A NEW GENERALIZATION OF THE *Y*-FUNCTION APPLIED TO MODEL THE ANOMALOUS DIFFUSION

by

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In this paper, we propose the W-, K-, τ-, U-, V-, and O-functions for the first time. The K series representations for the W-, τ-, U-, V-, and O-functions are discussed. The derivatives, and integral transforms and special cases for the obtained special functions are presented. The anomalous diffusion models via derivative operators associated with the τ- and L-functions are suggested. The obtained results are used to give the series representations for the I-, H-, and G-functions.

Key words: *anomalous diffusion, special function, Y-function, I-function, H-function, G-function*

Introduction

In the recent paper [1], author proposed the *Y*-function, given by the integral:

$$\mathbb{Y}_{p,q}^{m,n}[x;y;z] = \mathbb{Y}_{p,q}^{m,n}\left[x;y;z;\begin{matrix}\{\gamma_j,c_j\}_1^p\\ \{\lambda_j,\mathrm{d}_j\}_1^q\end{matrix}\right] = \frac{1}{2\pi i}\int_L \Phi(s)x^{-s}e^{sy}s^z\mathrm{d}s \tag{1}$$

in which:

$$\Phi(s) = \frac{\mathbf{A}(s)\mathbf{B}(s)}{\mathbf{C}(s)\mathbf{D}(s)} \tag{2}$$

where

$$\mathbf{A}(s) = \prod_{j=1}^{m}\Gamma(\lambda_j - \mathrm{d}_j s),\ \ \mathbf{B}(s) = \prod_{j=1}^{n}\Gamma(1-\gamma_j + c_j s) \tag{3a,b}$$

$$\mathbf{C}(s) = \prod_{j=m+1}^{q}\Gamma(1-\lambda_j + \mathrm{d}_j s),\ \ \mathbf{D}(s) = \prod_{j=n+1}^{p}\Gamma(\gamma_j - c_j s) \tag{4a,b}$$

with $x\in\mathbb{C}$, $y\in\mathbb{C}$, $z\in\mathbb{R}$, $q\geq 1$, $0\leq n\leq p$, $0\leq m\leq q$, $\{\gamma_j,\lambda_j\}\in\mathbb{C}$, and $\{\in\mathbb{R}_+\}$. Here, we denote $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{R}_+$ as the complex, real and positive real numbers, respectively.

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The *H*-function $\mathbb{H}_{p,q}^{m,n}(x)$, proposed by Fox [5], is defined by the Mellin-Barnes type integral [2]:

$$\mathbb{H}_{p,q}^{m,n}(x) = \mathbb{H}_{p,q}^{m,n}\left[x; \begin{matrix} \{\gamma_j, c_j\}_1^p \\ \{\lambda_j, d_j\}_1^q \end{matrix}\right] = \frac{1}{2\pi i}\int_L \Phi(s) x^s \mathrm{d}s \tag{5}$$

The *G*-function $\mathbb{G}_{p,q}^{m,n}(x)$, introduced by Meijer [6], is defined by the Mellin-Barnes type integral [3]:

$$\mathbb{G}_{p,q}^{m,n}(x) = \mathbb{G}_{p,q}^{m,n}\left[x; \begin{matrix} \{c_j\}_1^p \\ \{\mathrm{d}_j\}_1^q \end{matrix}\right] = \frac{1}{2\pi i}\int_L \phi(s) x^s \mathrm{d}s \tag{6}$$

where

$$\phi(s) = \frac{\prod_{j=1}^{m}\Gamma(\mathrm{d}_j - s)\prod_{j=1}^{n}\Gamma(1 - c_j + s)}{\prod_{j=m+1}^{q}\Gamma(1 - \mathrm{d}_j + s)\prod_{j=n+1}^{p}\Gamma(c_j - s)} \tag{7}$$

The *I*-function $\mathbb{I}_{p,q}^{m,n}(x)$, proposed by Rathie [4] based on the result of Innayat-Hussain [5], is defined by the Mellin-Barnes type integral [4]:

$$\mathbb{I}_{p,q}^{m,n}(x) = \mathbb{I}_{p,q}^{m,n}\left[x; \begin{matrix} \{\alpha_j, \gamma_j, c_j\}_1^p \\ \{\beta_j, \lambda_j, \mathrm{d}_j\}_1^q \end{matrix}\right] = \frac{1}{2\pi i}\int_L \Xi(s) x^s \mathrm{d}s \tag{9}$$

in which:

$$\Xi(s) = \frac{\mathbf{A}(s)\mathbf{B}(s)}{\mathbf{C}(s)\mathbf{D}(s)} \tag{10}$$

where

$$\mathbf{A}(s) = \prod_{j=1}^{m}\Gamma^{\alpha_j}(\lambda_j - \mathrm{d}_j s), \quad \mathbf{B}(s) = \prod_{j=1}^{n}\Gamma^{\beta_j}(1 - \gamma_j + c_j s) \tag{11a,b}$$

$$\mathbf{C}(s) = \prod_{j=m+1}^{q}\Gamma^{\alpha_j}(1 - \lambda_j + \mathrm{d}_j s), \quad \mathbf{D}(s) = \prod_{j=n+1}^{p}\Gamma^{\beta_j}(\gamma_j - c_j s) \tag{12a,b}$$

for $\alpha_j \in \mathbb{N}$ and $\beta_j \in \mathbb{N}$.

Based on the extened results of (1) and (9), the main target of the paper is to consider the *W*-function and beyond, and to propose the derivative operators associated with the special functions.

The series representations of the special functions

The W-function

The *W*-function $\mathbb{W}_{p,q}^{m,n}[x;y;z]$ is defined:

$$\mathbb{W}_{p,q}^{m,n}[x;y;z]=\mathbb{W}_{p,q}^{m,n}\left[x;y;z;\begin{matrix}\{\alpha_j,\gamma_j,c_j\}_1^p\\ \{\beta_j,\lambda_j,\mathrm{d}_j\}_1^q\end{matrix}\right]=\frac{1}{2\pi i}\int_L \Xi(s)x^{-s}e^{sy}s^z\mathrm{d}s \tag{13}$$

Thus,

$$\begin{aligned}\mathbb{W}_{p,q}^{m,n}[x;y;z]&=\frac{1}{2\pi i}\int_L \Xi(s)x^{-s}e^{sy}s^z\mathrm{d}s\\ &=\frac{1}{2\pi i}\int_L \Xi(s)e^{s(y-\log x)}s^z\mathrm{d}s\\ &=\frac{1}{2\pi i}\int_L \Xi(s)\sum_{k=0}^{\infty}\frac{s^{k+z}(y-\log x)^k}{k!}\mathrm{d}s\\ &=\sum_{k=0}^{\infty}\frac{(y-\log x)^k}{k!}\frac{1}{2\pi i}\int_L \Xi(s)s^{k+z}\mathrm{d}s\end{aligned} \tag{14}$$

When $\alpha_j=\beta_j=1,$ we get:

$$\mathbb{W}_{p,q}^{m,n}\left[x;y;z;\begin{matrix}\{1,\gamma_j,c_j\}_1^p\\ \{1,\lambda_j,\mathrm{d}_j\}_1^q\end{matrix}\right]=\mathbb{Y}_{p,q}^{m,n}\left[x;y;z;\begin{matrix}\{\gamma_j,c_j\}_1^p\\ \{\lambda_j,\mathrm{d}_j\}_1^q\end{matrix}\right] \tag{15}$$

where the *Y*-function is [1]:

$$\mathbb{Y}_{p,q}^{m,n}[x;y;z]=\mathbb{Y}_{p,q}^{m,n}\left[x;y;z;\begin{matrix}\{\gamma_j,c_j\}_1^p\\ \{\lambda_j,\mathrm{d}_j\}_1^q\end{matrix}\right]=\frac{1}{2\pi i}\int_L \Phi(s)x^{-s}e^{sy}s^z\mathrm{d}s \tag{16}$$

The K-function

The *K*-function $\mathbb{K}_{p,q}^{m,n}[z]$ is defined:

$$\mathbb{K}_{p,q}^{m,n}[z]=\mathbb{K}_{p,q}^{m,n}\left[z;\begin{matrix}\{\alpha_j,\gamma_j,c_j\}_1^p\\ \{\beta_j,\lambda_j,\mathrm{d}_j\}_1^q\end{matrix}\right]=\frac{1}{2\pi i}\int_L \Xi(s)s^z\mathrm{d}s \tag{17}$$

When $\alpha_j=\beta_j=1,$ we have:

$$\mathbb{K}_{p,q}^{m,n}\left[z;\begin{matrix}\{1,\gamma_j,c_j\}_1^p\\ \{1,\lambda_j,\mathrm{d}_j\}_1^q\end{matrix}\right]=\mathbb{J}_{p,q}^{m,n}\left[z;\begin{matrix}\{\gamma_j,c_j\}_1^p\\ \{\lambda_j,\mathrm{d}_j\}_1^q\end{matrix}\right] \tag{18}$$

where the *J*-function is [1]:

$$\mathbb{J}_{p,q}^{m,n}[z]=\mathbb{J}_{p,q}^{m,n}\left[z;\begin{matrix}\{\gamma_j,c_j\}_1^p\\ \{\lambda_j,\mathrm{d}_j\}_1^q\end{matrix}\right]=\frac{1}{2\pi i}\int_L \Phi(s)s^z\mathrm{d}s \tag{19}$$

By using eq. (17), the relationship between the *K*- and *W*-functions read:

$$\mathbb{W}_{p,q}^{m,n}[x;y;z]=\sum_{k=0}^{\infty}\frac{(y-\log x)^k}{k!}\frac{1}{2\pi i}\int_L \Xi(s)s^{k+z}\mathrm{d}s=\sum_{k=0}^{\infty}\frac{\mathbb{K}_{p,q}^{m,n}[k+z]}{k!}(y-\log x)^k \tag{20}$$

The τ-function

The *τ*-function $\tau_{p,q}^{m,n}[y]$ is defined:

$$\tau_{p,q}^{m,n}[y]=\tau_{p,q}^{m,n}\left[y;\begin{matrix}\{\alpha_j,\gamma_j,c_j\}_1^p\\ \{\beta_j,\lambda_j,\mathrm{d}_j\}_1^q\end{matrix}\right]=\frac{1}{2\pi i}\int_L \Xi(s)e^{sy}\mathrm{d}s \tag{21}$$

When $\alpha_j=\beta_j=1,$ we present:

$$\tau_{p,q}^{m,n}\left[y;\begin{matrix}\{1,\gamma_j,c_j\}_1^p\\ \{1,\lambda_j,\mathrm{d}_j\}_1^q\end{matrix}\right]=\mathbb{L}_{p,q}^{m,n}\left[y;\begin{matrix}\{\gamma_j,c_j\}_1^p\\ \{\lambda_j,\mathrm{d}_j\}_1^q\end{matrix}\right] \tag{22}$$

where the *L*-function is [1]:

$$\mathbb{L}_{p,q}^{m,n}[y]=\mathbb{L}_{p,q}^{m,n}\left[y;\begin{matrix}\{\gamma_j,c_j\}_1^p\\ \{\lambda_j,\mathrm{d}_j\}_1^q\end{matrix}\right]=\frac{1}{2\pi i}\int_L \Phi(s)e^{sy}\mathrm{d}s \tag{23}$$

From eqs. (20) and (21), the *K* series represenation for the *τ*-function can be expressed: as

$$\tau_{p,q}^{m,n}[y]=\tau_{p,q}^{m,n}\left[y;\begin{matrix}\{\alpha_j,\gamma_j,c_j\}_1^p\\ \{\beta_j,\lambda_j,\mathrm{d}_j\}_1^q\end{matrix}\right]=\sum_{k=0}^{\infty}\frac{y^k}{k!}\frac{1}{2\pi i}\int_L \Xi(s)s^k\mathrm{d}s=\sum_{k=0}^{\infty}\frac{\mathbb{K}_{p,q}^{m,n}[k]}{k!}y^k \tag{24}$$

The U-function

The *U*-function $\mathbb{U}_{p,q}^{m,n}[x;y]$ is defined:

$$\mathbb{U}_{p,q}^{m,n}[x;y]=\mathbb{U}_{p,q}^{m,n}\left[x;y;\begin{matrix}\{\alpha_j,\gamma_j,c_j\}_1^p\\ \{\beta_j,\lambda_j,\mathrm{d}_j\}_1^q\end{matrix}\right]=\frac{1}{2\pi i}\int_L \Xi(s)x^{-s}e^{sy}\mathrm{d}s \tag{25}$$

When $\alpha_j=\beta_j=1,$ we present:

$$\mathbb{U}_{p,q}^{m,n}\left[x;y;\begin{matrix}\left\{1,\gamma_j,c_j\right\}_1^p\\ \left\{1,\lambda_j,\mathrm{d}_j\right\}_1^q\end{matrix}\right]=\mathbb{D}_{p,q}^{m,n}\left[x;y;\begin{matrix}\left\{\gamma_j,c_j\right\}_1^p\\ \left\{\lambda_j,\mathrm{d}_j\right\}_1^q\end{matrix}\right] \tag{26}$$

where the *D*-function is [1]:

$$\mathbb{D}_{p,q}^{m,n}[x;y]=\mathbb{D}_{p,q}^{m,n}\left[x;y;\begin{matrix}\left\{\gamma_j,c_j\right\}_1^p\\ \left\{\lambda_j,\mathrm{d}_j\right\}_1^q\end{matrix}\right]=\frac{1}{2\pi i}\int_L \Phi(s)x^{-s}e^{sy}\mathrm{d}s \tag{27}$$

The *K* series represenation for the *U*-function can be given:

$$\mathbb{U}_{p,q}^{m,n}[x;y]=\frac{1}{2\pi i}\int_L \Xi(s)x^{-s}e^{sy}\mathrm{d}s=\sum_{k=0}^{\infty}\frac{\mathbb{K}_{p,q}^{m,n}[k]}{k!}(y-\log x)^k \tag{28}$$

The V-function

The *V*-function $\mathbb{V}_{p,q}^{m,n}[y;z]$ is defined:

$$\mathbb{V}_{p,q}^{m,n}[y;z]=\mathbb{V}_{p,q}^{m,n}\left[y;z;\begin{matrix}\left\{\alpha_j,\gamma_j,c_j\right\}_1^p\\ \left\{\beta_j,\lambda_j,\mathrm{d}_j\right\}_1^q\end{matrix}\right]=\frac{1}{2\pi i}\int_L \Xi(s)e^{sy}s^z\mathrm{d}s \tag{29}$$

When $\alpha_j=\beta_j=1,$ we get:

$$\mathbb{V}_{p,q}^{m,n}\left[y;z;\begin{matrix}\left\{1,\gamma_j,c_j\right\}_1^p\\ \left\{1,\lambda_j,\mathrm{d}_j\right\}_1^q\end{matrix}\right]=\mathbb{S}_{p,q}^{m,n}\left[y;z;\begin{matrix}\left\{\gamma_j,c_j\right\}_1^p\\ \left\{\lambda_j,\mathrm{d}_j\right\}_1^q\end{matrix}\right] \tag{30}$$

where the *S*-function is [1]:

$$\mathbb{S}_{p,q}^{m,n}[y;z]=\mathbb{S}_{p,q}^{m,n}\left[y;z;\begin{matrix}\left\{\gamma_j,c_j\right\}_1^p\\ \left\{\lambda_j,\mathrm{d}_j\right\}_1^q\end{matrix}\right]=\frac{1}{2\pi i}\int_L \Phi(s)e^{sy}s^z\mathrm{d}s \tag{31}$$

The *K* series represenation for the *V*-function can be suggested as:

$$\mathbb{V}_{p,q}^{m,n}[y;z]=\frac{1}{2\pi i}\int_L \Xi(s)e^{sy}s^z\mathrm{d}s=\sum_{k=0}^{\infty}\frac{\mathbb{K}_{p,q}^{m,n}[k+z]}{k!}y^k \tag{32}$$

The O-function

The *O*-function $\mathbb{O}_{p,q}^{m,n}[x;z]$ is defined:

$$\mathbb{O}_{p,q}^{m,n}[x;z]=\mathbb{O}_{p,q}^{m,n}\left[x;z;\begin{matrix}\left\{\alpha_j,\gamma_j,c_j\right\}_1^p\\ \left\{\beta_j,\lambda_j,\mathrm{d}_j\right\}_1^q\end{matrix}\right]=\frac{1}{2\pi i}\int_L \Xi(s)x^{-s}s^z\mathrm{d}s \tag{33}$$

When $\alpha_j=\beta_j=1,$ we show:

$$\mathbb{O}_{p,q}^{m,n}\left[x;z;\begin{matrix}\left\{1,\gamma_j,c_j\right\}_1^p\\ \left\{1,\lambda_j,\mathrm{d}_j\right\}_1^q\end{matrix}\right]=\mathbb{T}_{p,q}^{m,n}\left[x;z;\begin{matrix}\left\{\gamma_j,c_j\right\}_1^p\\ \left\{\lambda_j,\mathrm{d}_j\right\}_1^q\end{matrix}\right] \tag{34}$$

where the *T*-function is [1]:

$$\mathbb{T}_{p,q}^{m,n}[y;z]=\mathbb{T}_{p,q}^{m,n}\left[y;z;\begin{matrix}\left\{\gamma_j,c_j\right\}_1^p\\ \left\{\lambda_j,\mathrm{d}_j\right\}_1^q\end{matrix}\right]=\frac{1}{2\pi i}\int_L \Phi(s)x^{-s}s^z\mathrm{d}s \tag{35}$$

The *K* series represenation for the *O*-function can be suggested:

$$\mathbb{O}_{p,q}^{m,n}[x;z]=\frac{1}{2\pi i}\int_L \Xi(s)x^{-s}s^z\mathrm{d}s=\sum_{k=0}^{\infty}\frac{\mathbb{K}_{p,q}^{m,n}[k+z]}{k!}(\log x)^k \tag{36}$$

which leads to the *K* series represenation for the *I*-function, given:

$$\mathbb{I}_{p,q}^{m,n}(x)=\frac{1}{2\pi i}\int_L \Xi(s)x^s ds=\sum_{k=0}^{\infty}\frac{\mathbb{K}_{p,q}^{m,n}[k]}{k!}(\log x)^k \tag{37}$$

This is easy to see that:

$$\mathbb{V}_{p,q}^{m,n}[y;0]=\mathbb{U}_{p,q}^{m,n}[1;y]=\mathbb{I}_{p,q}^{m,n}(e^x)=\tau_{p,q}^{m,n}[y] \tag{38}$$

The Laplace transforms of some special functions can be given:

$$\int_0^\infty \tau_{p,q}^{m,n}[u]e^{-su}\mathrm{d}u=\Xi(s) \tag{39}$$

$$\int_0^\infty \mathbb{V}_{p,q}^{m,n}[u;z]e^{-su}\mathrm{d}u=\Xi(s)s^z \tag{40}$$

where the Laplace transform of the function is [6, 7]:

$$L\{\varpi(u)\}=\int_0^\infty \varpi(u)e^{-su}\mathrm{d}u \tag{41}$$

The derivative of the special functions

Here, we have the followings:

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{W}_{p,q}^{m,n}[x;y;z]=-\frac{1}{x}\sum_{k=0}^{\infty}\frac{\mathbb{K}_{p,q}^{m,n}[k+z+1]}{k!}(y-\log x)^k \tag{42}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\mathbb{W}_{p,q}^{m,n}[x;y;z]=\sum_{k=0}^{\infty}\frac{\mathbb{K}_{p,q}^{m,n}[k+z+1]}{k!}(y-\log x)^k \tag{33}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{U}_{p,q}^{m,n}[x;y]=-\frac{1}{x}\sum_{k=0}^{\infty}\frac{\mathbb{K}_{p,q}^{m,n}[k+1]}{k!}(y-\log x)^{k} \tag{44}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\mathbb{U}_{p,q}^{m,n}[x;y]=\sum_{k=0}^{\infty}\frac{\mathbb{K}_{p,q}^{m,n}[k+1]}{k!}(y-\log x)^{k} \tag{45}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\mathbb{V}_{p,q}^{m,n}[y;z]=\sum_{k=0}^{\infty}\frac{\mathbb{K}_{p,q}^{m,n}[k+z+1]}{k!}y^{k} \tag{46}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{O}_{p,q}^{m,n}[x;z]=-\frac{1}{x}\sum_{k=0}^{\infty}\frac{\mathbb{K}_{p,q}^{m,n}[k+z+1]}{k!}(-\log x)^{k} \tag{47}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}y}\tau_{p,q}^{m,n}[y]=\sum_{k=0}^{\infty}\frac{\mathbb{K}_{p,q}^{m,n}[k+1]}{k!}y^{k} \tag{48}$$

The integral transforms

Let:

$$\tau_{p,q}^{m,n}[\mu y]=\sum_{k=0}^{\infty}\frac{\mathbb{K}_{p,q}^{m,n}[k]}{k!}(\mu y)^{k} \tag{49}$$

and

$$\mathbb{L}_{p,q}^{m,n}[\mu y]=\sum_{k=0}^{\infty}\frac{\mathbb{J}_{p,q}^{m,n}[k]}{k!}(\mu y)^{k} \tag{50}$$

where $\mu > 0$.

The integral transform associated with the τ-function is:

$$\phi(u)=\int_{0}^{u}\tau_{p,q}^{m,n}[\mu(u-t)]g(t)\mathrm{d}t \tag{51}$$

which has:

$$L\{\phi(u)\}=\frac{1}{\mu}\Xi\left(\frac{s}{\mu}\right)g(s) \tag{52}$$

where

$$g(s)=\int_{0}^{\infty}g(t)e^{-su}\mathrm{d}u \tag{53}$$

The integral transform associated with the L-function is:

$$\psi(u)=\int_0^u \mathbb{L}_{p,q}^{m,n}[\mu(u-t)]g(t)\mathrm{d}t \tag{54}$$

which has:

$$L\{\psi(u)\}=\frac{1}{\mu}\Phi\left(\frac{s}{\mu}\right)g(s) \tag{55}$$

The derivative operators

Following [7], the derivative operator associated with the *τ*-function is:

$${}_0\mathrm{D}_u^{(1)}g(u)=\frac{\mathrm{d}}{\mathrm{d}u}\int_0^u \tau_{p,q}^{m,n}[\mu(u-t)]g(t)\mathrm{d}t \tag{56}$$

which has:

$$L\{{}_0\mathrm{D}_u^{(1)}g(u)\}=\frac{s}{\mu}\Xi\left(\frac{s}{\mu}\right)g(s) \tag{57}$$

As a special case of eq. (56), the derivative operator associated with the *L*-function is:

$${}_0\hat{\mathrm{D}}_u^{(1)}g(u)=\frac{\mathrm{d}}{\mathrm{d}u}\int_0^u \mathbb{L}_{p,q}^{m,n}[\mu(u-t)]g(t)\mathrm{d}t \tag{58}$$

which has:

$$L\{{}_0\hat{\mathrm{D}}_u^{(1)}g(u)\}=\frac{s}{\mu}\Phi\left(\frac{s}{\mu}\right)g(s) \tag{59}$$

The special cases

We present the following cases:

$$\mathbb{I}_{p,q}^{m,n}\left[x;\begin{matrix}\{\alpha_j,\gamma_j,c_j\}_1^p\\ \{\beta_j,\lambda_j,\mathrm{d}_j\}_1^q\end{matrix}\right]=\tau_{p,q}^{m,n}\left[[\log x];\begin{matrix}\{\alpha_j,\gamma_j,c_j\}_1^p\\ \{\beta_j,\lambda_j,\mathrm{d}_j\}_1^q\end{matrix}\right] \tag{60}$$

$$\mathbb{I}_{p,q}^{m,n}\left[e^x;\begin{matrix}\{\alpha_j,\gamma_j,c_j\}_1^p\\ \{\beta_j,\lambda_j,\mathrm{d}_j\}_1^q\end{matrix}\right]=\tau_{p,q}^{m,n}\left[x;\begin{matrix}\{\alpha_j,\gamma_j,c_j\}_1^p\\ \{\beta_j,\lambda_j,\mathrm{d}_j\}_1^q\end{matrix}\right] \tag{61}$$

$$\mathbb{H}_{p,q}^{m,n}\left[x;\begin{matrix}\{\gamma_j,c_j\}_1^p\\ \{\lambda_j,\mathrm{d}_j\}_1^q\end{matrix}\right]=\mathbb{I}_{p,q}^{m,n}\left[x;\begin{matrix}\{1,\gamma_j,c_j\}_1^p\\ \{1,\lambda_j,\mathrm{d}_j\}_1^q\end{matrix}\right]=\tau_{p,q}^{m,n}\left[[\log x];\begin{matrix}\{1,\gamma_j,c_j\}_1^p\\ \{1,\lambda_j,\mathrm{d}_j\}_1^q\end{matrix}\right] \tag{62}$$

$$\mathbb{H}_{p,q}^{m,n}\left[e^x;\begin{matrix}\{\gamma_j,c_j\}_1^p\\ \{\lambda_j,\mathrm{d}_j\}_1^q\end{matrix}\right]=\mathbb{I}_{p,q}^{m,n}\left[e^x;\begin{matrix}\{1,\gamma_j,c_j\}_1^p\\ \{1,\lambda_j,\mathrm{d}_j\}_1^q\end{matrix}\right]=\tau_{p,q}^{m,n}\left[x;\begin{matrix}\{1,\gamma_j,c_j\}_1^p\\ \{1,\lambda_j,\mathrm{d}_j\}_1^q\end{matrix}\right] \tag{63}$$

$$\mathbb{H}_{p,q}^{m,n}\left[x;\begin{matrix}\{1,c_j\}_1^p\\ \{1,\mathrm{d}_j\}_1^q\end{matrix}\right]=\mathbb{I}_{p,q}^{m,n}\left[x;\begin{matrix}\{1,1,c_j\}_1^p\\ \{1,1,\mathrm{d}_j\}_1^q\end{matrix}\right]=\tau_{p,q}^{m,n}\left[[\log x];\begin{matrix}\{1,1,c_j\}_1^p\\ \{1,1,\mathrm{d}_j\}_1^q\end{matrix}\right]=\mathbb{G}_{p,q}^{m,n}\left[x;\begin{matrix}\{c_j\}_1^p\\ \{\mathrm{d}_j\}_1^q\end{matrix}\right] \tag{64}$$

$$\mathbb{H}_{p,q}^{m,n}\left[e^x;\begin{matrix}\{1,c_j\}_1^p\\ \{1,\mathrm{d}_j\}_1^q\end{matrix}\right]=\mathbb{I}_{p,q}^{m,n}\left[e^x;\begin{matrix}\{1,1,c_j\}_1^p\\ \{1,1,\mathrm{d}_j\}_1^q\end{matrix}\right]=\tau_{p,q}^{m,n}\left[x;\begin{matrix}\{1,1,c_j\}_1^p\\ \{1,1,\mathrm{d}_j\}_1^q\end{matrix}\right]=\mathbb{G}_{p,q}^{m,n}\left[e^x;\begin{matrix}\{c_j\}_1^p\\ \{\mathrm{d}_j\}_1^q\end{matrix}\right] \tag{65}$$

$$\tau_{0,1}^{1,0}\left[[\log x];\begin{matrix}-\\ \{1,1,0\}\end{matrix}\right]=e^x \quad (x\in\mathbb{R}) \tag{66}$$

$$\sqrt{\pi}\tau_{0,2}^{1,0}\left[\log\left(\frac{x^2}{4}\right);\begin{matrix}-\\ \left\{1,1,\frac{1}{2}\right\},\{1,1,0\}\end{matrix}\right]=\cos x \quad (x\in\mathbb{R}) \tag{67}$$

$$\sqrt{\pi}\tau_{0,2}^{1,0}\left[\log\left(\frac{x^2}{4}\right);\begin{matrix}-\\ \{1,1,0\},\left\{1,1,\frac{1}{2}\right\}\end{matrix}\right]=\sin x \quad \left(-\frac{\pi}{2}<\arg x<\frac{\pi}{2}\right) \tag{68}$$

$$\sqrt{\pi}\tau_{0,2}^{1,0}\left[\log\left(-\frac{x^2}{4}\right);\begin{matrix}-\\ \left\{1,1,\frac{1}{2}\right\},\{1,1,0\}\end{matrix}\right]=\cosh x \quad (x\in\mathbb{R}) \tag{69}$$

$$-i\sqrt{\pi}\tau_{0,2}^{1,0}\left[\log\left(-\frac{x^2}{4}\right);\begin{matrix}-\\ \{1,1,0\},\left\{1,1,\frac{1}{2}\right\}\end{matrix}\right]=\sinh x \quad \left(-\frac{\pi}{2}<\arg x<\frac{\pi}{2}\right) \tag{70}$$

$$\tau_{2,2}^{1,2}\left[[\log x];\begin{matrix}\{1,1,1\},\{1,1,1\}\\ \{1,1,1\},\{1,1,0\}\end{matrix}\right]=\log(x+1) \quad (x\in\mathbb{R}) \tag{71}$$

$$\tau_{0,2}^{1,0}\left[\log\frac{x^2}{4};\begin{matrix}-\\ \left\{1,1,\frac{v}{2}\right\},\left\{1,1,-\frac{v}{2}\right\}\end{matrix}\right]=J_v(x) \quad \left(-\frac{\pi}{2}<\arg x<\frac{\pi}{2}\right) \tag{72}$$

$$\tau_{1,3}^{2,0}\left[\log\frac{x^2}{4};\begin{matrix}-\frac{v+1}{2}\\ \left\{1,1,\frac{v}{2}\right\},\left\{1,1,-\frac{v}{2}\right\},\left\{1,1,-\frac{v+1}{2}\right\}\end{matrix}\right]=Y_v(x) \quad \left(-\frac{\pi}{2}<\arg x<\frac{\pi}{2}\right) \tag{73}$$

$$i^{-v}\tau_{0,2}^{1,0}\left[\log\left(-\frac{x^2}{4}\right);\begin{matrix} - \\ \left\{1,1,\frac{v}{2}\right\},\left\{1,1,-\frac{v}{2}\right\}\end{matrix}\right]=I_v(x) \quad (-\pi<\arg x<0) \tag{74}$$

$$\tau_{0,2}^{2,0}\left[\log\frac{x^2}{4};\begin{matrix} -\dfrac{v+1}{2} \\ \left\{1,1,\frac{v}{2}\right\},\left\{1,1,-\frac{v}{2}\right\}\end{matrix}\right]=K_v(x) \quad \left(-\frac{\pi}{2}<\arg x<\frac{\pi}{2}\right) \tag{75}$$

where $J_v(x)$ is the Bessel function of the first kind, and $Y_v(x)$ is Bessel function of the second kind, and both $I_v(x)$ and $K_v(x)$ are the modified Bessel functions (for the more details of the above Bessel functions, see [8]).

Modelling the anomalous diffusion

We now consider two anomalous diffusion models based on the derivative operators associated with the τ- and L-functions.

Model 1 Let us consider the anomalous diffusion model based on the derivative operator associated with the τ-function:

$${}_0\partial_u^{(1)}\varphi(u,\sigma)=\frac{\partial^2\varphi(u,\sigma)}{\partial\sigma^2} \tag{76}$$

where the partial derivative associated with the τ-function is:

$${}_0\partial_u^{(1)}\varphi(u,\sigma)=\frac{\mathrm{d}}{\mathrm{d}u}\int_0^u \tau_{p,q}^{m,n}[\mu(u-t)]\varphi(t,\sigma)\mathrm{d}t \tag{77}$$

Model 2 We now consider the anomalous diffusion model based on the derivative operator associated with the L-function:

$${}_0\partial_u^{(1)}\varphi(u,\sigma)=\frac{\partial^2\varphi(u,\sigma)}{\partial\sigma^2} \tag{78}$$

where the partial derivative associated with the L-function is:

$${}_0\partial_u^{(1)}\varphi(u,\sigma)=\frac{\mathrm{d}}{\mathrm{d}u}\int_0^u \mathbb{L}_{p,q}^{m,n}[\mu(u-t)]\varphi(t,\sigma)\mathrm{d}t \tag{79}$$

Conclusion

In the work we have proposed the new special functions, such as the *W*-, *K*-, τ-, *U*-, *V*-, and *O*-functions, and given the relationships among them. The series, integrals and derivaitves of some special functions and their special cases were considered. The anomalous diffusion models involving the derivative operators based on the derivative operators associated with the τ and *L*-functions were suggested. The special formulas are used to give the series representations of the *I*-, *H*-, and *G*-functions.

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Nomenclature

u – time, [s]
$\varphi(u, \sigma)$ $\varphi(u,\sigma)$ – distribution, [–]
σ – co-ordinate, [m]

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