THE VARIATIONAL ITERATION METHOD FOR WHITHAM-BROER-KAUP SYSTEM WITH LOCAL FRACTIONAL DERIVATIVES

by

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The Whitham-Broer-Kaup equations are modified using local fractional derivatives, and the equations are then solved by the variational iteration method. Yang-Laplace transform method is adopted to make the solution process simpler.

Key words: *Whitham-Broer-Kaup system, fractal media, Yang-Laplace transform, variational iteration method*

Introduction

Many non-differentiable phenomena in applied sciences and engineering can be described by local fractional differential equations or fractal differential equations, especially for fractal media [1] including porous media like concrete [2], hierarchical structure [3], porous cells [4], crystals [5], and unsmooth boundaries [6, 7]. In recent years, the study of initial value problems modeled by local fractional partial differential equations has attracted many mathematicians and physicists [8-11]. In this paper, we consider the following local fractional Whitham-Broer-Kaup system in a fractal medium:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} + u\frac{\partial^{\beta} u}{\partial x^{\beta}} + \frac{\partial^{\beta} v}{\partial x^{\beta}} + b\frac{\partial^{2\beta} u}{\partial x^{2\beta}} = 0$$

$$\frac{\partial^{\alpha} v}{\partial t^{\alpha}} + u\frac{\partial^{\beta} v}{\partial x^{\beta}} + v\frac{\partial^{\beta} u}{\partial x^{\beta}} + a\frac{\partial^{3\beta} u}{\partial x^{3\beta}} - b\frac{\partial^{2\beta} v}{\partial x^{2\beta}} = 0 \tag{1}$$

with the initial conditions:

$$u(x,0) = f(x^{\alpha}), \quad v(x,0) = g(x^{\alpha}) \tag{2}$$

where $0 < \alpha \leq 1$, $0 < \beta \leq 1$, $\partial^{\alpha}/(\partial t^{\alpha})$, and $\partial^{\beta}/(\partial x^{\beta})$ denote the local fractional derivatives with respect to time and space, respectively, a and b are parameters, and both $f(x^{\alpha})$ and $g(x^{\alpha})$ are given functions.

The system of equations arises in various fields such as plasma physics and nuclear physics [12-15], however, the solution of the local fractional system is much involved. In general, it is difficult to obtain an exact solution for the system. Some useful techniques have

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been successfully applied to deal with the non-linear differential equations, for examples, Poincare–Lindstedt technique [16], He's frequency formulation [17-21], Li-He method [22-24], and variational theory [25-29], among which the variational iteration method is specially suitable for local differential equations, and the solution process becomes simpler when Yang-Laplace transform method [30] is used.

Preliminaries

In this section, we recall some basic definitions and essential facts about the local fractional calculus and Yang-Laplace transform. For more details, see [8, 30].

Assume the following relation exists:

$$\left| f(x) - f(x_0) \right| < \varepsilon^{\alpha} \tag{3}$$

with $|x - x_0| < \delta$ for $\varepsilon, \delta > 0$. Then $f(x)$ is local fractional continuous at x_0 which is denoted by:

$$\lim_{x \to x_0} f(x) = f(x_0)$$

If $f(x)$ is local fractional continuous on the interval (a, b), it is denoted by:

$$f(x) \in C_{\alpha}(a,b)$$

Let $f \in C_{\alpha}(a,b)$. The local fractional derivative of $f(x)$ at the point $x = x_0$ is given by:

$$D_x^{\alpha} f(x_0) = \frac{d^{\alpha}}{dx^{\alpha}} f(x)\bigg|_{x=x_0} = f^{(\alpha)}(x_0) = \lim_{x \to x_0} \frac{\Delta^{\alpha}[f(x) - f(x_0)]}{(x - x_0)^{\alpha}} \tag{4}$$

where $\Delta[f(x) - f(x_0)] \cong \Gamma(\alpha+1)[f(x) - f(x_0)]$.

A partition of the interval $[a, b]$ is denoted as (t_j, t_{j+1}), $j = 0, 1,\ldots, N - 1$, $t_0 = a$ and $t_N = b$ with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_0, \Delta t_1,\ldots, \Delta t_N\}$. The local fractional integral of $f(x)$ in the interval $[a, b]$ is given by:

$${}_a I_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(x)(dx)^{\alpha} = \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^{\alpha} \tag{5}$$

In the fractal space, the Mittag-Leffler function is given by:

$$E_{\alpha}(x^{\alpha}) = \sum_{n=0}^{\infty} \frac{x^{(n\alpha)}}{\Gamma(1+n\alpha)}, \quad 0 < \alpha \le 1 \tag{6}$$

Let:

$$\frac{1}{\Gamma(1+\alpha)} \int_0^{\infty} \left| f(x) \right| (dx)^{\alpha} < k < \infty, \quad 0 < \alpha \le 1$$

The Yang-Laplace transforms of $f(x)$ is given by:

$$L_{\alpha}\{f(x)\} = f_s^{L,\alpha}(s) = \frac{1}{\Gamma(1+\alpha)} \int_0^{\infty} E_{\alpha}(-s^{\alpha} x^{\alpha}) f(x)(dx)^{\alpha} \tag{7}$$

where the latter integral converges and $s^{\alpha} \in R^{\alpha}$.

The inverse transform of the Yang-Laplace transforms of $f(x)$ is given by:

$$L_\alpha^{-1}\{f_s^{L,\alpha}(s)\} = f(x) = \frac{1}{(2\pi)^\alpha}\int_{\beta-i\varpi}^{\beta+i\varpi} E_\alpha(s^\alpha x^\alpha) f_s^{L,\alpha}(S)(\mathrm{d}s)^\alpha \tag{8}$$

where $s^\alpha = \beta^\alpha + i^\alpha\varpi^\alpha$, fractal imaginary unit i^α and $\mathrm{Re}(s) = \beta > 0$.

Some useful formulas of local fractional derivative were summarized:

$$\frac{\mathrm{d}^\alpha(x^{n\alpha})}{\mathrm{d}x^\alpha} = \frac{\Gamma(1+n\alpha)x^{(n-1)\alpha}}{\Gamma[1+(n-1)\alpha]} \tag{9}$$

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b E_\alpha(x^\alpha)(\mathrm{d}x)^\alpha = E_\alpha(b^\alpha) - E_\alpha(a^\alpha) \tag{10}$$

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b x^{n\alpha}(\mathrm{d}x)^\alpha = \frac{\Gamma(1+n\alpha)[b^{(n+1)\alpha} - a^{(n+1)\alpha}]}{\Gamma[1+(n+1)\alpha]} \tag{11}$$

The following facts about local fractional Yang-Laplace Transform hold true.

Let $L_\alpha\{f(x)\} = f_s^{L,\alpha}(s)$ and $L_\alpha\{g(x)\} = g_s^{L,\alpha}(s)$, then:

$$L_\alpha\{af(x) + bg(x)\} = af_s^{L,\alpha}(s) + bg_s^{L,\alpha}(s) \tag{12}$$

$$L_\alpha\{E_\alpha(c^\alpha x^\alpha)f(x)\} = f_s^{L,\alpha}(s-c) \tag{13}$$

$$L_\alpha\{f^{(\alpha)}(x)\} = s^\alpha f_s^{L,\alpha}(s) - f(0) \tag{14}$$

$$L_\alpha\{x^{k\alpha}\} = \frac{\Gamma(1+k\alpha)}{s^{(k+1)\alpha}} \tag{15}$$

Analysis of the method

Consider the following general non-linear local fractional PDE system.

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) + P_{1\beta}[u(x,t),v(x,t)] + N_{1\beta}[u(x,t),v(x,t)] = 0 \tag{16}$$

$$\frac{\partial^\alpha}{\partial t^\alpha}v(x,t) + P_{2\beta}[u(x,t),v(x,t)] + N_{2\beta}[u(x,t),v(x,t)] = 0 \tag{17}$$

where $P_{k\beta}$ represents the general linear local fractional differential operator and $N_{k\beta}$ – the general non-linear local fractional differential operator ($k = 1, 2$).

Taking local fractional Yang-Laplace transform [30], we obtain:

$$L_\alpha\left[\frac{\partial^\alpha}{\partial t^\alpha}u(x,t)\right] + L_\alpha[P_{1\beta}(u,v)] + L_\alpha[N_{1\beta}(u,v)] = 0 \tag{18}$$

$$L_\alpha\left[\frac{\partial^\alpha}{\partial t^\alpha}v(x,t)\right] + L_\alpha[P_{2\beta}(u,v)] + L_\alpha[N_{2\beta}(u,v)] = 0 \tag{19}$$

By applying eq. (14), we have:

$$s^{\alpha}L_{\alpha}[u(x,t)]-u(x,0)=-L_{\alpha}[P_{1\beta}(u,v)]-L_{\alpha}[N_{1\beta}(u,v)] \tag{20}$$

$$s^{\alpha}L_{\alpha}[v(x,t)]-v(x,0)=-L_{\alpha}[P_{2\beta}(u,v)]-L_{\alpha}[N_{2\beta}(u,v)] \tag{21}$$

or

$$L_{\alpha}[u(x,t)]=\frac{1}{s^{\alpha}}u(x,0)-\frac{1}{s^{\alpha}}L_{\alpha}[P_{1\beta}(u,v)]-\frac{1}{s^{\alpha}}L_{\alpha}[N_{1\beta}(u,v)] \tag{22}$$

$$L_{\alpha}[v(x,t)]=\frac{1}{s^{\alpha}}u(x,0)-\frac{1}{s^{\alpha}}L_{\alpha}[P_{2\beta}u(x,t)]-\frac{1}{s^{\alpha}}L_{\alpha}[N_{2\beta}v(x,t)] \tag{23}$$

By the Yang-Laplace inverse transform [30], we obtain:

$$u(x,t)=u(x,0)+L_{\alpha}^{-1}\left\{\frac{1}{s^{\alpha}}L_{\alpha}[-P_{1\beta}(u,v)-N_{1\beta}(u,v)]\right\} \tag{24}$$

and

$$v(x,t)=v(x,0)+L_{\alpha}^{-1}\left\{\frac{1}{s^{\alpha}}L_{\alpha}[-P_{2\beta}u(x,t)-N_{2\beta}v(x,t)]\right\} \tag{25}$$

Finding the local fractional derivative in eqs. (24) and (25), we obtain:

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}u(x,t)-\frac{\partial^{\alpha}u}{\partial t^{\alpha}}(x,0)-\frac{\partial^{\alpha}}{\partial t^{\alpha}}L_{\alpha}^{-1}\left\{\frac{1}{s^{\alpha}}L_{\alpha}[-P_{1\beta}(u,v)-N_{1\beta}(u,v)]\right\}=0 \tag{26}$$

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}v(x,t)-\frac{\partial^{\alpha}v}{\partial t^{\alpha}}(x,0)-\frac{\partial^{\alpha}}{\partial t^{\alpha}}L_{\alpha}^{-1}\left\{\frac{1}{s^{\alpha}}L_{\alpha}[-P_{2\beta}(u,v)-N_{2\beta}(u,v)]\right\}=0 \tag{27}$$

According to the variational iteration method [31-35], the correction functions can be constructed:

$$u_{k+1}(x,t)=u_k(x,t)-{}_0I_t^{\alpha}\left\{\frac{\partial^{\alpha}u_k(x,\tau)}{\partial\tau^{\alpha}}-\frac{\partial^{\alpha}u_k(x,0)}{\partial\tau^{\alpha}}+\frac{\partial^{\alpha}}{\partial\tau^{\alpha}}L_{\alpha}^{-1}\left[\frac{1}{s^{\alpha}}L_{\alpha}\left(\Omega_1\right)\right]\right\} \tag{28}$$

$$v_{k+1}(x,t)=v_k(x,t)-{}_0I_t^{\alpha}\left\{\frac{\partial^{\alpha}v_k(x,\tau)}{\partial\tau^{\alpha}}-\frac{\partial^{\alpha}v_k(x,0)}{\partial\tau^{\alpha}}+\frac{\partial^{\alpha}}{\partial\tau^{\alpha}}L_{\alpha}^{-1}\left[\frac{1}{s^{\alpha}}L_{\alpha}\left(\Omega_2\right)\right]\right\} \tag{29}$$

where $\Omega_1=P_{1\beta}(u_k,v_k)+N_{1\beta}(u_k,v_k)$, $\Omega_2=P_{2\beta}(u_k,v_k)+N_{2\beta}(u_k,v_k)$.

Finally, the solutions are given by:

$$u(x,t)=\lim_{k\to\infty}u_k(x,t)$$

$$v(x,t)=\lim_{k\to\infty}v_k(x,t)$$

The solution of the fractional Whitham-Broer-Kaup system

Consider the following local fractional Whitham-Broer-Kaup system in a fractal medium:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} + u\frac{\partial^{\beta} u}{\partial x^{\beta}} + \frac{\partial^{\beta} v}{\partial x^{\beta}} + b\frac{\partial^{2\beta} u}{\partial x^{2\beta}} = 0$$

$$\frac{\partial^{\alpha} v}{\partial t^{\alpha}} + u\frac{\partial^{\beta} v}{\partial x^{\beta}} + v\frac{\partial^{\beta} u}{\partial x^{\beta}} + a\frac{\partial^{3\beta} u}{\partial x^{3\beta}} - b\frac{\partial^{2\beta} v}{\partial x^{2\beta}} = 0 \tag{30}$$

with the initial conditions:

$$u(x,0) = f(x^{\alpha}), \quad v(x,0) = g(x^{\alpha}) \tag{31}$$

Here, we have:

$$P_{1\beta}(u,v) = \frac{\partial^{\beta} v}{\partial x^{\beta}} + b\frac{\partial^{2\beta} u}{\partial x^{2\beta}}, \quad N_{1\beta}(u,v) = u\frac{\partial^{\beta} u}{\partial x^{\beta}}$$

$$P_{2\beta}(u,v) = a\frac{\partial^{3\beta} u}{\partial x^{3\beta}} - b\frac{\partial^{2\beta} v}{\partial x^{2\beta}}, \quad N_{2\beta}(u,v) = \frac{\partial^{\beta}(uv)}{\partial x^{\beta}}$$

By eqs. (22) and (23), we obtain:

$$u_{k+1}(x,t) = u_k(x,t) - {}_0I_t^{\alpha}\left\{\frac{\partial^{\alpha} u_k(x,\tau)}{\partial \tau^{\alpha}} - \frac{\partial^{\alpha} u_k(x,0)}{\partial \tau^{\alpha}} + \frac{\partial^{\alpha}}{\partial \tau^{\alpha}} L_{\alpha}^{-1}\left[\frac{1}{s^{\alpha}} L_{\alpha}\left(\Psi_1\right)\right]\right\} \tag{32}$$

$$v_{k+1}(x,t) = v_k(x,t) - {}_0I_t^{\alpha}\left\{\frac{\partial^{\alpha} v_k(x,\tau)}{\partial \tau^{\alpha}} - \frac{\partial^{\alpha} v_k(x,0)}{\partial \tau^{\alpha}} + \frac{\partial^{\alpha}}{\partial \tau^{\alpha}} L_{\alpha}^{-1}\left[\frac{1}{s^{\alpha}} L_{\alpha}\left(\Psi_2\right)\right]\right\} \tag{33}$$

where

$$\Psi_1 = \frac{\partial^{\beta} v}{\partial x^{\beta}} + b\frac{\partial^{2\beta} u}{\partial x^{2\beta}} + u\frac{\partial^{\beta} u}{\partial x^{\beta}}, \quad \Psi_2 = a\frac{\partial^{3\beta} u}{\partial x^{3\beta}} - b\frac{\partial^{2\beta} v}{\partial x^{2\beta}} + \frac{\partial^{\beta}(uv)}{\partial x^{\beta}}$$

Using the initial condition, we can select:

$$u_0(x,t) = f(x^{\alpha}), \quad v_0(x,t) = g(x^{\alpha})$$

From this selection, we can get the successive approximations solutions.

Example: Consider the following local fractional Whitham-Broer-Kaup system:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} + u\frac{\partial^{\beta} u}{\partial x^{\beta}} + \frac{\partial^{\beta} v}{\partial x^{\beta}} + \frac{\partial^{2\beta} u}{\partial x^{2\beta}} = 0$$

$$\frac{\partial^{\alpha} v}{\partial t^{\alpha}} + u\frac{\partial^{\beta} v}{\partial x^{\beta}} + v\frac{\partial^{\beta} u}{\partial x^{\beta}} + \frac{\partial^{3\beta} u}{\partial x^{3\beta}} - \frac{\partial^{2\beta} v}{\partial x^{2\beta}} = 0$$

with the initial conditions:

$$u(x,0) = -2\sqrt{2}[\tanh(x^{\beta}) + 1], \quad v(x,0) = -2\sqrt{2}(1+\sqrt{2})\sec h^2(x^{\beta})$$

Firstly, let $E = \exp(2x^\beta)$. From the initial condition, we select:

$$u_0(x,t) = \frac{-4\sqrt{2}E}{1+E}, \quad v_0(x,t) = \frac{-8\sqrt{2}(1+\sqrt{2})E}{(1+E)^2}$$

Then, by eqs. (32) and (33), we get the following approximations:

$$u_1(x,t) = \frac{-4\sqrt{2}E}{1+E} + \frac{-32E}{(1+E)^2}\frac{t^\alpha}{\Gamma(1+\alpha)}$$

$$v_1(x,t) = \frac{-8\sqrt{2}(1+\sqrt{2})E}{(1+E)^2} + \frac{64(1+\sqrt{2})(E^2-E)}{(1+E)^3}\frac{t^\alpha}{\Gamma(1+\alpha)}$$

$$u_2(x,t) = \frac{-4\sqrt{2}E}{1+E} + \frac{-32E}{(1+E)^2}\frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{128\sqrt{2}(E^2-E)}{(1+E)^3}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$

$$v_2(x,t) = \frac{-8\sqrt{2}(1+\sqrt{2})E}{(1+E)^2} + \frac{64(1+\sqrt{2})(E^2-E)t^\alpha}{(1+E)^3\Gamma(1+\alpha)} + \frac{256(2+\sqrt{2})(-E^3+4E^2-E)t^{2\alpha}}{(1+E)^4\Gamma(1+2\alpha)}$$

and so on.

When $\alpha = \beta = 1$, we obtain:

$$u(x,t) = \frac{-4\sqrt{2}E}{1+E} + \frac{-32E}{(1+E)^2}t + \frac{128\sqrt{2}(E^2-E)}{(1+E)^3}t^2 + \cdots$$

$$v(x,t) = \frac{-8\sqrt{2}(1+\sqrt{2})E}{(1+E)^2} + \frac{64(1+\sqrt{2})(E^2-E)t}{(1+E)^3\Gamma\left(1+\alpha\right)} + \frac{256(2+\sqrt{2})(-E^3+4E^2-E)t^2}{(1+E)^4\Gamma(1+2\alpha)} + \cdots$$

which represents the Taylor series of the exact solutions.

Discussion and conclusion

The couple of the variational iteration method and the Laplace transform is extremely suitable for solving the fractional or fractal differential equations, and the modification is called as He-Laplace variational iteration method in [33, 34]. Recently Ling and Wu [36] studied the Whitham-Broer-Kaup equation with fractal derivatives and established a variational principle, the present method can be easily extended to the fractal Whitham-Broer-Kaup equation. The variational principle and periodic property of eq. (1) can be studied in a similar way as discussed in [37-42].

In the present work, the approximate analytical solutions for the local fractional Whitham-Broer-Kaup system are obtained by the local fractional variational iteration method coupled the Yang-Laplace transform. The present method is very efficient for finding the approximate analytical solutions for the system of non-linear local fractional differential equations.

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