INTERNAL SOLITARY WAVES IN THE OCEAN BY SEMI-INVERSE VARIATIONAL PRINCIPLE

by

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Internal solitary waves are very common physical phenomena in the ocean, which play an important role in the transport of marine matter, momentum and energy. The non-linear Schrodinger equation is suitable for describing the deep-sea internal wave propagation. Firstly, by designing skillfully, the trial-Lagrange functional, variational principles are successfully established for the non-linear Schrodinger equation by the semi-inverse method. Secondly, the constructed variational principle is proved by minimizing the functionals with the calculus of variations. Finally, different kinds of internal solitary waves are obtained by the semi-inverse variational principle for the non-linear Schrodinger equation.

Key words: *internal solitary wave, variational principle, semi-inverse method, non-linear Schrodinger equation*

Introduction

Ocean internal waves [1-3] are a kind of physical motion that occurs in the interior of fluid, and they exist in the world ocean. The study of internal waves in the ocean is of great significance to the theoretical research of ocean science, utilization of marine resources, as well as marine military and engineering. Ocean internal waves play an important role in ocean dynamics, which affect the transport of marine matter, momentum, and energy. At present, the well-known KdV equation is only suitable for describing the propagation of small amplitude internal waves in shallow water [4-10], but there will be large errors in modeling large-amplitude internal waves in the deep sea. For deep-sea internal waves, the Benjamin-Ono equation is derived by Benjamin [11] and Ono [12], while the intermediate longwave equation is derived by Kubota *et al.* [13] and Choi and Camassa [14] obtained the fully non-linear evolution equation of the internal wave at the two-layer interface. The derived equation can be reduced to the intermediate longwave equation when it is weakly non-linear and propagates along one direction, and can be reduced to the Benjamin-Ono equation in infinite water depth. Song *et al.* [15] established the non-linear Schrodinger (NLS) equation under two-layer stratification, trying to develop a more accurate equation of the ocean internal wave characteristics in a specific environment, to minimize the gap between the real value and the NLS equation. Solving the non-linear PDE with integer or fractional orders is always an attractive and hot

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topic for many researchers in different scientific fields, because of their outstanding ability for modeling non-linear phenomena [16-22]. Numerous mathematical techniques have been developed to explore the approximate and exact solutions [23-26], of which variational-based methods have been very useful and effective, such as the variational iteration method [27-30], and variational approximation method [31-35], *etc.* When contrasted with other methods, variational ones show some advantages. In this paper, the NLS equation of the ocean internal wave model is studied by the semi-inverse method [36-51], which was first proposed in 1997 by Dr. Ji-Huan He, who is a famous Chinese mathematician. At first, by designing skillfully, the trial-Lagrange functional, variational principle are successfully established for the NLS equation by the semi-inverse method. Then, the constructed variational principle is proved correct by minimizing the functionals with the calculus of variations. Furthermore, different kinds of internal solitary waves are obtained by the semi-inverse variational principle for the NLS equation.

Variational principles for internal solitary waves

The NLS equation is one of the most active research topics and is a fundamental equation of a wide range of physical phenomena, such as quantum mechanics, hydrodynamics, plasma physics, non-linear optics, self-focusing in laser pulses, propagation of heat pulses in crystals, description of the dynamics of Bose-Einstein condensate at extremely low temperatures. For inviscid fluids, ignoring the influence of Coriolis force, if the fluid is selected as a two-layer structure, the NLS equation for deep-sea internal waves can be derived from the continuity equation and Bernoulli equation.

$$-iA_t + \alpha A_{xx} + \beta |A|^2 A + \gamma A = 0 \tag{1}$$

which can describe the propagation of internal solitary waves in the ocean. In eq. (1), A represent complex amplitude fields and $i=\sqrt{-1}$, α is the dispersion coefficient, and β and γ are parameters of constant values, which respectively indicate the effects of non-linearity and linearity. On substituting $A(x, t) = q_1(x, t) + iq_2(x, t)$ into eq. (1), where q_1 and q_2 are the real-valued functions of t and x, we obtain the following PDE for q_1 and q_2 in real space:

$$-\frac{\partial q_1}{\partial t} + \alpha\frac{\partial^2 q_2}{\partial x^2} + \beta(q_1^2 + q_2^2)q_2 + \gamma q_2 = 0 \tag{2}$$

$$\frac{\partial q_2}{\partial t} + \alpha\frac{\partial^2 q_1}{\partial x^2} + \beta(q_1^2 + q_2^2)q_1 + \gamma q_1 = 0 \tag{3}$$

The target is searching for variational formulations whose stationary conditions satisfy eqs. (2) and (3). With the help of He's semi-inverse method [36], a trial-functional is constructed in the following form:

$$J(q_1, q_2) = \int_{t_1}^{t_2} \mathrm{d}t \int_{x_1}^{x_2} L \mathrm{d}x = \int_{t_1}^{t_2} \mathrm{d}t \left[\int_{x_1}^{x_2} q_1\frac{\partial q_2}{\partial t} - \frac{\alpha}{2}\left(\frac{\partial q_1}{\partial x}\right)^2 + \frac{\beta}{4}(q_1^2 + q_2^2)^2 + \frac{\gamma}{2}q_1^2 + F(q_2)\right]\mathrm{d}x \tag{4}$$

where F is an unknown function of q_2 and their derivatives. There are various alternative approaches to the construction of trial-functional, illustrating examples can be found in [36], and detailed discussion about how to construct a suitable trial-functional is given in [36-51]. The main merit of the previous trial-functional lies on the fact that the stationary condition with respect to q_1 results in eq. (3).

Now calculating the variational derivative of the functional, in eq. (4), with respect to q_1, we obtain the following Euler equation:

$$-\frac{\partial q_1}{\partial t}+\beta(q_1^2+q_2^2)q_2+\frac{\delta F}{\delta q_2}=0 \tag{5}$$

where $\delta F/\delta q_2$ is called He's variational derivative with respect to q_2, defined as [31]:

$$\frac{\delta F}{\delta q_2}=\frac{\partial F}{\partial q_2}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial q_{2x}}\right)-\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial q_{2t}}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial q_{2xx}}\right)+\cdots$$

We search for such an F so that eq. (5) becomes eq. (2). Accordingly, we set:

$$\frac{\delta F}{\delta q_2}=\frac{\partial q_1}{\partial t}-\beta(q_1^2+q_2^2)q_2=\alpha\frac{\partial^2 q_2}{\partial x^2}+\gamma q_2 \tag{6}$$

from which the unknown F can be determined:

$$F=\frac{\alpha}{2}q_2\frac{\partial^2 q_2}{\partial x^2}+\frac{\gamma}{2}q_2^2 \tag{7}$$

or:

$$F=-\frac{\alpha}{2}\left(\frac{\partial q_2}{\partial x}\right)^2+\frac{\gamma}{2}q_2^2 \tag{8}$$

After embedding eqs. (7) or (8) into eq. (4), two variational principles in real space are established for the NLS eq. (1), respectively:

$$J(q_1,q_2)=\int_{t_1}^{t_2}\mathrm{d}t\int_{x_1}^{x_2}\left[q_1\frac{\partial q_2}{\partial t}-\frac{\alpha}{2}\left(\frac{\partial q_1}{\partial x}\right)^2-\frac{\alpha}{2}\left(\frac{\partial q_2}{\partial x}\right)^2+\frac{\beta}{4}(q_1^2+q_2^2)^2+\frac{\gamma}{2}(q_1^2+q_2^2)\right]\mathrm{d}x \tag{9}$$

and

$$J(q_1,q_2)=\int_{t_1}^{t_2}\mathrm{d}t\int_{x_1}^{x_2}\left[q_1\frac{\partial q_2}{\partial t}-\frac{\alpha}{2}\left(\frac{\partial q_1}{\partial x}\right)^2+\frac{\alpha}{2}q_2\frac{\partial^2 q_2}{\partial x^2}+\frac{\beta}{4}(q_1^2+q_2^2)^2+\frac{\gamma}{2}(q_1^2+q_2^2)\right]\mathrm{d}x \tag{10}$$

Proof. Making any one of the previous functionals stationary with respect to all independent functions q_1 and q_2 severally, the following Euler-Lagrange equations can be obtained:

$$\delta q_1:\ \ \frac{\partial q_2}{\partial t}+\alpha\frac{\partial^2 q_1}{\partial x^2}+\beta(q_1^2+q_2^2)q_1+\gamma q_1=0 \tag{11}$$

$$\delta q_2:\ \ -\frac{\partial q_1}{\partial t}+\alpha\frac{\partial^2 q_2}{\partial x^2}+\beta(q_1^2+q_2^2)q_2+\gamma q_2=0 \tag{12}$$

in which δq_1 and δq_2 are the first-order variations for q_1 and q_2. Obviously, eqs. (11) and (12) are totally equivalent to the field eqs. (3) and (2), in turn. Successfully, we proved the obtained two variational principles (9) and (10) correct.

Solitary wave solutions by the semi-inverse variational principle

To simplify the NLS equation, after inserting $A = A' \mathrm{e}^{-i\int \gamma(\tau)\mathrm{d}\tau}$ into eq. (1), and then omit superscript, we obtain the new equation:

$$-iA_t + \alpha A_{xx} + \beta |A|^2 A = 0 \tag{13}$$

where eq. (1) is a NLS equation in the standard form of internal solitary waves. Where α and β are dispersion coefficient and non-linear coefficient, respectively. They are related to the density and depth of the upper and lower fluids, h_1 and h_2 – the depths of the upper and lower fluids, respectively.

The initial hypothesis to analyze the NLS equation for deep ocean internal waves is:

$$A(x,t) = f(\xi)\mathrm{e}^{i(mx-nt)} \tag{14}$$

where the traveling wave transform is:

$$\xi = x - Et \tag{15}$$

where m and n are constants, f – an undetermined real function, and E – the wave velocity:

$$-i(-Ef' - inf) + \alpha(f'' + 2imf' - m^2 f) + \beta f^3 = 0 \tag{16}$$

$$(E + 2\alpha m) f' = 0 \tag{17}$$

where $f'' = (\mathrm{d}^2 f)/(\mathrm{d}\xi^2)$, and $f' = (\mathrm{d}f)/(\mathrm{d}\xi)$, and the speed of the soliton, from eq. (17), is:

$$E = -2\alpha m \tag{18}$$

Furthermore, from eq. (16), we can get:

$$\alpha f'' - (n + \alpha m^2) f + \beta f^3 = 0 \tag{19}$$

By using the semi-inverse method [36], the variational formulation of eq. (19) can be obtained:

$$J = \int_0^\infty \left[\frac{1}{2}\alpha (f')^2 + \frac{1}{2}(n + \alpha m^2) f^2 - \frac{1}{4}\beta f^4 \right] \mathrm{d}\xi \tag{20}$$

Now, f is assumed to have the following form:

$$f = p\,\mathrm{sech}(q\xi), \quad \xi = x - Et \tag{21}$$

where p and q are two parameters to be determined.

In order to obtain the two parameters function f, we insert eq. (21) into eq. (20), and after some manipulations, we get:

$$J=\int_0^\infty\left[\frac{1}{2}\alpha p^2q^2\tanh^2(q\xi)\operatorname{sec}h^2(q\xi)+\frac{1}{2}(n+\alpha m^2)p^2\operatorname{sec}h^2(q\xi)-\frac{1}{4}\beta p^4\operatorname{sec}h^4(q\xi)\right]\mathrm{d}\xi=$$
$$=\int_0^\infty\frac{1}{2}\alpha p^2q^2\tanh^2(q\xi)\operatorname{sec}h^2(q\xi)\mathrm{d}\xi+\int_0^\infty\frac{1}{2}(n+\alpha m^2)p^2\operatorname{sec}h^2(q\xi)\mathrm{d}\xi-$$
$$-\int_0^\infty\frac{1}{4}\beta p^4\operatorname{sec}h^4(q\xi)\mathrm{d}\xi=\frac{\alpha p^2q}{6}+\frac{(n+\alpha m^2)p^2}{2q}-\frac{\beta p^4}{6q} \tag{22}$$

In order to get the stagnation point of J on p and q, we are minimizing the previous functional with respect to two unknown parameters. The following equations are given:

$$\frac{\partial J}{\partial p}=\frac{\alpha pq}{3}+\frac{(n+\alpha m^2)p}{q}-\frac{2\beta p^3}{3q} \tag{23}$$

$$\frac{\partial J}{\partial q}=\frac{\alpha p^2}{6}-\frac{(n+\alpha m^2)p^2}{2q^2}+\frac{\beta p^4}{6q^2} \tag{24}$$

The equations are transformed into:

$$\alpha q^2+3(n+\alpha m^2)-2\beta p^2=0,\quad \alpha q^2-3(n+\alpha m^2)+\beta p^2=0 \tag{25}$$

After solving the algebraic equations, we can get:

$$p=\pm\sqrt{\frac{2(n+\alpha m^2)}{\beta}},\quad q=\pm\sqrt{\frac{n+\alpha m^2}{\alpha}} \tag{26}$$

Finally, the solitary wave solutions to eq. (13) are obtained:

$$A(x,t)=\pm\sqrt{\frac{2(n+\alpha m^2)}{\beta}}\operatorname{sech}\left[\pm\sqrt{\frac{n+\alpha m^2}{\alpha}}\xi\right]\mathrm{e}^{i(mx-nt)},\quad \xi=x-Et \tag{27}$$

Obviously, by giving different values to the parameters for α, β, m, n, and E, we will get different solitary wave solutions. If the parameters are set as $\alpha = 0.2$, $\beta = 0.2$, $m = 2$, $n = 2$, and $E = 2$, we can plot the solitary wave solution as fig. 1. From fig. 1, it is easy to show that the amplitude of wave solution is very local in space and has characteristics of soliton.

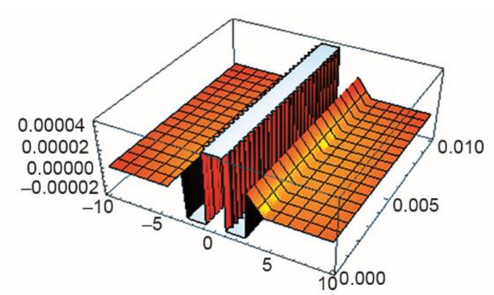

Figure 1. The shape of the solitary wave solution given by eq. (27)

Similarly, we can choose a different form of solution function:

$$f=p\operatorname{sech}^2(q\xi),\quad \xi=x-Et \tag{28}$$

The calculation procedure is similar to previous one, and p and q undetermined parameters.

In order to obtain the following two-parameter function, we insert eq. (18) into eq. (11):

$$J=\int_0^\infty\left[2\alpha p^2q^2\tanh^2(q\xi)\sec h^4(q\xi)+\frac{1}{2}(n+\alpha m^2)p^2\sec h^4(q\xi)-\frac{1}{4}\beta p^4\sec h^8(q\xi)\right]\mathrm{d}\xi=$$

$$=\int_0^\infty 2\alpha p^2q^2\tanh^2(q\xi)\sec h^4(q\xi)\mathrm{d}\xi+\int_0^\infty\frac{1}{2}(n+\alpha m^2)p^2\sec h^4(q\xi)\mathrm{d}\xi-$$

$$-\int_0^\infty\frac{1}{4}\beta p^4\sec h^8(q\xi)\mathrm{d}\xi=\frac{4}{15}\alpha p^2q+\frac{(n+\alpha m^2)p^2}{3q}-\frac{4\beta p^4}{35q} \quad (29)$$

In order to get the stagnation point of J on p and q, we set up the following equations:

$$\frac{\partial J}{\partial p}=\frac{8}{15}\alpha pq+\frac{2(n+\alpha m^2)p}{3q}-\frac{16\beta p^3}{35q}$$

$$\frac{\partial J}{\partial q}=\frac{4}{15}\alpha p^2-\frac{(n+\alpha m^2)p^2}{3q^2}+\frac{4\beta p^4}{35q^2} \quad (30)$$

Or simplify to get:

$$28\alpha q^2-35(n+\alpha m^2)+12\beta p^2=0,\quad 28\alpha q^2+35(n+\alpha m^2)-24\beta p^2=0 \quad (31)$$

After solving the algebraic equations, we can get:

$$p=\pm\sqrt{\frac{35(n+\alpha m^2)}{18\beta}},\quad q=\pm\sqrt{\frac{5(n+\alpha m^2)}{12\alpha}} \quad (32)$$

The result is:

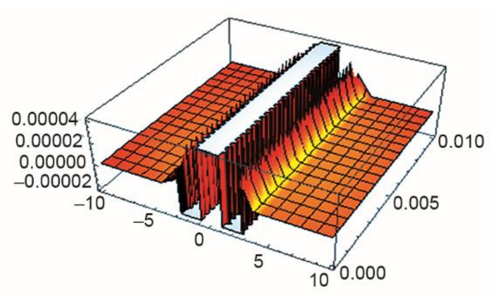

Figure 2. The shape of the solitary wave solution given by eq. (33)

$$A(x,t)=\pm\sqrt{\frac{35(n+\alpha m^2)}{18\beta}}\operatorname{sech}^2\left[\pm\sqrt{\frac{5(n+\alpha m^2)}{12\alpha}}\xi\right]\mathrm{e}^{i(mx-nt)},\quad \xi=x-Et \quad (33)$$

Obviously, by giving different values to the parameters for α, β, m, n, and E, we will get different solitary wave solutions. If the parameters are set as α = 0.2, β = 0.2, m = 2, n = 2, and E = 2, we can plot the solitary wave solution as fig. 2. From fig. 2, it is easy to show that the amplitude of wave solution is very local in space and has soliton characteristics.

Conclusion

Internal solitary waves are ubiquitous physical phenomena in the ocean, which play an essential role in the transport of marine matter, momentum and energy. The NLS equation is suitable for describing the deep-sea internal wave propagation. In this paper, variational principles have been successfully constructed for the NLS equation by the semi-inverse method [36-51] and designing skillfully trial-Lagrange functionals. Subsequently, the obtained variational principles have proved correct by minimizing the corresponding functionals. From the analysis results, it is concluded that the variational principle for the NLS equation studied in this paper has two different integral formulations, from which the same control equations can be derived. Then, different solution structures for solitary waves are obtained by the semi-inverse variational principle for the NLS equation. From the figures of solutions, it is easy to show that the amplitude of wave solution is very local in space and has characteristics of soliton. Our work in the future will focus on the dynamics of soliton in the NLS equation, by the variational approximation method using the established variational principles and methods in this paper.

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References

[1] Jiang, Z. H., *et al.*, Ocean Internal Waves Interpreted as Oscillation Travelling Waves in Consideration of Ocean Dissipation, *Chin. Phys. B*, *23* (2014), 5, 050302
[2] Lee, C. Y, Beardsley R. C., The Generation of Long Nonlinear Internal Waves in a Weakly Stratified Shear flow, *Journal of Geophysical Research*, *79* (1974), 3, pp. 453-462
[3] Wang, Z., Zhu Y. K., Theory, Modelling and Computation of Nonlinear Ocean Internal Waves, *Chinese Journal of Theoretical and Applied Mechanics*, *51* (2019), 6, pp. 1589-1604
[4] Karunakar, P., Chakraverty, S., Effect of Coriolis Constant on Geophysical Korteweg-de Vries Equation, *Journal of Ocean Engineering and Science*, *4* (2019), 2, pp. 113-121
[5] Kaya, D., Explicit and Numerical Solutions of Some Fifth-order KdV Equation by Decomposition Method, *Appl Math Comput*, *144* (2003), 2-3, pp. 353-363
[6] Kaya, D., EI-Sayed, S. M., On a Generalized Fifth-Order KdV Equations, *Phys. Lett. A*, *310* (2003), 1, pp. 44-51
[7] Zhang, Y., Chen, D. Y., The Novel Multi Solitary Wave Solution to the Fifth-Order KdV Equation, *Chin. Phys. B*, *10* (2004), 10, pp. 1606-1610
[8] Wazwaz, A. M., A Study on Compacton-Like Solutions for the Modified KdV and Fifth Order KdV-Like Equations, *Appl. Math. Comput.*, *147* (2004), 2, pp. 439-447
[9] He, J.-H. Variational Principle for the Generalized KdV-Burgers Equation with Fractal Derivatives for Shallow Water Waves, *J. Appl. Comput. Mech.*, *6* (2020), 4, pp. 735-740
[10] Li, J., *et al.*, Simulation Investigation on the Internal Wave *via* the Analytical Solution of Korteweg-de Vries Equation (in Chinese), *Marine Science Bulletin*, *30* (2011), 1, pp. 23-28
[11] Benjamin, B. T., Internal Waves of Permanent form in Fluids of Great Depth, *Journal of Fluid Mechanics*, *29* (1967), 3, pp. 559-592
[12] Ono, H. Algebraic Solitary Waves in Stratified Fluids, *Journal of the Physical Society of Japan*, *39* (1975), 4, pp. 1082-1091
[13] Kubota, T., *et al.*, Propagation of Weakly Non-linear Internal Waves in a Stratified Fluid of Finite Depth, *AIAA Journal of Hydronautics*, *12* (1978), 4, pp. 157-165
[14] Choi, W., Camassa, R., Fully Non-Linear Internal Waves in a Two-Fluid System, *Journal of Fluid Mechanics*, *396* (1999), Oct., pp. 1-36
[15] Song, S. Y., *et al.*, Non-Linear Schrödinger Equation for Internal Waves in Deep Sea, *Acta Physica Sinica*, *59* (2010), 2, pp. 1123-1129
[16] Wazwaz, A. M., *Partial Differential Equations and Solitary Waves Theory*, Higher Education Press, Beijing, 2009
[17] Wazwaz, A. M., *Linear and Nonlinear Integral Equations: Methods and Applications,* Higher Education Press, Beijing, 2011

[18] He, J. H., Exp-Function Method for Fractional Differential Equations, *Int. J. Nonlinear Sci. Numer. Simul.*, *14* (2013), 6, pp. 363-366
[19] He, J.-H., *et al.*, Homotopy Perturbation Method for the Fractal Toda Oscillator, *Fractal Fract*., *5* (2021), 3, 93
[20] Wu, Y., Variational Approach to Higher-Order Water-Wave Equations, *Chaos Solitons Fractals*, *32* (2007), 1, pp.195-203.
[21] Gazzola, F., *et al*., Variational Formulation of the Melan Equation, *Math. Methods Appl. Sci.*, *41* (2018), 3, pp. 943-951
[22] Durgun, D. D., Fractional Variational Iteration Method for Time-Fractional Nonlinear Functional Partial Differential Equation Having Proportional Delays, *Thermal Science*, *22* (2018), Suppl. 1, pp. S33-S46
[23] He, C. H., *et al.* Hybrid Rayleigh-van der Pol-Duffing Oscillator: Stability Analysis and Controller, *Journal of Low Frequency Noise Vibration and Active Control*, *41* (2022), 1, pp. 244-268
[24] Tian, D., *et al.,* Fractal N/MEMS: from Pull-In Instability to Pull-In Stability, *Fractals, 29* (2021), 2, 2150030
[25] Tian, D., He, C. H., A Fractal Micro-Electromechanical System and Its Pull-In Stability, *Journal of Low Frequency Noise Vibration and Active Control*, *40* (2021), 3, pp. 1380-1386
[26] Li, X. X., He, C. H., Homotopy Perturbation Method Coupled with the Enhanced Perturbation Method, *Journal of Low Frequency Noise Vibration and Active Control, 38* (2019), 3-4, pp. 1399-1403
[27] He, J. H. Variational Iteration Method - Some Recent Results and New Interpretations, *Journal of Computational and Applied Mathematics, 207* (2007), 1, pp. 3-7
[28] He, J. H., Wu, X. H., Variational Iteration Method: New Development and Applications, *Computers & Mathematics with Applications,* 5*4* (2007), 7-8, pp. 881-894
[29] Yang, X. J., Baleanu, D., Fractal Heat Conduction Problem Solved by Local Fractional Variation Iteration Method, *Thermal Science*, *17* (2013), 2, pp. 625-628
[30] He, J. H.; *et al.*, Dynamic Pull-In for Micro-Electromechanical Device with a Current-Carrying Conductor, *Journal of Low Frequency Noise Vibration and Active Control*, *40* (2021), 2, pp. 1059-1066
[31] Malomed, B. A., Variational Methods in Nonlinear fiber Optics and Related fields, *Prog. Opt.*, *43* (2002), 71, pp. 71-193
[32] Chong, C., Pelinovsky, D. E., Variational Approximations of Bifurcations of Asymmetric Solitons in Cubic-Quintic Nonlinear Schrödinger Lattices, *Discret. Contin. Dyn. Syst.*, *4* (2011), 5, pp. 1019-1031
[33] Kaup, D. J., Variational Solutions for the Discrete Nonlinear Schrödinger Equation, *Math. Comput. Simul.*, *69* (2005), 3-4, pp. 322-333
[34] Putri, N. Z., *et al*., Variational Approximations for Intersite Soliton in a Cubic-Quintic Discrete Nonlinear Schrödinger Equation, *J. Phys. Conf. Ser.*, *1317* (2019), 1, 012015
[35] He, J. H., Variational Principles for Some Nonlinear Partial Differential Equations with Variable Coefficients, *Chaos Solitons Fractals*, *19* (2004), 4, pp. 847-851
[36] He, J. H. Semi-Inverse Method of Establishing Generalized Variational Principles for Fluid Mechanics with Emphasis on Turbomachinery Aerodynamics, *International Journal of Turbo & Jet-Engines*, *14* (1997), 1, pp. 23-28
[37] He, J. H., *et al.* On a Strong Minimum Condition of a Fractal Variational Principle, *Applied Mathematics Letters*, *119* (2021), Sept., 107199
[38] Wang, K. J., Generalized Variational Principle and Periodic Wave Solution to the Modified Equal width-Burgers Equation in Nonlinear Dispersion Media, *Physics Letters A*, *419* (2021), Dec., 127723
[39] He, J. H., *et al.*, Variational Approach to Fractal Solitary Waves, *Fractals*, *29* (2021), 7, 2150199
[40] Cao, X. Q., *et al.*, Variational Theory for (2+1)-Dimensional Fractional Dispersive Long Wave Equations, *Thermal Science*, *25* (2021), 2, pp. 1277-1285
[41] Wang, K. J., Zhang, P. L., Investigation of the Periodic Solution of the Time-Space Fractional Sasa-Satsuma Equation Arising in the Monomode Optical Fibers, *EPL*, *137* (2021), 6, 62001
[42] Khan, Y., Fractal Higher-Order Dispersions Model and Its Fractal Variational Principle Arising in the Field of Physcial Process, *Fluctuation and Noise Letters*, *20* (2021), 4, 2150034
[43] He, J. H., Some Asymptotic Methods for Strongly Nonlinear wave Equation, *International Journal of Modern Physics B*, *20* (2006), 10, pp. 1141-1199
[44] Li, Y., He, C. H., A Short Remark on Kalaawy's Variational Principle for Plasma, *International Journal of Numerical Methods for Heat & Fluid Flow*, *27* (2017), 10, pp. 2203-2206
[45] He, J. H., *et al.*, A Fractal Modification of Chen-Lee-Liu Equation and Its Fractal Variational Principle, *International Journal of Modern Physics B*, *35* (2021), 21, 2150214

[46] He, J. H., Maximal Thermo-geometric Parameter in a Nonlinear Heat Conduction Equation, *Bulletin of the Malaysian Mathematical Sciences Society*, *39* (2016), 2, pp. 605-608
[47] Cao, X. Q., Variational Principles for Two Kinds of Extended Korteweg-de Vries Equations, *Chin. Phys. B*, *20* (2011), 9, pp. 94-102
[48] Cao, X. Q., Generalized Variational Principles for Boussinesq Equation Systems, *Acta Phys. Sin.*, *60* (2011), 8, pp. 105-113
[49] Wang, K. L., He, C. H., A Remark on Wang's Fractal Variational Principle, *Fractals, 27* (2019), 8, 1950132
[50] He, J. H., *et al.,* Solitary Waves Travelling Along an Unsmooth Boundary, *Results in Physics, 24* (2021), May, 104104
[51] Wang, K. L., *et al.,* Physical Insight of Local Fractional Calculus and Its Application to Fractional KdV-Burgers-Kuramoto Equation, *Fractals, 27* (2019), 7, 1950122

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