VARIATIONAL PRINCIPLE FOR ONE-DIMENSIONAL INVISCID FLOW

by

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A family of variational principles is obtained for the 1-D inviscid flow by Ji-Huan He's semi-inverse method. The invalidy of the Lagrange multiplier method, e. g., the Lagrange crisis, is also discussed to eliminate constraints of a constrained variational principle. Two approaches to the elimination of the crisis are elucidated.

Key words: *Bateman variational principle, trial functional, variational theory, Euler-Lagrange equation*

Introduction

An inviscid flow is an ideal flow without viscosity, it can be widely applied in engineering to figure out the main flow characters. The governing equations for 1-D inviscid flows are:

$$\rho_t + (\rho u)_x = 0 \tag{1}$$

$$\rho u_t + \rho u u_x + P_x = 0 \tag{2}$$

$$P = k\rho^{\gamma} \tag{3}$$

where ρ, u, and P are air density, air moving speed, and air pressure, respectively, k and γ are constants.

The system given in eqs. (1)-(3) is a special case of the Chaplygin-He gas [1-3]. A special function, φ, is defined:

$$\varphi_x = \rho \tag{4}$$

$$\varphi_t = -\rho u \tag{5}$$

Equation (1) is then equivalent to eqs. (4) and (5). In view of eq. (3), we convert eq. (2) to the following form:

$$u_t + \left(\frac{1}{2}u^2 + \frac{k\gamma}{\gamma - 1}\rho^{\gamma-1}\right)_x = 0 \tag{6}$$

In this paper, we will establish a variational formulation for the previous system.

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Variational formulation

In order to find a variational principle for the 1-D flow, we begin with a Bateman-like variational formulation [4-6]:

$$J(\varphi,u)=\iint\{P+F\}\mathrm{d}x\mathrm{d}t \tag{7}$$

where F is unknown yet. eq. (7) is called as the trial variational formulation, there are many alternative ways to construct a suitable energy-form functional [7-18].

The Lagrange function can be written in an alternative way:

$$L=P+F=k\rho^{\gamma}+F=k(\varphi_x)^{\gamma}+F \tag{8}$$

The trial variational formulation can be updated:

$$J(\varphi,u)=\iint\left\{k(\varphi_x)^{\gamma}+F\right\}\mathrm{d}x\mathrm{d}t \tag{9}$$

The stationary condition with respect to φ reads:

$$-[k\gamma(\varphi_x)^{\gamma-1}]_x+\frac{\delta F}{\delta\varphi}=0 \tag{10}$$

where $\delta F/\delta\varphi$ is defined:

$$\frac{\delta F}{\delta\varphi}=\frac{\partial F}{\partial\varphi}-\frac{\partial}{\partial t}\frac{\partial F}{\partial\varphi_t}-\frac{\partial}{\partial x}\frac{\partial F}{\partial\varphi_x} \tag{11}$$

Equation (11) is He's variational derivative. According to eq. (6), from eq. (10) we have:

$$\frac{\delta F}{\delta\varphi}=[k\gamma(\varphi_x)^{\gamma-1}]_x=(k\gamma\rho^{\gamma-1})_x=-(\gamma-1)(u_t+uu_x) \tag{12}$$

We, therefore, can identify F:

$$F=(\gamma-1)\left(u\varphi_t+\frac{1}{2}u^2\varphi_x\right)+F_1 \tag{13}$$

where F_1 is free of φ and its derivatives. The Lagrange function can be now updated:

$$L=k(\varphi_x)^{\gamma}+(\gamma-1)\left(u\varphi_t+\frac{1}{2}u^2\varphi_x\right)+F_1 \tag{14}$$

Now the stationary condition with respect to u is:

$$(\gamma-1)(\varphi_t+u\varphi_x)+\frac{\delta F_1}{\delta u}=0 \tag{15}$$

By eqs. (4) and (5), we have:

$$\frac{\delta F_1}{\delta u}=-(\gamma-1)(\varphi_t+u\varphi_x)=-(\gamma-1)(-\rho u+u\rho)=0 \tag{16}$$

So we can identify F_1:

$$F_1 = 0 \tag{17}$$

We finally obtain the following variational principle:

$$J(\varphi,u) = \iint \left[k(\varphi_x)^\gamma + (\gamma-1)\left(u\varphi_t + \frac{1}{2}u^2\varphi_x \right) \right] \mathrm{d}x\mathrm{d}t \tag{18}$$

which is under the constraint of eq. (4).

Proof. The Euler-Lagrange equations of eq. (18) are:

$$-k\gamma[(\varphi_x)^{\gamma-1}]_x - (\gamma-1)(u_t + uu_x) = 0 \tag{19}$$

$$(\gamma-1)(\varphi_t + u\varphi_x) = 0 \tag{20}$$

In view of the constraint, eq. (4), we can convert eqs. (19) and (20) to eqs. (6) and (5), respectively.

Submitting eq. (4) to eq. (18), we can obtain another constrained variational formulation:

$$J(\varphi) = \iint \left[k\rho^\gamma - \frac{1}{2}(\gamma-1)\rho u^2 \right] \mathrm{d}x\mathrm{d}t = \iint \left[P - \frac{1}{2}(\gamma-1)\rho u^2 \right] \mathrm{d}x\mathrm{d}t \tag{21}$$

Lagrange multiplier method

In order to eliminate the constraint of the variational principle given in eq. (18), we write down the following functional:

$$J(\varphi,u,\rho,\lambda) = \iint \left[k(\varphi_x)^\gamma + (\gamma-1)\left(u\varphi_t + \frac{1}{2}u^2\varphi_x \right) + \lambda(\varphi_x - \rho) \right] \mathrm{d}x\mathrm{d}t \tag{22}$$

where λ is the Lagrange multiplier. However, the stationary condition with respect to the density results in $\lambda = 0$. This phenomenon is called as Lagrange crisis [1, 9]. There are some approaches to the elimination of the crisis. In view of eq. (4), we convert eq. (18) to the form:

$$J(\varphi,u) = \iint \left[k\rho^\gamma + (\gamma-1)\left(u\varphi_t + \frac{1}{2}u^2\varphi_x \right) \right] \mathrm{d}x\mathrm{d}t \tag{23}$$

which is subject to eq.(4).

Now Lagrange multiplier method results in:

$$J(\varphi,u,\rho,\lambda) = \iint \left[k\rho^\gamma + (\gamma-1)\left(u\varphi_t + \frac{1}{2}u^2\varphi_x \right) + \lambda(\varphi_x - \rho) \right] \mathrm{d}x\mathrm{d}t \tag{24}$$

We, therefore, can easily determine the multiplier, which reads:

$$\lambda = k\gamma\rho^{\gamma-1} \tag{25}$$

The generalized variational principle is:

$$J(\varphi,u,\rho) = \iint \left[k\rho^\gamma + (\gamma-1)\left(u\varphi_t + \frac{1}{2}u^2\varphi_x \right) + k\gamma\rho^{\gamma-1}(\varphi_x - \rho) \right] \mathrm{d}x\mathrm{d}t \tag{26}$$

Proof. The Euler-Lagrange equations of eq. (26) are:

$$-(\gamma-1)(u_t+uu_x)-k\gamma(\rho^{\gamma-1})_x=0 \tag{27}$$

$$(\gamma-1)(\varphi_t+u\varphi_x)=0 \tag{28}$$

$$k\gamma\rho^{\gamma-1}+k\gamma(\gamma-1)\rho^{\gamma-2}(\varphi_x-\rho)-k\gamma\rho^{\gamma-1}=0 \tag{29}$$

Equations (27)-(29) can be converted to eqs. (6), (5), and (4), respectively.

Another approach to the elimination of the crisis is the semi-inverse method. We replace the term involving the Lagrange multiplier in eq. (22) by f:

$$f=\lambda(\varphi_x-\rho) \tag{30}$$

Equation (22) becomes:

$$J(\varphi,u,\rho)=\iint\left[k(\varphi_x)^\gamma+(\gamma-1)\left(u\varphi_t+\frac{1}{2}u^2\varphi_x\right)+f\right]\mathrm{d}x\mathrm{d}t \tag{31}$$

After identifying f, we finally have:

$$J(\varphi,u,\rho)=\iint\left[k(\varphi_x)^\gamma+(\gamma-1)\left(u\varphi_t+\frac{1}{2}u^2\varphi_x\right)+\beta(\varphi_x-\rho)^2\right]\mathrm{d}x\mathrm{d}t \tag{32}$$

Proof. The stationary conditions of eq. (32) are:

$$-k\gamma[(\varphi_x)^{\gamma-1}]_x-(\gamma-1)(u_t+uu_x)-2\beta(\varphi_x-\rho)_x=0 \tag{33}$$

$$(\gamma-1)(\varphi_t+u\varphi_x)=0 \tag{34}$$

$$-2\beta(\varphi_x-\rho)=0 \tag{35}$$

It is easy to prove that eqs. (33)-(35) are equivalent to eqs. (6), (5), and (4), respectively.

Conclusions

This paper shows a new method to find a variational formulation for the discussed problem. Generally, a variational formulation has an integral form, so a trial variational formulation can be in kinetic form:

$$J(\varphi,u)=\iint\left(\frac{1}{2}\rho u^2+F\right)\mathrm{d}x\mathrm{d}t \tag{36}$$

Hinted by the Bernoulli equation or Cauchy-Lagrange integral [1], we can convert eq. (36) into eq. (1). We can also begin with a more general form:

$$J(\varphi,u)=\iint\left(\frac{1}{2}a\rho u^2+bP+F\right)\mathrm{d}x\mathrm{d}t \tag{37}$$

where a and b are constants.

Variational theory is powerful mathematical tool to both the numerical and analytical analysese. The porous electrodes can be optimized by variational theory [15], and computer simulation can be designed by establishing a suitable functional [16, 17].

To be concluded, the article finds some variational formulations for the 1-D ideal flow, and the present approach can be easily extended to the Chaplygin gas and its modifications with fractal derivatives [1, 2] or the fractional derivatives [19-21], we will discuss it in a forthcoming paper.

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