VARIATIONAL PRINCIPLES FOR TWO KINDS OF NON-LINEAR GEOPHYSICAL KdV EQUATION WITH FRACTAL DERIVATIVES

by

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It is an important and difficult inverse problem to construct variational principles from complex models directly, because their variational formulations are theoretical bases for many methods to solve or analyze the non-linear problems. At first, this paper extends two kinds of non-linear geophysical KdV equations in continuum mechanics to their fractional partners in fractal porous media or with irregular boundaries. Then, by designing skillfully, the trial-Lagrange functional, variational principles are successfully established for the non-linear geophysical KdV equation with Coriolis term, and the high-order extended KdV equation with fractal derivatives, respectively. Furthermore, the obtained variational principles are proved to be correct by minimizing the functionals with the calculus of variations.

Key words: *variational principle, geophysical KdV equation, fractal dimension, high-order extended KdV equation, semi-inverse method*

Introduction

Solving PDE with integer or fractional orders is always an attractive and hot topic for many researchers in different scientific fields because of their outstanding ability to model non-linear phenomena [1-5]. Investigating solutions of such non-linear PDE is a critical research area, and numerous mathematical techniques have been developed to explore the approximate and exact solutions, of which variational-based methods have been very useful and effective, such as Ritz technique [6-9], variational iteration method [10-13], and variational approximation method [14-20], *etc.* When contrasted with other methods, variational ones show some advantages. For example, they can provide physical insight into the nature of the solutions and investigate practical problems from a global perspective. The obtained solutions are the best among all possible trial-functions. We require much less strong local differentiability of variables than those for PDE, such as the finite difference method, the finite volume method, *etc.* Because variational principles are so important for obtaining the approximate or exact solutions [14-20], it is of great significance to seek explicit variational formulations for the non-linear PDE. It is also an inverse problem to find variational principles directly from a

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set of known equations by the calculus of variations. The semi-inverse method [21-30] was firstly proposed in 1997 by Dr. Ji-Huan He, who is a famous Chinese mathematician. The semi-inverse method has been widely used to establish variational principles from the governing equations directly, and has become a significant and effective tool in the variational theory far beyond the well-known Lagrange multiplier method [21-30]. Because it is not necessary to introduce Lagrange multipliers, the Lagrange crisis frequently encountered in constructing variational principles can be avoided naturally [21-30]. Recently, many scientists have made a lot of efforts and great successes for constructing variational principles in different kinds of fields such as fluid dynamics, meteorology, ocean, mathematical biology, solid-state physics, optics, and plasma physics, and so forth [21-35]. Non-linear PDE are also used widely and commonly to model internal solitary waves, solitons, tsunami wave and so on in the ocean and sea [21-35]. In this paper, variational principles are established by the semi-inverse method [21-35] for the non-linear geophysical KdV (gKdV) equation with Coriolis term, and the high-order extended KdV (EKdV) equation, in fractal space and time derivatives [36-42], respectively. Although both KdV-type equations in this paper have been extensively studied for a long time by some scientists [43-53], but, up to now, variational principles for them with fractal derivatives have not been dealt with. Therefore, finding variational principles for them is of great value, and might find lots of applications in numerical simulations and scientific researches.

The fractional partners

Usually, we can view physical motions and phenomena using two distinctly different scales [30, 36, 37]. One is a large scale, where Newton's calculus is approximately valid and the traditional mechanics can be roughly applied. The other scale is a much smaller one, a scale of molecule size. Under such a small scale, the media becomes discontinuous, and the fractal calculus [38-42] has to be adopted. Equations (1) and (2) are very useful models to describe oceanic internal solitary waves in continuous media [49-53], however an unsmooth boundary will greatly affect the properties of non-linear waves. Therefore, the smooth space (X, T) should be replaced by a fractal space (X^β, T^α), where β and α are fractal dimensions in space and time, respectively. In the fractal space, the high-order EKdV equation [44-48], and the non-linear gKdV equation with Coriolis term [43] can be modified as following, respectively:

$$\frac{\partial u}{\partial T^\alpha}+c\frac{\partial u}{\partial X^\beta}+b_1\frac{\partial^3 u}{\partial X^{3\beta}}+b_2u\frac{\partial u}{\partial X^\beta}+b_3\frac{\partial^4 u}{\partial X^{4\beta}}+b_4u^2\frac{\partial u}{\partial X^\beta}+$$

$$+b_5u\frac{\partial^3 u}{\partial X^{3\beta}}+b_5\frac{\partial u}{\partial X^\beta}\frac{\partial^2 u}{\partial X^{2\beta}}=0 \tag{1}$$

$$\frac{\partial \eta}{\partial T^\alpha}-\omega_0\frac{\partial \eta}{\partial X^\beta}+\frac{3}{2}\eta\frac{\partial \eta}{\partial X^\beta}+\frac{1}{6}\frac{\partial^3 \eta}{\partial X^{3\beta}}=0 \tag{2}$$

where the He's fractal derivatives are defined as [38-41]:

$$\frac{\partial u}{\partial T^\alpha}(T_0,X)=\Gamma(1+\alpha)\lim_{\substack{T-T_0\to\Delta T\\ \Delta T\neq 0}}\frac{u(T,X)-u(T_0,X)}{(T-T_0)^\alpha} \tag{3}$$

$$\frac{\partial u}{\partial X^{\beta}}(T,X_0)=\Gamma(1+\beta)\lim_{\substack{X-X_0\to\Delta X\\ \Delta X\neq 0}}\frac{u(T,X)-u(T,X_0)}{(X-X_0)^{\beta}} \tag{4}$$

The similar definitions of eqs. (3) and (4) can also be given for the solution function $\eta(T,X)$ in eq. (2), which is the gKdV equation with Coriolis term, in fractal space. For the fractal derivatives, we have the following chain rules:

$$\frac{\partial^2}{\partial X^{2\beta}}=\frac{\partial}{\partial X^{\beta}}\frac{\partial}{\partial X^{\beta}} \tag{5}$$

$$\frac{\partial^3}{\partial X^{3\beta}}=\frac{\partial}{\partial X^{\beta}}\frac{\partial}{\partial X^{\beta}}\frac{\partial}{\partial X^{\beta}} \tag{6}$$

$$\frac{\partial^4}{\partial X^{4\beta}}=\frac{\partial}{\partial X^{\beta}}\frac{\partial}{\partial X^{\beta}}\frac{\partial}{\partial X^{\beta}}\frac{\partial}{\partial X^{\beta}} \tag{7}$$

In the fractal space, all variables depend upon the scales used for observation and the fractal dimensions of the discontinuous boundary. In the definitions given in eqs. (3) and (4), ΔX and ΔT are the smallest spatial scale for discontinuous boundary and the smallest temporal scale, respectively, for watching the physical phenomena. For example, when we watch the solitary wave on a scale larger than ΔT, a smooth wave morphology is predicted, however, when we observe the wave on the scale of ΔT, discontinuous wave morphology can be found [36-37]. The fractal derivatives are widely used in applications [36-42] for discontinuous media.

Variational orinciples for high-order EKdV equation with fractal derivatives

According to the basic properties of previously given fractal calculus, we have the following time and space scale transforms [36-42]:

$$t=T^{\alpha} \tag{8}$$

$$x=X^{\beta} \tag{9}$$

The high-order EKdV eq. (1) in fractal space becomes:

$$u_t+cu_x+b_1u_{xxx}+b_2uu_x+b_3u_{xxxx}+b_4u^2u_x+b_5uu_{xxx}+b_5u_xu_{xx}=0 \tag{10}$$

Ocean internal wave is a common natural phenomenon, which is investigated by researchers using some non-linear internal wave models such as the KdV equation [49-53]. However, in practical application, some internal solitary waves often have strong non-linearity, which is inconsistent with the weak non-linear assumption of the KdV equation. In addition, due to the change of water depth and stratification in some sea areas, the KdV and mKdV equations have great limitations. So, it is more practical to use the high-order EKdV eq. (10) to simulate the internal solitary wave in the ocean on the free surface [44-50]. At the same time, the high-order EKdV has the ability to simulate the propagation and fission process of large amplitude and strong non-linear internal solitary waves. The exact traveling wave and soliton solutions, in particular, have been studied extensively [41, 44, 45, 48, 49]. In eq. (10), c, b_1, b_2, b_3, b_4 and b_5 are parameters of constant values, which indicate the effects of long wave phase speed, dispersion, non-linearity, high-order non-linearity, and dissipation, respectively.

In order to find its variational principles, eq. (10) can be transformed into the following form:

$$u_t + \left(cu + b_1 u_{xx} + \frac{b_2}{2}u^2 + b_3 u_{xxx} + \frac{b_4}{3}u^3 + b_5 u u_{xx} \right)_x = 0 \tag{11}$$

It is obvious that finding Lagrangian representations for the above high-order EKdV equation is a non-trivial problem. Additionally, it is necessary to replace the physical field $u(x,t)$ by it's derivatives of potential fields. According to eq. (11), a potential function Φ can be introduced:

$$\Phi_x = u$$

$$\Phi_t = -\left(cu + b_1 u_{xx} + \frac{b_2}{2}u^2 + b_3 u_{xxx} + \frac{b_4}{3}u^3 + b_5 u u_{xx} \right) \tag{12}$$

Thus, eq. (11) will be automatically satisfied. It is hoped to construct different variational principles, according to eq. (11) and the field eqs. (12).

For establishing the variational principles, whose Euler-Lagrange equations will be equivalent to the high-order EKdV equation, we can firstly set a trial-functional in the following form:

$$J(u,\Phi) = \iint L(u, u_{xx}, u_{xxx}, \Phi_t, \Phi_x)\mathrm{d}x\mathrm{d}t \tag{13}$$

where L is the trial-Lagrange functional. In view of eqs. (11) and (12), we design by the semi-inverse method [21-30], that the L can be written:

$$L = u\Phi_t + \left(cu + b_1 u_{xx} + \frac{b_2}{2}u^2 + b_3 u_{xxx} + \frac{b_4}{3}u^3 + b_5 u u_{xx} \right)\Phi_x + F \tag{14}$$

where F is an unknown functional of only variable u and its derivatives, to be determined later. There are many alternative methods for constructing the trial-functional, [21-30]. The great merit of the above trial-Lagrange functional eq, (14) is whose stationary condition with respect to Φ leads to the following Euler-Lagrange equation:

$$\frac{\partial L}{\partial \Phi} - \frac{\partial}{\partial x}\frac{\partial L}{\partial \Phi_x} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \Phi_t} = 0 \tag{15}$$

After introducing eq. (14), eq. (15) is identical to the high-order EKdV eq. (10). Subsequently, by calculating the stationary conditions of eq. (14) with respect to u, we obtain:

$$\frac{\partial L}{\partial u} + \frac{\partial}{\partial x^2}\frac{\partial L}{\partial u_{xx}} - \frac{\partial}{\partial x^3}\frac{\partial L}{\partial u_{xxx}} + \frac{\delta F}{\delta u} = 0 \tag{16}$$

where $\delta F/\delta u$ is called He's variational derivative [21] of F. By using eq. (14), eq. (16) can be rewritten:

$$\Phi_t + c\Phi_x + b_2 u\Phi_x + b_4 u^2\Phi_x + b_5 u_{xx}\Phi_x + b_1\Phi_{xxx} - b_3\Phi_{xxxx} + b_5(u\Phi_x)_{xx} + \frac{\delta F}{\delta u} = 0 \tag{17}$$

It is hoped to find such an F, so that eq. (17) turns out to be the field eq. (12). Accordingly, after substituting the eq. (12) into eq. (17), we get:

$$\frac{\delta F}{\delta u}=-\frac{b_2u^2}{2}-\frac{2b_4}{3}u^3+2b_3u_{xxx}-b_5(u^2)_{xx} \tag{18}$$

From eq. (18), unfortunately, we cannot identify F through the calculus of variations, because of existing the term $2b_3u_{xxx}$, so we have to modify the trial-Lagrange function, L, into a new form [25]:

$$L=Au\Phi_t+B\Phi_x\Phi_t+\left(cu+\frac{b_2}{2}u^2+\frac{b_4}{3}u^3+b_1u_{xx}+b_3u_{xxx}+b_5uu_{xx}\right)\Phi_x+F \tag{19}$$

Again, by calculating the variational derivatives of L with respect to Φ and u, respectively, the new Euler-Lagrange equations can be obtained:

$$\frac{\delta L}{\delta \Phi}:\quad -(A+2B)u_t-\left(cu+\frac{b_2}{2}u^2+\frac{b_4}{3}u^3+b_1u_{xx}+b_3u_{xxx}+b_5uu_{xx}\right)_x+\frac{\delta F}{\delta \Phi}=0 \tag{20}$$

$$\frac{\delta L}{\delta u}:\quad A\Phi_t+c\Phi_x+b_2u\Phi_x+b_4u^2\Phi_x+b_5u_{xx}\Phi_x+b_1\Phi_{xxx}-b_3\Phi_{xxxx}+b_5(u\Phi_x)_{xx}+\frac{\delta F}{\delta u}=0 \tag{21}$$

In view of eq. (12) and $\delta F/\delta\Phi = 0$, eq. (20) becomes:

$$-(A+2B)u_t-\left(cu+\frac{b_2}{2}u^2+\frac{b_4}{3}u^3+b_1u_{xx}+b_3u_{xxx}+b_5uu_{xx}\right)_x=0 \tag{22}$$

Because eq. (22) should be identical to eq. (11), we must set the coefficient of u_t to one. That is:

$$A+2B=1 \tag{23}$$

After substituting eq. (12) into eq. (21), we obtain:

$$(1-A)(cu+b_1u_{xx}+b_5uu_{xx})+\left(1-\frac{A}{2}\right)b_2u^2+\left(1-\frac{A}{3}\right)b_4u^3-$$

$$-(1+A)b_3u_{xxx}+b_5(u^2)_{xx}+\frac{\delta F}{\delta u}=0 \tag{24}$$

In order to determine the unknown function F successfully, it is necessary to eliminate the term u_{xxx}, whose coefficient must be set to zero in eq. (24). At the same time, according to the variational calculus and $b_3\neq 0$, we get:

$$A+1=0 \tag{25}$$

From eqs. (23) and (25), we obtain A = –1 and B = 1. Furthermore:

$$\frac{\delta F}{\delta u}=-2(cu+b_1u_{xx}+b_5uu_{xx})-\frac{3b_2}{2}u^2-\frac{4b_4}{3}u^3-b_5(u^2)_{xx} \tag{26}$$

From eq. (26), F can be identified easily:

$$F=-cu^2-b_1uu_{xx}-\frac{b_2u^3}{2}-\frac{b_4u^4}{3}+2b_5uu_x^2 \tag{27}$$

or

$$F=-cu^2+b_1u_x^2-\frac{b_2u^3}{2}-\frac{b_4u^4}{3}+2b_5uu_x^2 \tag{28}$$

Finally, we obtain the variational formulations for the high-order EKdV eq. (10), which reads:

$$J(u,\Phi)=\iint\left[\Phi_x\Phi_t-u\Phi_t+\left(cu+\frac{b_2}{2}u^2+\frac{b_4}{3}u^3+b_1u_{xx}+b_3u_{xxx}+b_5uu_{xx}\right)\Phi_x-\right.$$
$$\left.-cu^2-b_1uu_{xx}-\frac{b_2u^3}{2}-\frac{b_4u^4}{3}+2b_5uu_x^2\right]\mathrm{d}x\mathrm{d}t \tag{29}$$

and

$$J(u,\Phi)=\iint\left[\Phi_x\Phi_t-u\Phi_t+\left(cu+\frac{b_2}{2}u^2+\frac{b_4}{3}u^3+b_1u_{xx}+b_3u_{xxx}+b_5uu_{xx}\right)\Phi_x-\right.$$
$$\left.-cu^2+b_1u_x^2-\frac{b_2u^3}{2}-\frac{b_4u^4}{3}+2b_5uu_x^2\right]\mathrm{d}x\mathrm{d}t \tag{30}$$

both of which are subject to the constraint equation $\Phi_x = u$. The established variational principles are firstly discovered by the semi-inverse method [21-30], and may find many applications in numerical simulations and researches of the high-order EKdV equation. In the following, we will prove the obtained variational principles correct. By making anyone of the previous functionals, eqs. (29) and (30), stationary with respect to independent functions u and Φ severally, we can obtain two different Euler-Lagrange equations:

$$\delta\Phi:\quad u_t-\left(cu+b_1u_{xx}+\frac{b_2}{2}u^2+\frac{b_4}{3}u^3+b_3u_{xxx}+b_5uu_{xx}\right)_x-2\Phi_{xt}=0 \tag{31}$$

$$\delta u:\quad -\Phi_t+(c+b_2u+b_4u^2+b_5u_{xx})\Phi_x+b_1\Phi_{xxx}-b_3\Phi_{xxxx}+b_5(u\Phi_x)_{xx}-$$
$$-2cu-2b_1u_{xx}-\frac{3}{2}b_2u^2-\frac{4}{3}b_4u^3-2b_5(u_x^2+2uu_{xx})=0 \tag{32}$$

where $\delta\Phi$ and δu is the first-order variation for Φ and u. Substituting $\Phi_x = u$ into eq. (31) leads to the high-order EKdV equation, obviously. After substituting $\Phi_x = u$ into eq. (32), we can get that:

$$-\Phi_t-cu-b_1u_{xx}-\frac{1}{2}b_2u^2-\frac{1}{3}b_4u^3-b_3u_{xxx}-b_5uu_{xx}=0$$

which is identical to the second one of eqs. (12). Hence, successfully, we proved the obtained variational principles (29) and (30) correct. In the fractal space (X^β, T^α), the variational formulation can be written into the new forms:

$$J(u,\Phi)=\iint\Bigg[-u\frac{\partial\Phi}{\partial T^{\alpha}}+\left(cu+\frac{b_2}{2}u^2+\frac{b_4}{3}u^3+b_1\frac{\partial^2 u}{\partial X^{2\beta}}+b_3\frac{\partial^3 u}{\partial X^{3\beta}}+b_5 u\frac{\partial^2 u}{\partial X^{2\beta}}\right)\frac{\partial\Phi}{\partial X^{\beta}}+$$

$$+\frac{\partial\Phi}{\partial X^{\beta}}\frac{\partial\Phi}{\partial T^{\alpha}}-cu^2-b_1u\frac{\partial^2 u}{\partial X^{2\beta}}-\frac{b_2u^3}{2}-\frac{b_4u^4}{3}+2b_5u\left(\frac{\partial u}{\partial X^{\beta}}\right)^2\Bigg]\mathrm{d}X^{\beta}\mathrm{d}T^{\alpha} \tag{33}$$

and

$$J(u,\Phi)=\iint\Bigg[-u\frac{\partial\Phi}{\partial T^{\alpha}}+\left(cu+\frac{b_2}{2}u^2+\frac{b_4}{3}u^3+b_1\frac{\partial^2 u}{\partial X^{2\beta}}+b_3\frac{\partial^3 u}{\partial X^{3\beta}}+b_5 u\frac{\partial^2 u}{\partial X^{2\beta}}\right)\frac{\partial\Phi}{\partial X^{\beta}}+$$

$$+\frac{\partial\Phi}{\partial X^{\beta}}\frac{\partial\Phi}{\partial T^{\alpha}}-cu^2+b_1\left(\frac{\partial u}{\partial X^{\beta}}\right)^2-\frac{b_2u^3}{2}-\frac{b_4u^4}{3}+2b_5u\left(\frac{\partial u}{\partial X^{\beta}}\right)^2\Bigg]\mathrm{d}X^{\beta}\mathrm{d}T^{\alpha} \tag{34}$$

Variational principles for gKdV equation with fractal derivatives

Similarly, using the time and space scale transforms (8) and (9), the gKdV eq. (2) in fractal space is transformed into:

$$\eta_t-\omega_0\eta_x+\frac{3}{2}\eta\eta_x+\frac{1}{6}\eta_{xxx}=0 \tag{35}$$

The KdV-type equations have numerous applications in various branches of science and engineering. Various research works have been reported over the recent years related to KdV like equations by different authors [43-51]. Here we aimed to study the results of gKdV equation with Coriolis term, which represents the impact of the Earth's rotation on the fluid, and is admissible for large-scale ocean waves [43]. Inclusion of Coriolis term $\omega_0\eta_x$ in eq. (35) is helpful to study the effect of Earth's rotation on the propagation of tsunami waves [43]. In order to construct variational principles, eq. (35) can be transformed into the following conservative form:

$$\eta_t-\left(\omega_0\eta-\frac{3}{4}\eta^2-\frac{1}{6}\eta_{xx}\right)_x=0 \tag{36}$$

It is obvious that finding Lagrangian representations for eq. (36) is not a trivial problem. Firstly, it is essential to replace original variable of wave height with the derivative of potential field. According to eq. (36), a potential function Φ can be introduced:

$$\Phi_x=\eta$$

$$\Phi_t=\omega_0\eta-\frac{3}{4}\eta^2-\frac{1}{6}\eta_{xx} \tag{37}$$

so that eq. (35) or eq. (36) is automatically satisfied. We will construct the variational principle, directly from the original eq. (35) and field eqs. (37).

Secondly, we can build a trial-functional in the following form by the semi-inverse method [26-34]:

$$J(\eta,\Phi)=\iint L(\eta,\eta_{xx},\Phi_t,\Phi_x)\mathrm{d}x\mathrm{d}t \tag{38}$$

where L is the trial-Lagrange functional. In view of eq. (38), it is designed that the L is written:

$$L=\eta\Phi_t-\left(\omega_0\eta-\frac{3}{4}\eta^2-\frac{1}{6}\eta_{xx}\right)\Phi_x+G \tag{39}$$

Specially, G is an unknown functional only of u and it’s derivatives. The remarkable merit of the previous trial-Lagrange functional (39) is whose stationary condition with respect to Φ leads to the following Euler-Lagrange equation:

$$\frac{\partial L}{\partial \Phi}-\frac{\partial}{\partial x}\frac{\partial L}{\partial \Phi_x}-\frac{\partial}{\partial t}\frac{\partial L}{\partial \Phi_t}=0 \tag{40}$$

In view of eq. (39), eq. (40) is equivalent to the non-linear gKdV eq. (36). Subsequently, by calculating the stationary conditions of eq. (39) with respect to η, it leads to:

$$\frac{\partial L}{\partial \eta}+\frac{\partial}{\partial x^2}\frac{\partial L}{\partial \eta_{xx}}+\frac{\delta G}{\delta \eta}=0 \tag{41}$$

where $\delta G/\delta\eta$ is called the Frechet's variational derivative [8-35] of G. By using eq. (39), eq. (41) can be rewritten:

$$\Phi_t-\omega_0\Phi_x+\frac{3}{2}\eta\Phi_x+\frac{1}{6}\Phi_{xxx}+\frac{\delta G}{\delta \eta}=0 \tag{42}$$

It is hoped to find such a G, so that eq. (42) turns out to be the field eq. (37). Accordingly, after substituting eq. (37) into eq. (42), we get:

$$\frac{\delta G}{\delta \eta}=-\frac{3}{4}\eta^2 \tag{43}$$

From eq. (43), we can identify G successfully by the variational calculus, in the following form:

$$G=-\frac{\eta^3}{4} \tag{44}$$

At last, we obtain the following variational principle for the non-linear gKdV equation [43], which reads:

$$J(\eta,\Phi)=\iint\left[\eta\Phi_t-\left(\omega_0\eta-\frac{3}{4}\eta^2-\frac{1}{6}\eta_{xx}\right)\Phi_x-\frac{1}{4}\eta^3\right]\mathrm{d}x\mathrm{d}t \tag{45}$$

which is subject to the constraint of $\Phi_x=\eta$. The established variational principles by the semi-inverse method [21-35] provide conservation laws and may find lots of applications in numerical simulation and scientific analysis of eq. (35). In the following, we will prove the obtained variational principles correct. By making the functional (45), stationary with respect to two independent functions η and Φ severally, two Euler-Lagrange equations can be obtained:

$$\delta\Phi: \quad -\eta_t + \left(\omega_0\eta - \frac{3}{4}\eta^2 - \frac{1}{6}\eta_{xx}\right)_x = 0 \tag{46}$$

$$\delta\eta: \quad \Phi_t - \omega_0\Phi_x + \frac{3}{2}\eta\Phi_x + \frac{1}{6}\Phi_{xxx} - \frac{3}{4}\eta^2 = 0 \tag{47}$$

where $\delta\Phi$ and $\delta\eta$ is the first-order variation of Φ and η. Equation (46) is the original non-linear gKdV equation [43], obviously. By substituting $\Phi_x = \eta$ into eq. (47), we get that:

$$\Phi_t - \omega_0\eta + \frac{3}{4}\eta^2 + \frac{1}{6}\eta_{xx} = 0$$

which is identical to the second one in eq. (37). Hence, successfully, we proved the obtained variational principles of the non-linear gKdV equation correct. In the fractal space (X^β, T^α), the variational formulation can be written in a new form:

$$J(\eta,\Phi) = \iint\left[\eta\frac{\partial\Phi}{\partial T^\alpha} - \left(\omega_0\eta - \frac{3}{4}\eta^2 - \frac{1}{6}\frac{\partial^2\eta}{\partial X^{2\beta}}\right)\frac{\partial\Phi}{\partial X^\beta} - \frac{1}{4}\eta^3\right]\mathrm{d}X^\beta\mathrm{d}T^\alpha \tag{48}$$

Conclusion

In the third and fourth parts, variational principles have been successfully constructed for the high-order EKdV equation and gKdV equation, respectively, by the semi-inverse method [21-35] and designing skillfully trial-Lagrange functionals. Subsequently, the obtained variational principles have proved correct by minimizing the corresponding functionals. From the results of analysis, it is concluded that the variational principle for the high-order EKdV equation studied in this paper has two different integral formulations, from which the same control equations can be derived. The procedure also reveals that the semi-inverse method [21-35] is effective and powerful. According to the obtained variational principles, on the one hand, we can study possible solution structures for solitary waves. On the other hand, they also provide hints for numerical algorithms, so eqs. (1) and (2) can be solved numerically by the variational-based methods. In numerical simulations and ocean engineering, it is of great importance to choose an appropriate variational principle according to practical applications. Our work in the future will focus on the dynamics of soliton in the high-order EKdV equation and gKdV equation, by the variational approximation method using the established variational principles in this paper.

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