx} + 30uu_{xxx} + 30u_x u_{xx} + 180u^2 u_x = 0 \tag{1}$$

where the fractional derivative $u_t^\alpha (0 < \alpha \le 1)$ is defined by:

$$u_t^\alpha = \frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{t_0}^{t} (s-t)^{n-\alpha-1} [u_0(x,s) - u(x,s)\mathrm{d}s \tag{2}$$

with a known function $u_0(x, t)$ [2, 3]. When $\alpha = 1$, the fractional eq. (1) reduces to the fifth-order CDGSKE [4]. Equation (1) with $\alpha = 1$ was also called as CDG equation [5, 6] or SK equation [7]. The physical understanding of CDGSKE was illustrated in [4].

The CDGSKE was widely applied in the area of fluid dynamics [4-7]. In the past decades, many different solutions to CDGSKE were developed by the analytical or numerical methods, including the dressing method [7], Darboux transformation [4], Backlund transformation in bilinear forms [8], Hirota's bilinear method [9], the exp-function method [10], the exp[$-\varphi(z)$]-expansion method [11], the Riemann theta function method [12], the variational approach [13, 14], the Fourier spectral method [15], and the reproducing kernel method [16], and so on. There are also some research results about the non-linear (2+1)-D CDGKSE, which can be seen as the general form of CDGSKE [17]. Recently, Baleanu *et al.* [1] investigated the exact solutions of FCDGSKE by the Lie symmetry analysis. In this paper, we aim to

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consider the numerical behavior of FCDGSKE. Motivated by the idea of the fractional complex transform [18-22], we propose two analytical approaches based upon the efficient methods, including the variational iteration method [23-29] and the homotopy perturbation method [30-35]. We name these two methods as FCT-VIM and FCT-HPM, respectively. Due to the efficiency of FCT-VIM and FCT-HPM, the approximations can be given with high accuracy. Numerical experiments with an initial value problem are presented to confirm the efficiency.

Fractional complex transform (FCT)

We consider a fractional PDE:

$$f(u, u_t^{\alpha}, u_x^{\beta}, u_x^{2\alpha}, u_x^{2\beta}, \ldots) = 0 \tag{3}$$

where $u_t^{\alpha} = [\partial^{\alpha} u(x,t)]/\partial t^{\alpha}$ denotes He's fractional derivation [2, 3] defined by eq. (2), the function $u(x, t)$ is continuous (but not necessarily differentiable), and $0 < \alpha < 1$, $0 < \beta < 1$. It is difficult to give the exact solutions of the fractional PDE. In order to give the analytical or numerical solutions of eq. (3), He and Li proposed a fractional complex transform [18-21]:

$$T = \frac{pt^{\alpha}}{\Gamma(1+\alpha)}, \quad X = \frac{qx^{\beta}}{\Gamma(1+\beta)} \tag{4}$$

with non-zero constants p and q [18-21]. In view of eq. (4), we have:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = p\frac{\partial u}{\partial T} \tag{5}$$

$$\frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} = q\frac{\partial u}{\partial X} \tag{6}$$

The fractional complex transform can be explained by the two-scale fractal theory [36-44], and it is also called the two-scale transform.

Then the original fractional partial differential eq. (3) can be transformed to an ordinary partial differential equation.

Analysis of the variational iteration method

We consider the following differential equation:

$$Lu + Nu = g(x) \tag{7}$$

where L is a linear operator, N – a non-linear operator, and $g(x)$ – the inhomogeneous term. Then we can construct a correct functional:

$$u_{n+1}(x) = u_n(x) - \int_0^x \lambda\{Lu_n(\xi) - N\tilde{u}_n(\xi) - g(\xi)\}\mathrm{d}\xi \tag{8}$$

where λ is a general Lagrange multiplier [23-29] which can be identified optimally *via* variational theory. The second term on the right is called the correction and $\tilde{u}_n$ is considered as a restricted variation, *i. e.* $\widetilde{\delta u_n} = 0$.

Analysis of the homotopy perturbation method

To illustrate the idea of the homotopy perturbation method [30-35], we consider the following non-linear differential equation:

$$A(u) - f(r) = 0, \; r \in \Omega \tag{9}$$

with boundary conditions;

$$B\left(u, \frac{\mathrm{d}u}{\mathrm{d}n}\right) = 0, \quad r \in \Gamma \tag{10}$$

where A is a general differential operator, B – a boundary operator, u – a known analytic function, and Γ – the boundary of the domain.

The operator A can be divided into two parts, L and N, where L is linear and N non-linear. Therefore eq. (9) can be rewritten:

$$L(u) + N(u) - f(r) = 0 \tag{11}$$

By the homotopy perturbation method [30-35], we can construct a homotopy $v(r, p)$: $\Omega\times[0, 1] \rightarrow R$ which satisfies:

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0 \tag{12}$$

or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \tag{13}$$

where $r \in \Gamma$ and $p \in [0, 1]$ is an embedding parameter, u_0 – an initial approximation of eq. (9), which satisfies the boundary conditions.

Assume that the solution of eq. (12) can be expressed as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + \ldots \tag{14}$$

Then the approximate solution of eq. (9) can be given by:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \ldots \tag{15}$$

Numerical experiments

We consider the initial value problem of FCDGSK eq. (1) with the following initial condition:

$$u(x,0) = \frac{1}{4}k^2 \operatorname{sech}^2\left(\frac{1}{2}kx + c\right) \tag{16}$$

where k and c are arbitrary constants. The single soliton solution to the classical CDGSK eq. (1) is given by [9]:

$$u(x,t) = \frac{1}{4}k^2 \operatorname{sech}^2\left(\frac{1}{2}kx - \frac{1}{2}k^5 t + c\right) \tag{17}$$

Application of FCT

By FCT technique with $T = t^\alpha/[\Gamma(1 + \alpha)]$, the previous initial value problem can be equivalently transformed to the ordinary PDE:

$$u_T + u_{xxxxx} + 30uu_{xxx} + 30u_xu_{xx} + 180u^2u_x = 0 \tag{18}$$

with the initial condition (16).

Application of FCT-VIM

By using VIM, it is easy to obtain the iteration formulae:

$$u_{n+1}(x,T) = u_n(x,T) + \int_0^T \lambda\{u_{n\xi}(x,\xi) + u_{nxxxxx}(x,\xi) + 30u_nu_{nxxx}(x,\xi) +$$
$$+30u_{nx}u_{nxx}(x,\xi) + 180u_n^2(x,\xi)u_{nx}(x,\xi)\}\mathrm{d}\xi \tag{19}$$

The stationary conditions are given by:

$$1 + \lambda = 0, \quad \lambda'\big|_{\xi=T} = 0 \tag{20}$$

which yields that $\lambda = -1$. Therefore, we have the following iteration formula:

$$u_{n+1}(x,T) = u_n(x,T) + \int_0^T \{u_{n\xi}(x,\xi) + u_{nxxxxx}(x,\xi) + 30u_nu_{nxxx}(x,\xi) +$$
$$+30u_{nx}u_{nxx}(x,\xi) + 180u_n^2(x,\xi)u_{nx}(x,\xi)\}\mathrm{d}\xi \tag{21}$$

We call this analytical approach based upon fractional complex transform and variational iteration method as FCT-VIM.

We begin with the initial approximation:

$$u_0 = \frac{1}{4}k^2\mathrm{sech}^2\left(\frac{1}{2}kx + c\right)$$

with $k = 0.5$ and $c = 0.5$, and obtain the first-order approximation:

$$u_1(x, T) = \mathrm{sech}^2(0.25x + 0.5)\{0.0625 + 0.00195312\tanh(0.25x + 0.5)T\cdot$$
$$\cdot[\mathrm{sech}^4(0.25x + 0.5)] + \tanh^4(0.25x + 0.5) +$$
$$+0.00390625T\mathrm{sech}^2(0.25x + 0.5)$$
$$\tanh^3(0.25x + 0.5)\} \tag{22}$$

The rest approximations can be given by eq. (21). Recalling the fractional complex transform $T = t^\alpha/[\Gamma(1 + \alpha)]$, we have the following approximation to eq. (1):

$$u_1(x,t) = \mathrm{sech}^2(0.25x + 0.5)\{0.0625 + 0.00195313\tanh(0.25x + 0.5)\frac{t^\alpha}{\Gamma(1+\alpha)}$$
$$\cdot[\mathrm{sech}^4(0.25x + 0.5) + \tanh^4(0.25x + 0.5)] +$$
$$+0.00390625\frac{t^\alpha}{\Gamma(1+\alpha)}\mathrm{sech}^2(0.25x + 0.5)\tanh^3(0.25x + 0.5)\} \tag{23}$$

The formulation of $u_2(x, t)$ is omitted here, due to the complexity of the expression.

Application of FCT-HPM

By HPM for (18), it follows the homotopy $v(r, p)$: $\Omega\times[0, 1] \to R$ satisfying that:

$$v_t(x, T) - v_{0t}(x, T) + pv_{0t}(x, T) + p[v_{xxxxx}(x, T) + 30v(x, T)v_{xxx}(x, T) + \\ + 30v_x(x, T)v_{xx}(x, T) + 180v^2(x, T)v_x(x, T)] = 0 \tag{24}$$

with the initial approximation:

$$v_0(x,T) = u(x,0) = \frac{1}{4}k^2\text{sech}^2\left(\frac{1}{2}kx + c\right)$$

Assume that the solution to eq. (18) is defined by:

$$v(x, T) = v_0(x, T) + pv_1(x, T) + p^2v_2(x, T) + \ldots \tag{25}$$

Substituting eq. (25) to eq. (24), and equating the terms of the same power of p, we have that:

$$p^1: v_{1T} + v_{0T} + v_{0xxxxx} + 30v_0v_{0xxx} + 30v_0v_{0xx} + 180v_0^2v_{0x} = 0, \quad v_1(x,0) = 0$$

$$p^2: v_{2T} + v_{1xxxxx} + 30v_0v_{1xxx} + 30v_1v_{0xxx} + 30v_{0x}v_{1xx} + 30v_{1x}v_{0xx} + 180v_0^2v_{1x} = 0, \quad v_2(x,0) = 0$$

$$\ldots$$

We name the above approach FCT-HPM. By the above equations with the given constants $k = c = 0.5$, it is easy to obtain the first order HPM solution:

$$u_1(x, T) = \text{sech}^2(0.25x + 0.5)\{0.0625 + 0.00195312\tanh(0.25x + 0.5)T\cdot \\ \cdot[\text{sech}^4(0.25x + 0.5) + \tanh^4(0.25x + 0.5)] + \\ + 0.00390625T\text{sech}^2(0.25x + 0.5)\tanh^3(0.25x + 0.5)\} \tag{26}$$

It is easy to find that the first-order approximation is the same as that by FCT-VIM. Finally, we have the fractional FCT-HPM solution defined by eq. (23). We remark that the second-order approximation can be given after the calculation of v_2.

Numerical comparisons

In this section, we consider the efficiency of FCT-VIM and FCT-HPM for solving eq. (1). For simplicity, the second-order approximations obtained by FCT-VIM and FCT-HPM are denoted by u_{VIM} and u_{HPM}, respectively.

We first give the numerical results for the classical CDGSK equation. Figure 1 shows the numerical behavior of the second-order approximated solutions and the exact solutions $u(x, t)$ when $\alpha = 1$. The approximated solutions given by FCT-VIM or FCT-HPM agree well with the exact solutions to CDGSKE. We then consider the behavior of the solutions to eq. (1) when the time t is set as $t = 5$. Figure 2 shows the curves of the approximations and the exact solutions, respectively. The absolute errors of u_{VIM} and u_{HPM} are given in fig. 3. The FCT-HPM performs slightly better than FCT-VIM in this example. We remark that the accuracy of the approximated solutions can be further improved by considering more iteration steps of FCT-VIM or FCT-HPM.

In order to further illustrate the effectiveness of FCT-VIM and FCT-HPM for FCDGSKE, we provide the numerical results of the approximations with different α and time t. The numerical solutions can be obtained without linearization, perturbation or complicated iterations. Figure 4 shows the behavior of the approximated solutions obtained by FCT-VIM and FCT-HPM for FCDGSKE with $\alpha = 0.5$. The numerical results for FCDGSKE with $\alpha = 0.8$ are plotted in fig. 5.

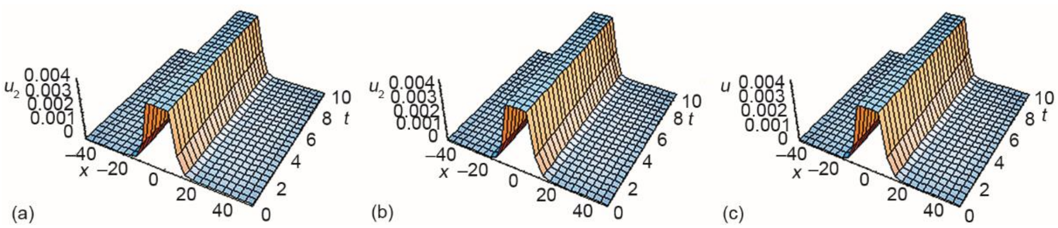

Figure 1. Results of (a) u_{VIM}, (b) u_{HPM}, and (c) u_{Exact} with $-50 \le x \le 50$ and $0 \le t \le 10$

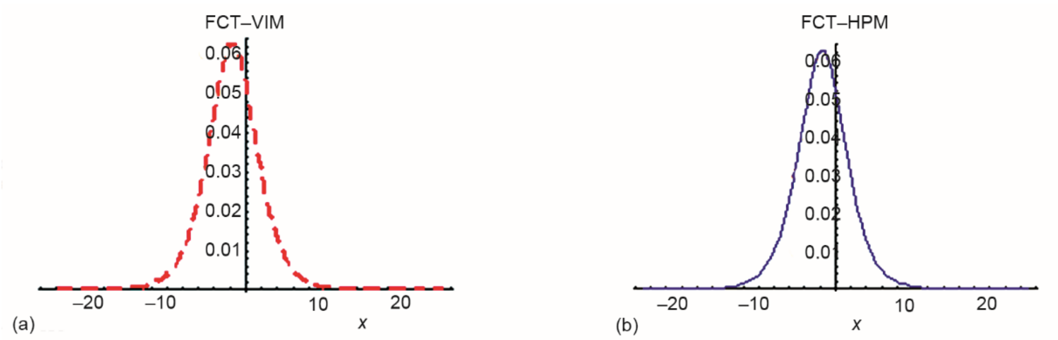

Figure 2. Results of (a) u_{VIM} and (b) u_{HPM} when $t = 5$

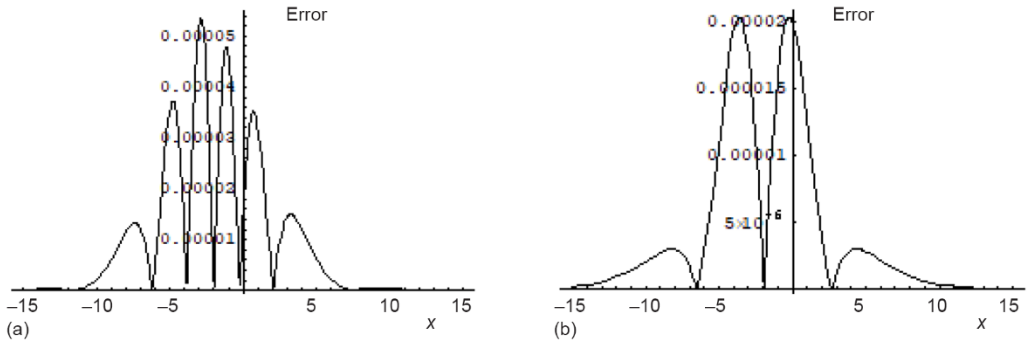

Figure 3. Numerical errors of (a) u_{VIM} and (b) u_{HPM} when $t = 5$

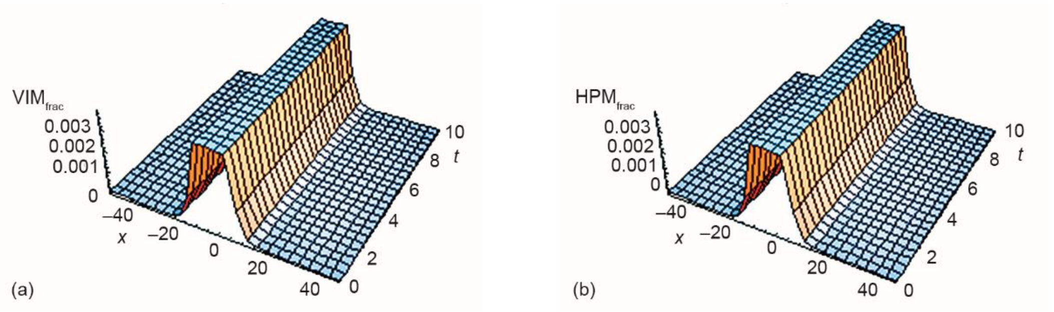

Figure 4. Fractional solutions of (a) u_{VIM} and (b) u_{HPM} with α = 0.5

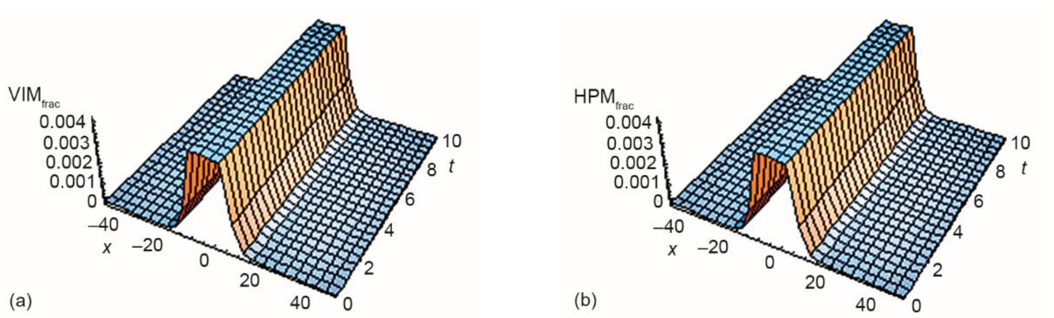

Figure 5. Fractional solutions of (a) u_{VIM} and (b) u_{HPM} with α = 0.8

Conclusion

This paper proposed two analytical approaches based on the fractional complex transform, the variational iteration method and the homotopy perturbation method. The initial value problem associated with the time FGDGSKE was used as an example to show the efficiency of these two methods. In future work, we will extend these analytical approaches to fractal differential equations [36-44].

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