THE FIRST SOLUTION FOR THE HELICAL FLOWS OF GENERALIZED MAXWELL FLUID WITH LONGITUDINAL TIME DEPENDENT SHEAR STRESSES ON THE BOUNDARY

by

Ping WANG[a], Jin-Ling LIU[b], and Fang WANG[c,*]

[a] Data Science and Intelligent Engineering School, Xiamen Institute of Technology, Xiamen, Fujian, China
[b] College of Mathematics and Science, Yongzhou Normal College, Yongzhou, Hunan, China
[c] School of Mathematics and Statistics, Changsha University of Science and Technology, Changsha, Hunan, China

Original scientific paper
https://doi.org/10.2298/TSCI2202113W

Helical flows of generalized Maxwell fluid is researched between two infinite coaxial circular cylinders. The velocity field and the adequate shear stress corresponding to the flow of a Maxwell fluid with fractional derivative model, between two infinite coaxial cylinders, are determined by means of the Laplace and finite Hankel transforms. The first solutions that have been obtained, presented under integral and series form in terms of the generalized G- and R-functions, satisfy all imposed initial and boundary conditions. The similar solutions for ordinary Maxwell and Newtonian fluid can be also obtained as the limit of the solution of generalized Maxwell fluid.

Key words: *helical flows, Maxwell fluid, finite Hankel transforms, exact solutions*

Introduction

The generalized Maxwell fluid as one of the non-Newtonian fluid is now considered to be more important and appropriate in technological application than Newtonian fluids. For Newtonian fluids, the transient velocity distribution for the flow within a circular cylinder can be found in [1]. The first exact solutions for flows of non-Newtonian fluids in such a domain seem to be those of Ting [2], corresponding to second grade fluids and Srivastava [3] for Maxwell fluids. Later, Casarella *et al.* [4] obtained an exact solution for the motion of a second grade fluid due to both longitudinal and torsional oscillations of the rod. Rajagopal [5] found two simple but elegant solutions for the flow of the same fluid induced by the longitudinal and torsional oscillations of an infinite rod. These solutions have been already extended to Oldroyd-B fluids by Rajagopal *et al.* [6]. During recent years, many papers of this type have been obtained by Khan *et al.* [7], Rajagopal [8], Fetecau [9], and Yang [10, 11]. The helical flow in an annular region between two coaxial circular cylindrical surfaces due to a combination of their rotation and the flow along the axis. In general, the streamlines are helices [12]. Such a motion is very important to study the mechanism of viscoelastic fluids flow in many industry fields, such as oil exploitation, chemical and food industry, bio-engineering

* Corresponding author, e-mail: wangfang1209@csust.edu.cn

and lubrication studies [13]. Such a flow includes simple shear, channel, Couette, Poiseuille, and pipe flows as special cases. The term helical flow was first introduced by Rivlin [14] who derived the velocity distribution for fluids of the differential type in a concentric annular space. Coleman [15] also studied the concentric helical flow and gave the fundamental theory for a general fluid. Using Taylor series expansion of the velocity profiles, Wood [16] has considered the helical flow of an Oldroyd-B fluid due to the combined action of rotating cylinders and a constant pressure gradient. Fetecau *et al.* [17] studied some helical flows of Maxwell and Oldroyd-B fluids between two infinite coaxial cylinders and within an infinite cylinder by means of the expansion theorem of Steklov. The velocity fields and the associated tangential stresses are determined in form of series with Bessel functions.

Motivated by the introduction, this paper researches the Helical flows of Maxwell fluid between two infinite coaxial circular cylinders. The inner cylinder begins to rotate around its axis and to slide along the same axis due to the torsional and longitudinal time dependent shear stresses. The exact solution of velocity field and shear stress are obtained by the generalized *G*- and *R*-functions.

Basic governing equations

The conservation and constitutive equations of an incompressible Maxwell fluid with fractional derivative are given by [18]:

$$T = -pI + S$$
$$S + \lambda(\mathrm{D}_t^{\alpha} S + V\nabla S - LS - SL^T) = \mu A \tag{1}$$

where T is the Cauchy stress tensor, pI – the indeterminate spherical stress, S – the extra-stress tensor, λ – the material constant, μ – the dynamic viscosity of the fluid, $A = L + L^T$ – the first Rivlin-Ericksen tensor with L the velocity gradient, V – the velocity vector, ∇ – the gradient operator, the superscript T – the transpose operation, and the fractional differential operators D_t^{α} are defined as [19]. This model can be reduced to ordinary Maxwell model when $\alpha \to 1$. Furthermore, this model reduces to the classical Newtonian model for $\alpha \to 1$ and $\lambda \to 0$.

In cylindrical co-ordinates (r, θ , z), the helical flow velocity is:

$$V = V(r,t) = \omega(r,t)e_\theta + v(r,t)e_z \tag{2}$$

where e_θ and e_z are the unit vectors in the θ- and z-directions. For such flows, the constraint of incompressibility is automatically satisfied. Since the velocity field (2) depends only on r and t, so the extra stress tensor S is also independent of θ and z. If the fluid is assumed to be at rest at the moment $t = 0$, then $V(r,0) = 0$, $S(r,0) = 0$, introducing (2) into the constitutive equation (1), we find that:

$$(1+\lambda \mathrm{D}_t^{\alpha})S_{rr} = 0, \quad (1+\lambda \mathrm{D}_t^{\alpha})\tau_1(r,t) = \mu\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\omega(r,t), \quad (1+\lambda \mathrm{D}_t^{\alpha})\tau_2(r,t) = \mu\frac{\partial v(r,t)}{\partial r} \tag{3}$$

where $\tau_1 = S_{r\theta}$ and $\tau_2 = S_{rz}$ are the shear stresses, which are different of zero. In the absence of body forces and pressure gradient in the axial direction, the balance of the linear momentum leads to the relevant and meaningful equation:

$$(1+\lambda \mathrm{D}_t^\alpha)\tau_2(r,t)=\mu\frac{\partial v(r,t)}{\partial r},\quad \rho\frac{\partial v(r,t)}{\partial t}=\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\tau_2(r,t) \tag{4}$$

where ρ is the constant density of the fluid.

In this paper, we are interested into the helical flow [20] of a generalized Maxwell fluid between two infinite coaxial circular cylinders of radius R_1 and R_2 $(R_2 > R_1)$. Suppose that an incompressible Maxwell fluid at rest is situated in the annular region between two infinite coaxial circular cylinders. At time $t=0^+$ the inner cylinder begins to rotate around its axis due to a time dependent shear $\tau_1(r,t)$ and to slide along the same axis due to a time dependent shear $\tau_2(R_1,t)$:

$$\tau_1(R_1,t)=\frac{f}{\lambda}R_{\alpha,-2}\left(\frac{-1}{\lambda},t\right),\quad \tau_2(R_1,t)=\frac{g}{\lambda}R_{\alpha,-2}\left(\frac{-1}{\lambda},t\right),\quad 0<\alpha<1 \tag{5}$$

where f and g are constants and the generalized $R_{a,b}(c, t)$ function is defined by [21]:

$$R_{a,b}(c,t)=\sum_{n=0}^{\infty}\frac{c^n t^{(n+1)a-b-1}}{\Gamma[(n+1)a-b]} \tag{6}$$

Due to the shear, the fluid is gradually moved. Its velocity is of the form (2) and the governing equations are given by eqs. (3) and (4). The appropriate initial and boundary conditions are:

$$\omega(r,0)=v(r,0)=\frac{\partial\omega(r,0)}{\partial t}=\frac{\partial v(r,0)}{\partial t}=0,\quad \tau_1(r,0)=\tau_2(r,0)=0,\quad r\in[R_1,R_2] \tag{7}$$

$$(1+\lambda \mathrm{D}_t^\alpha)\tau_1(r,t)|_{r=R_1}=ft,\quad t>0 \tag{8}$$

$$(1+\lambda \mathrm{D}_t^\alpha)\tau_2(r,t)|_{r=R_1}=gt,\quad t>0 \tag{9}$$

$$\omega(R_2,t)=0, v(R_2,t)=0,\quad t>0 \tag{10}$$

Calculation of the velocity field

Applying the Laplace transform to (4) and using the Laplace transform formula for sequential fractional derivatives, we obtain:

$$(1+\lambda s^\alpha)\bar{\tau}_1(r,s)=\mu\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\bar{\omega}(r,s),\quad (1+\lambda s^\alpha)\bar{\tau}_2(r,s)=\mu\frac{\partial\bar{v}(r,s)}{\partial r} \tag{11}$$

$$\rho s\bar{\omega}(r,s)=\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)\bar{\tau}_1(r,s),\quad \rho s\bar{v}(r,s)=\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\bar{\tau}_2(r,s) \tag{12}$$

Eliminating τ_1 and τ_2 among (11)-(12), we obtain the ODE:

$$(s+\lambda s^{\alpha+1})\bar{\omega}(r,s)=\nu\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)\bar{\omega}(r,s) \tag{13}$$

$$(s+\lambda s^{\alpha+1})\bar{v}(r,s)=\nu\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\bar{v}(r,s) \tag{14}$$

where $\nu=\mu/\rho$ is the kinematic viscosity of the fluid, the image functions have to satisfy the conditions:

$$\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\bar{\omega}(r,s)|_{r=R_1}=\frac{f}{\mu s^2},\quad \frac{\partial\bar{v}(r,s)}{\partial r}|_{r=R_1}=\frac{f}{\mu s^2} \tag{15}$$

$$\bar{\omega}(R_2,s)=0,\quad \bar{v}(R_2,s)=0 \tag{16}$$

In the following, let us denote the Hankel transforms of $\omega(r,s)$ and $v(r,s)$ [22]:

$$\bar{\omega}_H(r_p,s)=\int_{R_1}^{R_2} r\bar{\omega}(r,s)B_\omega(rr_p)\mathrm{d}r,\quad p=1,2,3,\cdots \tag{17}$$

$$\bar{v}_H(r_q,s)=\int_{R_1}^{R_2} r\bar{v}(r,s)B_v(rr_q)\mathrm{d}r,\quad q=1,2,3,\cdots \tag{18}$$

where r_p and r_q are the positive roots of the transcendental equation $B_\omega(R_{2r})=0$ and $B_v(R_{2r})=0$, and:

$$B_\omega(rr_p)=J_1(rr_p)Y_2(R_1r_p)-J_2(R_1r_p)Y_1(rr_p) \tag{19}$$

$$B_v(rr_q)=J_0(rr_q)Y_1(R_1r_q)-J_1(R_1r_q)Y_0(rr_q) \tag{20}$$

where $J_v(\cdot),Y_v(\cdot)$ are the Bessel functions of the first and second kind of order v. Multiplying both sides of (13) and (14) by $rB_\omega(rr_p)$ and $rB_v(rr_q)$, integrating with respect to r from R_1 to R_2 and taking into account the conditions (16), we can obtain:

$$\int_{R_1}^{R_2} r\left(\frac{\partial^2\bar{\omega}}{\partial r^2}+\frac{1}{r}\frac{\partial\bar{\omega}}{\partial r}-\frac{\bar{\omega}}{r^2}\right)rB_\omega(rr_p)\mathrm{d}r=\frac{2}{\pi r_p}\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\bar{\omega}(r,s)|_{r=R_1}-r_p^2\bar{\omega}_H \tag{21}$$

$$\int_{R_1}^{R_2} r\left(\frac{\partial^2\bar{v}}{\partial r^2}+\frac{1}{r}\frac{\partial\bar{v}}{\partial r}\right)B_v(rr_q)\mathrm{d}r=\frac{2}{\pi r_q}\frac{\partial\overline{v(R_1,s)}}{\partial r}-r_q^2\bar{v}_H \tag{22}$$

and

$$\int_{R_1}^{R_2} r(s+\lambda s^{\alpha+1})\bar{\omega}(r,s)B_\omega(rr_p)\,\mathrm{d}r=(s+\lambda s^{\alpha+1})\bar{\omega}_H \tag{23}$$

$$\int_{R_1}^{R_2} r(s+\lambda s^{\alpha+1})\bar{v}(r,s)B_v(rr_q)\mathrm{d}r=(s+\lambda s^{\alpha+1})\bar{v}_H \tag{24}$$

So we can get:

$$(s+\lambda s^{\alpha+1})\bar{\omega}_H=\frac{2\nu}{\pi r_p}\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\bar{\omega}(r,s)|_{r=R_1}-\nu r_p^2\bar{\omega}_H=\frac{2\nu}{\pi r_p}\frac{f}{\mu s^2}-\nu r_p^2\bar{\omega}_H \tag{25}$$

$$(s+\lambda s^{\alpha+1})\bar{v}_H = \frac{2\nu}{\pi r_q}\frac{\partial \overline{v(R_1,s)}}{\partial r} - \nu r_q^2 \bar{v}_H = \frac{2\nu}{\pi r_q}\frac{g}{\mu s^2} - \nu r_q^2 \bar{v}_H \tag{26}$$

in order to determine $\bar{\omega}(r,s)$ and $\bar{v}(r,s)$ from $\omega_H(s)$ and $v_H(s)$, we first write (25) and (26) under the suitable form:

$$\bar{\omega}_H = \frac{R_2^2}{R_1^2}\frac{4}{\pi r_p^3}\frac{fR_1^2}{2R_2^2 \mu s^2} - \frac{2f}{\pi \mu r_p^3}\frac{1+\lambda s^\alpha}{s(s+\lambda s^{\alpha+1}+\nu r_p^2)}$$

$$\bar{v}_H = \frac{2}{\pi R_1 r_q^3}\frac{gR_1}{\mu s^2} - \frac{2g}{\pi \mu r_q^3}\frac{1+\lambda s^\alpha}{s(s+\lambda s^{\alpha+1}+\nu r_q^2)} \tag{27}$$

and use the inverse Hankel transform formula [23], one gets the expression for $\bar{\omega}$ and $\bar{v}$ in the form:

$$\bar{\omega}(r,s) = \frac{\pi^2}{2}\sum_{p=1}^{\infty}\frac{r_p^2 J_1^2(R_2 r_P) B_\omega(r r_P)}{J_2^2(R_1 r_P) - J_1^2(R_2 r_P)}\bar{\omega}_H, \quad \bar{v}(r,s) = \frac{\pi^2}{2}\sum_{q=1}^{\infty}\frac{r_q^2 J_1^2(R_2 r_q) B_\omega(r r_q)}{J_2^2(R_1 r_q) - J_1^2(R_2 r_q)}\bar{v}_H \tag{28}$$

Due to:

$$\int_{R_1}^{R_2} J_1(r r_p)\mathrm{d}r = \frac{1}{r_p}[J_0(R_1 r_p) - J_0(R_2 r_p)], \quad \int_{R_1}^{R_2} Y_1(r r_p)\mathrm{d}r = \frac{1}{r_p}[J_0(R_1 r_p) - J_0(R_2 r_p)] \tag{29}$$

and

$$\int_{R_1}^{R_2} r^2 Y_1(r r_p)\mathrm{d}r = \frac{R_2^2}{r_p}Y_2(R_2 r_p) - \frac{R_1^2}{r_p}Y_2(R_1 r_p) \tag{30}$$

so

$$\int_{R_1}^{R_2} \frac{r^2 - R_2^2}{r}B_\omega(r r_p)\mathrm{d}r = \frac{R_2^2}{R_1^2}\frac{4}{\pi r_p^3} \tag{31}$$

Since:

$$\int_{R_1}^{R_2} r J_0(r r_q)\mathrm{d}r = \frac{R_2}{r_q}J_1(R_2 r_q) - \frac{R_1}{r_q}J_1(R_1 r_q) \tag{32}$$

$$\int_{R_1}^{R_2} r Y_0(r r_p)\mathrm{d}r = \frac{R_2}{r_q}Y_1(R_2 r_p) - \frac{R_1}{r_q}Y_1(R_1 r_p) \tag{33}$$

$$\int_{R_1}^{R_2} r\ln r J_0(r r_q)\mathrm{d}r = \frac{R_2 \ln R_2}{r_q}J_1(R_2 r_q) - \frac{R_1 \ln R_1}{r_q}J_1(R_1 r_q) + \frac{1}{r_q^2}[J_0(R_2 r_q) - J_0(R_1 r_q)] \tag{34}$$

$$\int_{R_1}^{R_2} r\ln rY_0(rr_p)\mathrm{d}r = \frac{R_2\ln R_2}{r_q}Y_1(R_2r_q) - \frac{R_1\ln R_1}{r_q}Y_1(R_1r_q) + \frac{1}{r_q^2}[Y_0(R_2r_q) - Y_0(R_1r_q)] \quad (35)$$

that is

$$\int_{R_1}^{R_2} r\ln\left(\frac{r}{R_2}\right)B_v(rr_q)\mathrm{d}r = \frac{2}{\pi R_1 r_q^3} \quad (36)$$

Indeed, taking into account (30) and (35), we can obtain:

$$\bar{\omega}(r,s) = \frac{r^2 - R_2^2}{r}\frac{fR_1^2}{2R_2^2\mu s^2} - \frac{\pi f}{\mu}\sum_{q=1}^{\infty}\frac{J_1^2(R_2r_p)B_\omega(rr_p)}{r_p[J_2^2(R_1r_p) - J_1^2(R_2r_p)]}\frac{1+\lambda s^\alpha}{s(s+\lambda s^{\alpha+1} + \nu r_p^2)} \quad (37)$$

and

$$\bar{v} = \frac{gR_1}{\mu s^2}\ln\left(\frac{r}{R_2}\right) - \frac{\pi g}{\mu}\sum_{q=1}^{\infty}\frac{J_1^2(R_2r_p)B_\omega(rr_p)}{r_p[J_2^2(R_1r_p) - J_1^2(R_2r_p)]}\frac{1+\lambda s^\alpha}{s(s+\lambda s^{\alpha+1} + \nu r_p^2)} \quad (38)$$

In order to avoid the burdsome calculations of residues and contour integrals, we apply the discrete inversion Laplace transform method, we can get:

$$\frac{1}{s(s+\lambda s^{\alpha+1} + \nu r_q^2)} = \frac{1}{\lambda}\sum_{k=0}^{\infty}\left(-\frac{\nu r_q^2}{\lambda}\right)^k \frac{s^{-k-2}}{(\lambda^{-1} + s^\alpha)^{k+1}}$$

$$\frac{\lambda s^\alpha}{s(s+\lambda s^{\alpha+1} + \nu r_q^2)} = \sum_{k=0}^{\infty}\left(-\frac{\nu r_q^2}{\lambda}\right)^k \frac{s^{\alpha-k-2}}{(\lambda^{-1} + s^\alpha)^{k+1}}$$

For this we use the expansion:

$$\frac{1+\lambda s^\alpha}{s(s+\lambda s^{\alpha+1} + \nu r_p^2)} = \frac{1}{\lambda}\sum_{k=0}^{\infty}\left(-\frac{\nu r_p^2}{\lambda}\right)^k\left[\frac{s^{-k-2}}{(\lambda^{-1}+s^\alpha)^{k+1}} + \frac{\lambda s^{\alpha-k-2}}{(\lambda^{-1}+s^\alpha)^{k+1}}\right] \quad (39)$$

$$\frac{1+\lambda s^\alpha}{s(s+\lambda s^{\alpha+1} + \nu r_q^2)} = \frac{1}{\lambda}\sum_{k=0}^{\infty}\left(-\frac{\nu r_q^2}{\lambda}\right)^k\left[\frac{s^{-k-2}}{(\lambda^{-1}+s^\alpha)^{k+1}} + \frac{\lambda s^{\alpha-k-2}}{(\lambda^{-1}+s^\alpha)^{k+1}}\right] \quad (40)$$

Introducing (37) and (38) into (39) and (40), applying the discrete inverse Laplace transform and using the known result:

$$L^{-1}\left\{\frac{s^b}{(s^a - \mathrm{d})^c}\right\} = G_{a,b,c}(\mathrm{d},t)$$

where the generalized *G*-function and $(c)_j$ is the Pochhammer polynomial [24]:

$$G_{a,b,c}(\mathrm{d},t) = \sum_{j=0}^{\infty}\frac{\Gamma(c+j)\mathrm{d}^j}{\Gamma(c)\Gamma(j+1)}\frac{t^{(j+c)a-b-1}}{\Gamma[(j+c)a-b]} \quad (41)$$

we obtain the following expressions for the velocity field:

$$\omega(r,t)=\frac{r^2-R_2^2}{r}\frac{R_1^2}{2R_2^2}\frac{ft}{\mu}-\frac{\pi f}{\lambda\mu}\sum_{p=1}^{\infty}\frac{J_1^2(R_2r_p)B_\omega(rr_p)}{r_p[J_2^2(R_1r_p)-J_1^2(R_2r_p)]}\sum_{k=0}^{\infty}\left(-\frac{\nu r_p^2}{\lambda}\right)^k$$

$$[G_{\alpha,-k-2,k+1}(-\lambda^{-1},t)+\lambda G_{\alpha,\alpha-k-2,k+1}(-\lambda^{-1},t)] \tag{42}$$

and

$$v(r,t)=\frac{gR_1t}{\mu}\ln\left(\frac{r}{R_2}\right)-\frac{\pi g}{\lambda\mu}\sum_{q=1}^{\infty}\frac{J_0^2(R_2r_q)B_v(rr_q)}{r_q[J_1^2(R_1r_q)-J_0^2(R_2r_q)]}\sum_{k=0}^{\infty}\left(-\frac{\nu r_q^2}{\lambda}\right)^k$$

$$[G_{\alpha,-k-2,k+1}(-\lambda^{-1},t)+\lambda G_{\alpha,\alpha-k-2,k+1}(-\lambda^{-1},t)] \tag{43}$$

In view of the definitions (41) of the generalized function *G*, the shear stress $\omega(r,t)$ and $v(r,t)$ clearly satisfy the initial condition (7). We can easily obtain that $\omega(R_2,t)=0$, $v(R_2,t)=0$.

Calculation of the shear stress

Applying the Laplace transform to (3)-(4), and using the initial condition (7), we find that:

$$\bar{\tau}_1(r,s)=\frac{\mu}{1+\lambda s^\alpha}\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\bar{\omega}(r,s) \tag{44}$$

$$\bar{\tau}_2(r,s)=\frac{\mu}{1+\lambda s^\alpha}\frac{\partial\bar{v}(r,s)}{\partial r} \tag{45}$$

so

$$\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\bar{\omega}(r,s)=\frac{2R_2^2}{r^2}\frac{fR_1^2}{2R_2^2\mu s^2}-\frac{\pi f}{\mu}\sum_{p=1}^{\infty}\frac{J_1^2(R_2r_p)[r_p\tilde{B}_\omega(rr_p)-\frac{2}{r}B_\omega(rr_p)]}{r_p[J_2^2(R_1r_p)-J_1^2(R_2r_p)]}$$

$$\frac{1+\lambda s^\alpha}{s(s+\lambda s^{\alpha+1}+\nu r_p^2)} \tag{46}$$

and

$$\frac{\partial\bar{v}(r,s)}{\partial r}=\frac{R_1}{r}\frac{g}{\mu s^2}+\frac{\pi g}{\mu}\sum_{q=1}^{\infty}\frac{J_0^2(R_2r_q)\tilde{B}_\omega(rr_q)}{J_1^2(R_1r_q)-J_0^2(R_2r_q)}\frac{1+\lambda s^\alpha}{s(s+\lambda s^{\alpha+1}+\nu r_q^2)} \tag{47}$$

Introducing (46) and (47) into (44) and (45), we can obtain:

$$\bar{\tau}_1(r,s)=\frac{R_1^2}{r^2}\frac{f}{s^2}\frac{1}{1+\lambda s^\alpha}-\pi f\sum_{p=1}^{\infty}\frac{J_1^2(R_2r_p)[r_p\tilde{B}_\omega(rr_p)-\frac{2}{r}B_\omega(rr_p)]}{r_p[J_2^2(R_1r_p)-J_1^2(R_2r_p)]}\frac{1+\lambda s^\alpha}{s(s+\lambda s^{\alpha+1}+\nu r_p^2)} \tag{48}$$

and

$$\bar{\tau}_2(r,s)=\frac{R_1}{r}\frac{g}{s^2}\frac{1}{1+\lambda s^\alpha}+\pi g\sum_{q=1}^{\infty}\frac{J_0^2(R_2 r_q)\tilde{B}_\omega(r r_q)}{J_1^2(R_1 r_q)-J_0^2(R_2 r_q)}\frac{1+\lambda s^\alpha}{s(s+\lambda s^{\alpha+1}+\nu r_q^2)} \tag{49}$$

Applying again the discrete inversion Laplace transform to the obtained results, we find the expression of shear stress in the following form:

$$\tau_1(r,t)=\frac{R_1^2}{\lambda r^2}fR_{\alpha,-2}(-\lambda^{-1},t)-\frac{\pi f}{\lambda}\sum_{p=1}^{\infty}\frac{J_1^2(R_2 r_p)[r_p\tilde{B}_\omega(r r_p)-\frac{2}{r}B_\omega(r r_p)]}{r_p[J_2^2(R_1 r_p)-J_1^2(R_2 r_p)]}$$

$$\sum_{k=0}^{\infty}\left(-\frac{\nu r_p^2}{\lambda}\right)^k G_{\alpha,-k-2,k+1}(-\lambda^{-1},t) \tag{50}$$

and

$$\tau_2(r,t)=\frac{R_1}{\lambda r}gR_{\alpha,-2}(-\lambda^{-1},t)+\frac{\pi g}{\lambda}\sum_{q=1}^{\infty}\frac{J_0^2(R_2 r_q)\tilde{B}_\omega(r r_q)}{J_1^2(R_1 r_q)-J_0^2(R_2 r_q)}\sum_{k=0}^{\infty}\left(-\frac{\nu r_q^2}{\lambda}\right)^k G_{\alpha,-k-2,k+1}(-\lambda^{-1},t) \tag{51}$$

In view of the definitions (11) and (41) of the generalized functions *R* and *G*, the shear stress $\tau_1(r,t)$ and $\tau_2(r,t)$ clearly satisfy the initial condition (7) and the boundary condition (8)-(10).

In the special case when $\alpha \to 1$, eqs. (42), (43), (50), and (51) can be simplified as the Classical Maxwell fluid. By letting $\lambda \to 0$ into eqs. (52) and (55), we can also obtain the corresponding conclusion for the Newtonian fluid.

Acknowledgment

Ping Wang is supported by the educational research projects of young and middle-aged teachers in Fujian Province in 2020 (Science and technology JAT200837). Fang Wang is supported by the National Natural Science Foundation of China (No. 12001064), the Hunan Provincial Education Department Project (No. 20B006).

Nomenclature

t – time, [s]
T – Cauchy stress tensor, [Pa]
x, y, z – co-ordinates, [m]
V – velocity vector, [ms^{-1}]

References

[1] Batchelor, G. K., *An Introduction to fluid dynamics*, Cambridge University Press, Cambridge, UK, 1967
[2] Ting, T. W., Certain Non-Steady flows of Second-Order fluids, Archive for Rational Mechanics and Analysis, *14* (1963), 1, pp. 1-26
[3] Srivastava, P. N., Non-Steady Helical Flow of a Viscoelastic Liquid, *Archiwum Mechaniki Stosowanej*, *18* (1966), 1, pp. 145-150
[4] Casarella, M. J., *et al.*, Drag on an Oscillating Rod with Longitudinal and Torsional Motion, *Journal of Hydronautics*, *3* (2012), 4, pp. 180-183
[5] Rajagopal, K. R., Longitudinal and Torsional Oscillations of a Rod in a Non-Newtonian Fuid, *Acta Mechanica*, *49* (1983), 3, pp. 281-285

[6] Rajagopal, K. R., *et al.*, Exact Solutions for Some Simple flows of an Oldroyd-B fluid, *Acta Mechanica*, *113* (1995), 1-4, pp. 233-239
[7] Khan, M., *et al.*, Oscillating Flow of a Burgers Fluid in a Pipe, *Abdus Salam International Center for Theoretical Physics*, No. IC/2005/071
[8] Rajagopal, K. R., A Note on Unsteady Unidirectional flows of a Non-Newtonian Fluid, *International Journal of Non-Linear Mechanics*, *17* (1982), 5-6, pp. 369-373
[9] Fetecau, C., *et al.*, Starting Solutions for the Motion of a Second Grade Fluid Due to Longitudinal and Torsional Oscillations of a Circular Cylinder, *International Journal of Engineering Science*, *44* (2006), 11-12, pp. 788-796
[10] Yang, X.-J., New Rheological Problems Involving General Fractional Derivatives with Non-Singular Power-Law Kernels, *Proceedings of the Romanian Academy-Series A*, *19* (2018), 1, pp. 45-52
[11] Yang, X.-J., *et al.*, *General Fractional Derivatives with Applications in Viscoelasticity*, Academic Press, New York, USA, 2020
[12] Huilgol, R. R., *et al.*, *Fluid Mechanics of Viscoelasticity*, Elsevier, New York, USA, 1997
[13] Yu, Z. S., *et al.*, Numerical Research on the Coherent Structure in the Viscoelastic Second-Order Mixing Layers, *Applied Mathmatics and Mechanics*, *19* (1998), 8, pp. 717-723
[14] Rivlin, R. S., Solution of Some Problems in the Exact Theory of Visco-Elasticity, *Archive for Rational Mechanics and Analysis*, *5* (1956), 2, pp. 179-188
[15] Coleman, B.-N. W., Helical Flow of General fluid, *Journal of Applied Physic*s, *30* (1959), 2, pp. 1508-1512
[16] Wood, W. P., Transient Viscoelastic Helical flows in Pipes of Circular and Annular Cross-Section, *Journal of Non-Newtonian Fluid Mechanics*, *100* (2001), 1-3, pp. 115-126
[17] Fetecau, C., *et al.*, Unsteady Helical flows of a Maxwell Fluid, *Proceedings of the Romanian Academy Series A*, *5* (2004), 1, pp. 13-19
[18] Tong, D., *et al.*, Exact Solutions for the Flow of Non-Newtonian Fluid with Fractional Derivative in an Annular Pipe, *Science in China Series G (Physics, Mechanics and Astronomy), 48* (2005), 4, pp. 485-495
[19] Hilfer, R., *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000
[20] Truesdell, C., *et al.*, *The Non-Linear Field Theories of Mechanics*, Encyclopedia of Physics. III/3, Springer, Heidelberg, Berlin, Germany, 1965
[21] Lorenzo, C. F., *et al.*, Generalized Functions for the Fractional Calculus, NASA/TP-1999-209424/Rev1, 1999
[22] Polyanin, A. D., *et al.*, *Handbook of Integral Equations*, CRC Press, Boca Raton, Fla., USA, 1998
[23] Sneddon, I. N., *Functional analysis, Encyclopedia of physics*, vol. II., Springer-Heidelberg, Berlin, Germany, 1955
[24] Wang, F., *et al.*, The Analytic Solutions for the Unsteady Rotation Flows of the Generalized Maxwell Fluid Between Coaxial Cylinders, *Thermal Science*, *24* (2020), 6B, pp. 4041-4048

Paper submitted: June 19, 2021
Paper revised: July 26, 2021
Paper accepted: July 29, 2021