A FRACTIONAL MODEL AND ITS APPLICATION TO HEAT PREVENTION COATING WITH COCOON-LIKE HIERARCHY

by

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In this paper, a fractional model is established by using the variational iteration method to elucidate the thermal properties of building prevention coating with a cocoon-like hierarchy. The fractal hierarchical structure of heat prevention coating makes the building wall mathematically adapted for an extreme temperature environment. This work has inspired the bionic design of protective suits and extreme temperature clothing.

Key words: *cocoon, fractional calculus, hierarchical structure, heat prevention*

Introduction

Many natural biomaterials exhibit structures with more than one length scale, in which the elements themselves have structural features to form a hierarchical architecture. This structural hierarchy plays an important role in determining bulk materials properties. Thus, understanding the effects of hierarchical structure can guide the design of new materials with physical properties targeted for specific applications [1]. A cocoon is a natural protein polymer composite with unique hierarchical porous micro-structures [2] and has significantly superior mechanical performances [3, 4], thus serving as a source of inspiration for high-performance material designs. Compared with other natural or man-made fibers, silkworm cocoons have excellent heat and moisture transfer ability, so they are considered breathable [5-8]. In addition, wild silkworm pupa can survive in an extreme temperature environment, at either –40 °C or +40 °C, due to some special functions and the configurations of the cocoons [9]. The fractal model for heat transfer in layered cocoons has been analyzed with the aid of the fractal derivative model [10-12]. Fractional differential models can also model various discontinuous problems, Tian and Liu [13] found some interesting properties of the fractional Fokas equation. Tian and Wan [14] revealed the solitary wave travelling in a fractal space. Tian and Liu [15] found some exact solutions of the fractional differential equations. Wang [16] revealed the basic properties of solitary waves travelling through a Cantor set. Wang and Zhang [17] studied the periodic solution of the fractional Sasa-Satsuma equation. He *et al.* [18] established a variational principle in a fractal space. Han *et al.* [19] and Dan *et al.* [20] suggested some effective methods to solve fractional differential equations.

In this paper, a new fractional derivative is defined through the variational iteration method (VIM) [21], and adopted for elucidating the excellent thermal properties of heat pre-

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vention coating with cocoon-like hierarchy for the building's wall. There will be helpful for developing and designing multi-functional shoes and clothing.

Definition on fractional derivative through the variational iteration method

There are many definitions of fractional derivatives, this paper adopts the variational iteration algorithm-based definition, and it is called as He's fractional derivative [22] or the fractional derivative in Ji Huan He's sense [23] in the literature. The VIM was first used to solve fractional differential equations in [24], and it has been proved to be effective, easy, and accurate to solve a lot of non-linear differential problems with the approximate values converging rapidly to the exact solutions.

According to the VIM [24-27], we get the variational iteration algorithm:

$$u_{m+1}(t)=u_m(t)+(-1)^n\int_{t_0}^{t}\tfrac{1}{(n-1)!}(s-t)^{n-1}[u_m^{(n)}(s)-f_m(s)]\,\mathrm{d}s \tag{1}$$

We introduce an integration operator I^n defined by Ji-Huan He [21]:

$$I^n f=\int_{t_0}^{t}\frac{1}{(n-1)!}(s-t)^{n-1}[u_0^{(n)}(s)-f(s)]\mathrm{d}s=\frac{1}{\Gamma(n)}\int_{t_0}^{t}(s-t)^{n-1}[f_0(s)-f(s)]\,\mathrm{d}s \tag{2}$$

where $f_0(t)=u_0^{(n)}(t)$.

We can define the following fractional derivative [21]:

$$\mathrm{D}_t^\alpha f=\mathrm{D}_t^\alpha\frac{\mathrm{d}^n}{\mathrm{d}t^n}(I^n f)=\frac{\mathrm{d}^n}{\mathrm{d}t^n}(I^{n-\alpha}f)=\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_{t_0}^{t}(s-t)^{n-\alpha-1}[f_0(s)-f(s)]\,\mathrm{d}s \tag{3}$$

where $f_0(t)$ is a known function and its physical explanation will be given in the next section. His fractional derivative can be widely used to model various problems arising in porous media.

An application

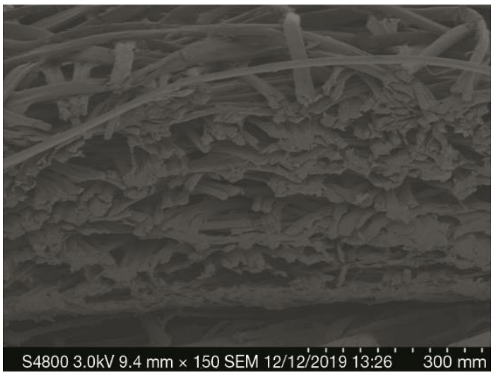

Figure 1. The SEM micrograph of the cross-sections of the domestic cocoon

The cross-section image of the silkworm cocoon is presented in fig. 1. From the SEM micrograph, it reveals that the cocoon has hierarchical and porous structures. Silk fibers are arranged more closely in the inner layer. The outer loose-inner tight structures of silkworm cocoons highly guarantee their survival in the extreme temperature environment.

The building is exposed to the natural environment all year round, and its outer layer must have thermal protection to resist high or low temperatures. As an application of the new fractional derivative, we consider heat prevention coating with a cocoon-like hierarchy for the building's wall. Heat-insulating coating is superior properties such as the excellent thermal protection. Its cocoon-like hierarchy plays a key role in resisting harsh environments.

Using Fourier's Law of thermal conduction in a fractal porous medium, we have:

$$\frac{\partial^\beta}{\partial x^\beta}\left(D\frac{\partial^\beta \theta}{\partial x^\beta}\right)=0 \tag{4}$$

with boundary conditions:

$$\theta(0)=\theta_0, \quad \theta(L)=\theta_L \tag{5}$$

where θ is the temperature, D – the thermal conductivity of heat flux in the fractal medium, β – the fractional dimensions of the fractal medium, and $\partial^\beta/\partial x^\beta$ – the fractional derivative defined as [21] from the eq. (3):

$$\frac{\partial^\beta \theta}{\partial x^\beta}=\frac{1}{\Gamma(n-\beta)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_{t_0}^{t}(s-x)^{n-\beta-1}[\theta_0(s)-\theta(s)]\mathrm{d}s \tag{6}$$

where $\theta_0(x)$ can be the solution of its continuous partner of the problem with the same boundary/initial conditions of the fractal partner.

By the fractional complex transform [28-30]:

$$\frac{\partial^\beta \theta}{\partial x^\beta}=\frac{\partial \theta}{\partial s}\frac{\partial^\beta s}{\partial x^\beta}=\frac{\partial \theta}{\partial s} \tag{7}$$

where

$$s=\frac{x^\beta}{\Gamma(1+\beta)} \tag{8}$$

Equation (4) is transformed into a partial differential equation, which reads:

$$\frac{\partial}{\partial s}\left(D\frac{\partial \theta}{\partial s}\right)=0 \tag{9}$$

The solution of eq. (9) is:

$$\theta=a+bs=a+\frac{bx^\beta}{\Gamma(1+\beta)} \tag{10}$$

After incorporating the boundary conditions of eq. (5), we obtain:

$$\theta=\theta_0+\frac{(\theta_L-\theta_0)}{L^\beta}x^\beta \tag{11}$$

Obviously, this solution has the following notable features:

$$\frac{\mathrm{d}\theta}{\mathrm{d}x}(x=0)=\begin{cases}0, & \beta>1\\ \dfrac{(\theta_L-\theta_0)}{L}, & \beta=1\\ \infty, & \beta<1\end{cases} \tag{12}$$

The progressive relationship between continuous and porous media is shown in the upper part of fig. 2. The schematic of heat transfer in the lower part of fig. 2. It is found that the continuous media can not withstand extreme environments very well due to linear temperature variation. However, the hierarchical porous media has fractal heat transfer property, figs. 2(b)-2(d). The slope at $x = 0$ depends strongly upon the value of the fractal dimensions β. The temperature of its surface for the building wall should be unchanged as much as possible, and it requires $\beta > 1$. Heat prevention coating with a cocoon-like hierarchy can guarantee $\beta > 1$, fig. 3. line (c).

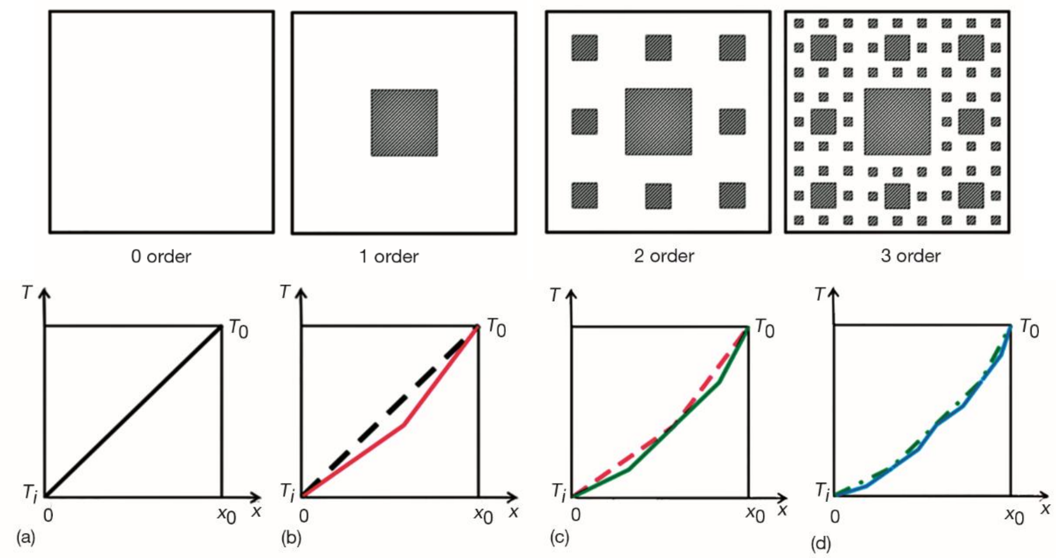

Figure 2. The schematic of heat transfer in the continuous and hierarchic porous media; curves (a) black, (b) red, (c) green, and (d) blue represent continuous media, fractal media with one, two, and three iterations, respectively

We fit the temperature changes in the porous medium to get the heat conduction curves. The straight-line y_1, fig. 4, represents the heat conduction in the continuous medium, and $\alpha = 1$. The curves y_2, y_3, and y_4, fig. 4, mean the heat conduction in the fractal porous medium, respectively. As the order is higher, the heat conduction curve changes more slowly, that is, the temperature approaching the inside of the building changes extremely slow. According to eq. (12), it can be seen that the fractal dimensions of porous media of different or-

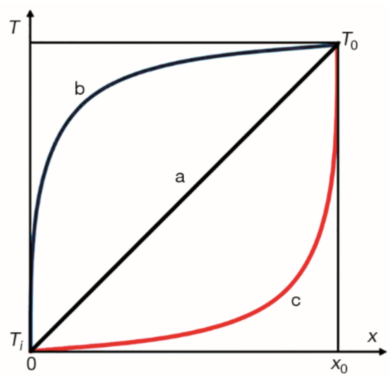

Figure 3. The curves a, b, and c mean $\beta = 1$, $\beta < 1$, and, $\beta > 1$, respectively

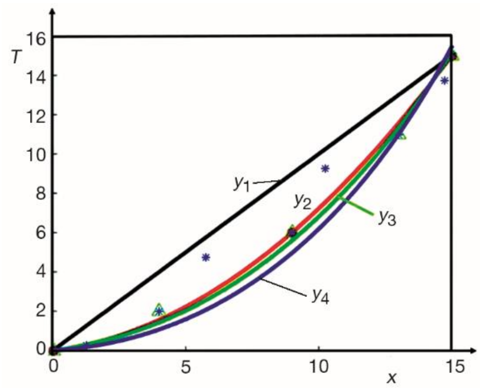

Figure 4. Fitting curves for temperature changes in both continuous, y_1, and porous media, y_2 for 1 order, y_3 for 2 order, and y_4 for 3 order

ders has the following relationship: $\beta_4 > \beta_3 > \beta_2 > 1$. Therefore, when designing the exterior wall coating of a building, increasing the effective number of layers and the fractal dimension $\beta > 1$ of the hierarchical porous media, no matter how the environment temperature changes, the temperature close to the interior will change super slowly.

Conclusion

A more generalized fractional derivative was derived using the VIM. The fractional derivative is an effective method to deal with hierarchic porous media's complicated heat transfer problems. The slope at the boundary largely depends upon the fractal structure of heat prevention coating with a cocoon-like hierarchy for the surface of the buildings. Increasing the effective number of layers and the fractal dimension of the hierarchical porous media can make the temperature close to the interior change super slowly. The establishment of heat transfer mechanisms for the coating could be beneficial to the bionic design, such as bio-materials, functional textiles and the aviation industry.

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References

[1] Huang, W. W., *et al.*, Silkworm Silk-Based Materials and Devices Generated Using Bio-nano-technology, *Chemical Society Review*, *47* (2018), 17, pp. 6486-6504
[2] Chen, F. J., *et al.*, Silk Cocoon (Bombyx Mori): Multi-layer Structure and Mechanical Properties, *Acta Biomaterialia*, *8* (2012), 7, pp. 2620-2627
[3] Vollrath, F., Knight, D. P., Liquid Crystalline Spinning of Spider Silk, *Nature*, *410* (2001), 6828, pp. 541-548
[4] Zhang, K., *et al.*, Hierarchical, Multilayered Cell Walls Reinforced by Recycled Silk Cocoons Enhance the Structural Integrity of Honeybee Combs, *Proceedings of the National Academy of Sciences of the United States of America*, *107* (2010), 21, pp. 9502-9506
[5] Blossman-Myer, B., Burggren, W. W., The Silk Cocoon of the Silkworm, Bombyx Mori: Macro Structure and its Influence on Transmural Diffusion of Oxygen and Water Vapor, *Comparative Biochemistry and Physiology A-Molecular & Integrative Physiology*, *155* (2010), 2, pp. 259-263
[6] Chen, R. X., *et al.*, Waterproof and Dustproof of Wild Silk: a Theoretical Explanation, *Journal of Nano Research*, *22* (2013), May, pp. 61-63
[7] Tao, H., *et al.*, Silk Materials – a Road to Sustainable High Technology, *Advanced Materials*, *24* (2012), 21, pp. 2824-2837
[8] Wegst, U. G. K., *et al.*, Bioinspired Structural Materials, *Nature Materials*, *14* (2015), 1, pp. 23-36
[9] Omenetto, F. G., Kaplan, D. L., New Opportunities for an Ancient Material, *Science*, *329* (2010), 5991, pp. 528-531
[10] Liu, F. J., *et al.*, A Fractional Model for Insulation Clothings with Cocoon-like Porous Structure, *Thermal Science*, *20* (2016), 3, pp. 779-784
[11] Liu, F. J., *et al.*, He's Fractional Derivative for Heat Conduction in a Fractal Medium Arising in Silkworm Cocoon Hierarchy, *Thermal Science*, *19* (2015), 4, pp. 1155-1159
[12] Fei, D. D., *et al.*, Fractal Approach to Heat Transfer in Silkworm Cocoon Hierarchy, *Thermal Science*, *17* (2013), 5, pp. 1546-1548
[13] Tian, Y., Liu, J., Direct Algebraic Method for Solving Fractional Fokas Equation, *Thermal Science*, *25* (2021), 3, pp. 2235-2244
[14] Tian, Y., Wan, J. X., Exact Solutions of Space-Time Fractional 2+1 Dimensional Breaking Soliton Equation, *Thermal Science*, *25* (2021), 2, pp. 1229-1235

[15] Tian, Y., Liu, J., A Modified Exp-Function Method for Fractional Partial Differential Equations, *Thermal Science, 25* (2021), 2, pp. 1237-1241
[16] Wang, K. J., On New Abundant Exact Traveling Wave Solutions to the Local Fractional Gardner Equation Defined on Cantor Sets, *Mathematical Methods in the Applied Sciences*, *45* (2022), 4, pp. 1904-1915
[17] Wang, K. J., Zhang, P. L., Investigation of the Periodic Solution of the Time-Space Fractional Sasa-Satsuma Equation Arising in the Monomode Optical Fibers, *EPL, 137* (2022), 6, ID 62001
[18] He, J. H., *et al.,* Variational Approach to Fractal Solitary Waves, *Fractals, 29* (2021), 7, 2150199
[19] Han, C., *et al.*, Numerical Solutions of Space Fractional Variable-Coefficient KdV-modified KdV Equation by Fourier Spectral Method, *Fractals, 29* (2021), 8, 2150246
[20] Dan, D. D., *et al.* Using Piecewise Reproducing Kernel Method and Legendre Polynomial for Solving a Class of the Time Variable Fractional Order Advection-Reaction-Diffusion Equation, *Thermal Science*, *25* (2021), 2B, pp. 1261-1268
[21] He, J. H., A Tutorial Review on Fractal Spacetime and Fractional Calculus, *International Journal of Theoretical Physics*, *53* (2014), 11, pp. 3698-3718
[22] Pandey, R. K., Mishra, H. K., Semi- Analytic Numerical Method for Solution of Time-Space Fractional Heat and Wave Type Equations with Variable Coefficients, *Open Physics*, *15* (2017), 1, pp. 74-86
[23] Sayevand, K., Arjang, F., A Reliable Implicit Difference Scheme for Treatments of Fourth-Order Fractional Sub-Diffusion Equation, *Scientia Iranica, 24* (2017), 3, pp. 1100-1107
[24] He, J. H., Approximate Analytical Solution for Seepage Flow with Fractional Derivatives in Porous Media, *Computer Methods in Applied Mechanics and Engineering*, *167* (1998), 1-2, pp. 57-68
[25] He, J. H., *et al.*, Dynamic Pull-in for Micro-Electromechanical Device with a Current-Carrying Conductor, *Journal of Low Frequency Noise Vibration and Active Control, 40* (2021), 2, pp. 1059-1066
[26] He, J. H., Maximal Thermo-geometric Parameter in a Non-Linear Heat Conduction Equation, *Bulletin of the Malaysian Mathematical Sciences Society, 39* (2016), 2, pp. 605-608
[27] Skrzypacz, P., *et al.,* A Simple Approximation of Periodic Solutions to Microelectromechanical System Model of Oscillating Parallel Plate Capacitor, *Mathematical Methods in the Applied Sciences*, On-line first, https://doi.org/10.1002/mma.6898, 2020
[28] Li, Z. B., He, J. H., Fractional Complex Transform for Fractional Differential Equations, *Mathematical and Computation Applications, 15* (2010), 5, pp. 970-973
[29] He, J. H., Li, Z. B., Converting Fractional Differential Equations into Partial Differential Equations, *Thermal Science, 16* (2012), 2, pp. 331-334
[30] Li, Z. B., et al., Exact Solutions of Time-Fractional Heat Conduction Equation by the Fractional Complex Transform, *Thermal Science, 16* (2012), 2, pp. 335-338

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