# THE NON-DARCY LAW FOR THE SCALING LAW FLOW IN POROUS MEDIUM

by

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*In this article the non-Darcy law for the scaling law flow in porous medium associated with the scaling law calculus with respect to the Mandelbrots scaling law is suggested for the first time.*

Key words: *non-Darcy law, scaling law flow, scaling law calculus, Mandelbrots scaling law, porous medium*

## Introduction

The Darcy law, discovered in 1856 by a French engineer Henry Philibert Gaspard Darcy [1], is an equation that describes the transport processes for the flow of the fluid in the porous medium [2, 3]. The different forms for the Darcy law was considered in [4-6] and considered in the study of the ground-water motion (laminar flow) [7]. The theoretical derivation of Darcy's law for the porous media was investigated in the different works [8-10]. The Darcy law for the single-phase fluid flow in a porous medium can be written [4]:

$$\mathbf{q}(x,y,z,t) = -\frac{\mathbf{k}}{\boldsymbol{\mu}}\nabla\psi(x,y,z,t) \tag{1}$$

where $\psi(x,y,z,t)$ is the hydraulic head, $\mathbf{q}(x,y,z,t)$ – the Darcy velocity vector (specific discharge), $\mathbf{k}$ – the permeability tensor, and $\boldsymbol{\mu}$ – the dynamic viscosity of the fluid. In many practical case, the Darcy law for the flow of fluids in porous media reads [4, 11]:

$$\mathbf{q}(x,y,z,t) = -\frac{\mathbf{k}}{\boldsymbol{\mu}}[\nabla\psi(x,y,z,t) - \rho\mathbf{g}] \tag{2}$$

where $\rho$ denotes the mass density and $\mathbf{g}$ represents the gravity vector.

Since the scaling law behaviors in the porous media [12, 13], the classical theory of the flow of fluids in porous media are invalid. Based on the scaling law behaviors in the porous media, we suggested the Darcy-like law for the scaling law flow of the fluid in porous medium, given [14]:

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$$q(x,t) = \phi\kappa D x^{-(D+1)} \frac{\partial \psi(x,t)}{\partial x} \tag{3}$$

where $\phi = k/\mu$ is the hydraulic conductivity, $\kappa$ – the normalization constant, and $D$ – the scaling exponent.

The fractal scaling law calculus associated with the Mandelbrots scaling law was proposed by Yang [15] to describe the telegraph equation. The fractal scaling law vector calculus associated with the Mandelbrots scaling law was developed by Yang *et al.* [16] to model the fractal scaling law elasticity. The fractal scaling law flow with the Mandelbrots scaling law was considered in [17]. The scaling law heat conduction process was suggested in [18]. The main aim of the paper is to develop the non-Darcy law for the scaling law flow in porous medium with the aid of the fractal scaling law calculus.

## Theory of the fractal scaling law calculus

The Mandelbrot scaling law is defined [19]:

$$\tau(x) = \ell x^{1-D} \tag{4}$$

where $\ell > 0$, $x > 0$, and $0 \le D \le 1$ is the fractional dimension.

Let:

$$\Phi(x) = (\Phi \circ \tau)(x) = [\Phi \circ (\ell x^{1-D})](x) = \Phi(\ell x^{1-D})$$

where $\ell > 0$, $x > 0$, and $0 \le D \le 1$ is the fractional dimension.

The scaling law derivative of $\Phi(x)$ of order $n$ associated with the Mandelbrot scaling law (4), denoted by ${}^{\mathrm{MSL}}\mathrm{D}_x^{(n)}\Phi(x)$, is defined [15]:

$$ {}^{\mathrm{MSL}}\mathrm{D}_x^{(n)}\Phi(x) = \left(\Im x^D \frac{\mathrm{d}}{\mathrm{d}x}\right)^n \Phi(x) \tag{5}$$

where $n \in \mathbb{N}$, and $\Im = [(1-D)\ell]^{-1}$.

The scaling law integral of the function $\phi(x)$ associated with the Mandelbrot scaling law, denoted by ${}^{\mathrm{MSL}}_{\ a}I_x^{(1)}\phi(x)$, is defined [15, 16]:

$$ {}^{\mathrm{MSL}}_{\ \mathrm{a}}I_x^{(1)}\phi(x) = \Re \int_a^x \phi(x) x^D \mathrm{d}x \tag{6}$$

where $n \in \mathbb{N}$, and $\Re = [(1-D)\ell]$.

The scaling law partial derivatives of the function $\Theta(x, y, z, t)$ associated with the Mandelbrot scaling law reads [15-18]:

$$ {}^{\mathrm{MSL}}\partial_x^{(1)}\Theta(x,y,z,t) = \Im_1 x^{D_1} \frac{\partial \Theta(x,y,z,t)}{\partial x} \tag{7}$$

$$ {}^{\mathrm{MSL}}\partial_y^{(1)}\Theta(x,y,z,t) = \Im_1 y^{D_2} \frac{\partial \Theta(x,y,z,t)}{\partial y} \tag{8}$$

$$ {}^{\mathrm{MSL}}\partial_z^{(1)}\Theta(x,y,z,t) = \Im_3 z^{D_3} \frac{\partial \Theta(x,y,z,t)}{\partial z} \tag{9}$$

$$ {}^{\mathrm{MSL}}\partial_t^{(1)}\Theta(x,y,z,t)=\Im_0 t^{D_0}\frac{\partial\Theta(x,y,z,t)}{\partial t} \tag{10}$$

in which $\Im_0=[(1-D_0)\ell_0]^{-1}, \Im_1=[(1-D_1)\ell_1]^{-1}, \Im_2=[(1-D_2)\ell_2]^{-1}$, and $\Im_3=[(1-D_3)\ell_3]^{-1}$, where $\ell_0$, $\ell_1$, $\ell_2$, and $\ell_3$ are the positive constants, and $0 < D_0, D_1, D_2, D_3 < 1$ are the fractional dimensions.

The total scaling law differential form of the Mandelbrot scaling law scalar field $\Xi=\Xi(x,y,z)$ is defined [15-18]:

$$\begin{aligned} \mathrm{d}\Xi &= \left[\ell_1(1-D_1)x^{-D_1}\,{}^{\mathrm{MSL}}\partial_x^{(1)}\Xi\right]\mathrm{d}x + \\ &+\left[\ell_2(1-D_2)y^{-D_2}\,{}^{\mathrm{MSL}}\partial_y^{(1)}\Xi\right]\mathrm{d}y + \\ &+\left[\ell_3(1-D_3)z^{-D_3}\,{}^{\mathrm{MSL}}\partial_z^{(1)}\Xi\right]\mathrm{d}z = \\ &= \Im_1 x^{-D_1}\,{}^{\mathrm{MSL}}\partial_x^{(1)}\Xi\mathrm{d}x + \Im_2 y^{-D_2}\,{}^{\mathrm{MSL}}\partial_y^{(1)}\Xi\mathrm{d}y + \Im_3 z^{-D_3}\,{}^{\mathrm{MSL}}\partial_z^{(1)}\Xi\mathrm{d}z \end{aligned} \tag{11}$$

The scaling law gradient operator with respect to the Mandelbrot scaling law in a Cartesian co-ordinate system is given by [15-18]:

$$\begin{aligned} \nabla^{\binom{D_1,D_2,D_3}{\ell_1,\ell_2,\ell_3}} &= \mathbf{i}\left[\ell_1(1-D_1)x^{-D_1}\right]{}^{\mathrm{MSL}}\partial_x^{(1)} + \mathbf{j}\left[\ell_2(1-D_2)y^{-D_2}\right]{}^{\mathrm{MSL}}\partial_y^{(1)} + \\ &+\mathbf{k}\left[\ell_3(1-D_3)z^{-D_3}\right]{}^{\mathrm{MSL}}\partial_z^{(1)} = \\ &= \mathbf{i}\Im_1 x^{-D_1}\,{}^{\mathrm{MSL}}\partial_x^{(1)} + \mathbf{j}\Im_2 y^{-D_2}\,{}^{\mathrm{MSL}}\partial_y^{(1)} + \mathbf{k}\Im_3 z^{-D_3}\,{}^{\mathrm{MSL}}\partial_z^{(1)} \end{aligned} \tag{12}$$

where **i**, **j**, and **k** are unite vectors in a Cartesian co-ordinate system.

Therefore, eq. (11) may be rewritten as [15-18]:

$$ \mathrm{d}\Xi = \nabla^{\binom{D_1,D_2,D_3}{\ell_1,\ell_2,\ell_3}}\Xi\mathbf{n}\mathrm{d}r = \nabla^{\binom{D_1,D_2,D_3}{\ell_1,\ell_2,\ell_3}}\Xi\mathrm{d}\mathbf{r} \tag{13}$$

where **n** denotes the unit vector, d$r$ – a distance measured along the normal direction, and d**r** = **n**d$r$ = **i**d$x$ + **j**d$y$ + **k**d$z$ with d**r** = **n**d$r$.

The scaling law volume integral of the fractal scaling law scalar field $\Xi=\Xi(x,y,z)$ is defined by [15-18]:

$$ \mathfrak{V}(\Xi)=\iiint_{\Omega}\Xi\mathrm{d}V \tag{14}$$

where

$$\begin{aligned} \mathrm{d}V &= \left[\ell_1(1-D_1)x^{-D_1}\right]\left[\ell_2(1-D_2)y^{-D_2}\right]\left[\ell_3(1-D_3)z^{-D_3}\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z = \\ &= \Im_1 x^{-D_1}\Im_2 y^{-D_2}\Im_3 z^{-D_3}\mathrm{d}x\mathrm{d}y\mathrm{d}z \end{aligned} \tag{15}$$

Therefore, we see that [15-18]:

$$\begin{aligned}\mathfrak{V}(\Xi) &= \iiint_{\Omega} \Xi \mathrm{d}V \\ &= \aleph \int_{a_3}^{b_3} z^{-D_3} \mathrm{d}z \int_{a_2}^{b_2} y^{-D_2} \mathrm{d}y \int_{a_1}^{b_1} \Xi x^{-D_1} \mathrm{d}x \\ &= \aleph \int_{a_1}^{b_1} x^{-D_1} \mathrm{d}x \int_{a_3}^{b_3} z^{-D_3} \mathrm{d}z \int_{a_2}^{b_2} \Xi y^{-D_2} \mathrm{d}y \\ &= \aleph \int_{a_2}^{b_2} y^{-D_2} \mathrm{d}y \int_{a_1}^{b_1} x^{-D_1} \mathrm{d}x \int_{a_3}^{b_3} \Xi z^{-D_3} \mathrm{d}z\end{aligned} \tag{16}$$

where $x \in [a_1, b_1], y \in [a_2, b_2], z \in [a_3, b_3]$, and $\aleph = \Im_1 \Im_2 \Im_3$.

For the more results on the scaling law calculus, see [20].

## The non-Darcy law for the scaling law flow in porous medium

With the aid of the scaling law gradient operator with respect to the Mandelbrot scaling law in a Cartesian co-ordinate system, the non-Darcy law for the scaling law flow in porous medium can be expressed by:

$$\tilde{\mathbf{q}}(x, y, z, t) = -\frac{\mathbf{k}}{\boldsymbol{\mu}} \nabla^{\left(\substack{D_1, D_2, D_3 \\ \ell_1, \ell_2, \ell_3}\right)} \tilde{\psi}(x, y, z, t) \tag{17}$$

where $\tilde{\psi}(x, y, z, t)$ is the scaling law hydraulic head, $\tilde{\mathbf{q}}(x, y, z, t)$ – the non-Darcy velocity vector, $\mathbf{k}$ – the permeability tensor, and $\boldsymbol{\mu}$ – the dynamic viscosity.

When $\ell_0 = \ell_1 = \ell_2 = \ell_3 = 1$ and $D_0 = D_1 = D_2 = D_3 = 1$, eq. (14) implies eq. (1).

In 1-D case, the non-Darcy law for the single-phase scaling law flow in porous medium can be suggested:

$$\tilde{q}(x, t) = -\frac{k}{\mu} {}^{\mathrm{MSL}}\partial_x^{(1)} \tilde{\psi}(x, t) = -\frac{k}{\mu} \Im_1 x^D \frac{\mathrm{d}}{\mathrm{d}x} \tilde{\psi}(x, t) \tag{18}$$

where $\tilde{\psi}(x, t)$ is the scaling law hydraulic head, $\tilde{q}(x, t)$ – the non-Darcy velocity, $k$ – the permeability, and $\mu$ – the dynamic viscosity.

By using eq. (14), the mass density $\tilde{\rho}$ for the scaling law flow in porous medium is defined:

$$M = \iiint_{\Omega} \tilde{\rho} dV = \aleph \iiint_{\Omega} \tilde{\rho} x^{-D_1} y^{-D_2} z^{-D_3} \mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{19}$$

where $M$ represents the mass, and $\tilde{\rho}$ denotes the mass density.

If we now consider the gravity field, the non-Darcy law for the scaling law flow in porous medium can be given:

$$\tilde{\mathbf{q}}(x,y,z,t) = -\frac{\mathbf{k}}{\boldsymbol{\mu}}\left[\nabla^{\begin{pmatrix} D_1,D_2,D_3 \\ \ell_1,\ell_2,\ell_3 \end{pmatrix}}\tilde{\psi}(x,y,z,t) - \tilde{\rho}\mathbf{g}\right] \tag{20}$$

where $\tilde{\psi}(x,y,z,t)$ is the scaling law hydraulic head, $\tilde{\mathbf{q}}(x,y,z,t)$ – the non-Darcy velocity vector, $\mathbf{k}$ – the permeability tensor, and $\boldsymbol{\mu}$ – the dynamic viscosity, $\tilde{\rho}$ – the mass density, and $\mathbf{g}$ – the gravity vector.

In 1-D case, the non-Darcy law for the single-phase scaling law flow in porous medium takes the form:

$$\tilde{q}(x,t) = -\frac{k}{\mu}\left[{}^{\mathrm{MSL}}\partial_x^{(1)}\tilde{\psi}(x,t) - \tilde{\rho}\mathrm{g}\right] = -\frac{k}{\mu}\left[\Im_1 x^D \frac{\mathrm{d}}{\mathrm{d}x}\tilde{\psi}(x,t) - \tilde{\rho}\mathrm{g}\right] \tag{21}$$

where $\tilde{\psi}(x,t)$ is the scaling law hydraulic head, $\tilde{q}(x,t)$ – the non-Darcy velocity, $k$ – the permeability, $\mu$ – the dynamic viscosity, $\tilde{\rho}$ – the mass density, and g – the gravity.

## Conclusion

In the present work we suggested the non-Darcy law for the scaling law flow in porous medium inolving the Mandelbrots scaling law. This is considered in the sense of the scaling law calculus associated with the Mandelbrots scaling law. The obtained result maybe play an important role in the study of the scaling law flow in porous medium involving the Mandelbrots scaling law phenomena.

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## Nomenclature

$t$ – time, [s]
$x, y, z$ – space co-ordinates, [m]
$\tilde{\rho}$ – density, [kgm$^{-3}$]
$\tilde{q}(x,t)$ – non-Darcy velocity, [ms$^{-1}$]
$\tilde{\psi}(x,t)$ – scaling law hydraulic head, [m]

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