A DECOUPLED HIGH ACCURACY LINEAR DIFFERENCE SCHEME FOR SYMMETRIC REGULARIZED LONG WAVE EQUATION WITH DAMPING TERM

by

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In this paper, the initial boundary value problem of the dissipative symmetric regularized long wave equation with a damping term is studied numerically, and a decoupled linearized difference scheme with a theoretical accuracy of $O(\tau^2 + h^4)$ *is proposed. Because the scheme removes the coupling between the variables in the original equation, the linearized difference scheme and the explicit difference scheme can be used to solve the two variables in parallel, which greatly improves the efficiency of numerical solutions. To obtain the maximum norm estimation of numerical solutions, the mathematical induction and the discrete functional analysis methods are introduced directly to prove the convergence and the stability of the scheme. Numerical experiments have also verified the reliability of the proposed scheme.*

Key words: *damping, dissipation, symmetric regularized long wave equation, decoupled linearized difference scheme, convergence, stability*

Introduction

In many physical problems, viscous damping is inevitable, and the dissipation term should be considered during the propagation of non-linear waves. In this paper, we focus on the finite difference scheme for the following dissipative symmetric regularized long wave (SRLW) equation with a damping term:

$$u_{xxt} - u_t + \upsilon u_{xx} = \rho_x + u u_x, \quad (x,t) \in (x_L, x_R) \times (0,T] \tag{1}$$

$$\rho_t + u_x + \gamma\rho = 0, \quad (x,t) \in (x_L, x_R) \times (0,T] \tag{2}$$

where $\upsilon > 0$, $\gamma > 0$ [1]. Some researchers discussed the well-posedness of the solutions of the periodic boundary value problems and initial boundary value problems (1) and (2), such as the global existence and the uniqueness, and the long-term behavior of the solutions [2-5]. Here we mainly consider the following initial and boundary value problems of eqs. (1) and (2):

$$u(x,0) = u_0(x), \quad \rho(x,0) = \rho_0(x), \quad x \in [x_L, x_R] \tag{3}$$

$$u(x_L,t) = u(x_R,t) = 0, \quad \rho(x_L,t) = \rho(x_R,t) = 0, \quad t \in [0,T] \tag{4}$$

in which $u_0(x)$, $\rho_0(x)$ are known functions.

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There have been related works of numerical investigations for the SRLW model (1)-(4), for example, the finite element method [6], the finite difference method [7, 8]. Moreover, the finite difference method also is extended for the generalized SRLW equation with damping term [9]. However, aforementioned methods only have second-order accuracy in space, and often lead to a coupled non-linear system, with a large amount of iterations. As far as the numerical dispersion is concerned, the coupled relationship between variables u and ρ in (1) and (2) could be removed by the extrapolation technique [8] and an uncoupled linear difference scheme with second-order theoretical accuracy is proposed for the initial boundary value problem (1)-(4). Therefore, a linear difference scheme with theoretical accuracy of $O(\tau^2 + h^4)$ is proposed for the initial boundary value problem (1)-(4) based on the decoupled variables u and ρ by using the Richardson extrapolation in the spatial direction, which would further improve the efficiency of numerical solutions. On the other hand, the maximum norm estimation of the numerical solutions, the convergence and the stability of the scheme are proved directly by the mathematical induction and the discrete functional analysis method. Finally, we carry out some numerical experiments to confirm the robustness and accuracy of the proposed scheme.

The finite difference scheme

For the domain $[x_L, x_R]\times[0,T]$, let $h=(x_R - x_L)/J$ be the step size for the spatial grid, and τ be the step size for the temporal direction such that $x_j = x_L + jh(0\le j\le J)$, $t_n = n\tau(n=0,1,2,\cdots,N, N=[T/\tau])$..

Denote: $u_j^n = u(x_j,t_n)$, $\rho_j^n=\rho(x_j,t_n)$, $U_j^n \approx u(x_j,t_n)$, $\phi_j^n \approx \rho(x_j,t_n)$. Let $\{U_j^n\}$ and $\{V_j^n\}$ be the grid function. Define the following notations:

$$(U_j^n)_x = \frac{U_{j+1}^n - U_j^n}{h},\ (U_j^n)_{\bar{x}} = \frac{U_j^n - U_{j-1}^n}{h},\ (U_j^n)_{\hat{x}} = \frac{U_{j+1}^n - U_{j-1}^n}{2h}$$

$$(U_j^n)_{\ddot{x}} = \frac{U_{j+2}^n - U_{j-2}^n}{4h},\ (U_j^n)_t = \frac{U_j^{n+1} - U_j^n}{\tau},\ U_j^{n+\frac{1}{2}} = \frac{U_j^{n+1} + U_j^n}{2}$$

$$\langle U^n, V^n\rangle = h\sum_{j=1}^{J-1} U_j^n V_j^n,\ \langle U^n, U^n\rangle = \left\|U^n\right\|^2,\ \left\|U^n\right\|_\infty = \max_{1\le j\le J-1}\left|U_j^n\right|$$

$$Z_h^0 = \{U = (U_j) \big| U_{-1} = U_0 = U_J = U_{J+1} = 0,\ j=-1,0,1,\cdots,J-1,J,J+1\}$$

Consider the following finite difference scheme for the initial boundary value problem (1)-(4):

$$(U_j^n)_t - \frac{4}{3}(U_j^n)_{x\bar{x}t} + \frac{1}{3}(U_j^n)_{\hat{x}\hat{x}t} + \frac{4}{3}\left(\frac{3}{2}\phi_j^n - \frac{1}{2}\phi_j^{n-1}\right)_{\hat{x}} - \frac{1}{3}\left(\frac{3}{2}\phi_j^n - \frac{1}{2}\phi_j^{n-1}\right)_{\ddot{x}} -$$

$$-\frac{4}{3}\upsilon(U_j^{n+\frac{1}{2}})_{x\bar{x}} + \frac{1}{3}\upsilon(U_j^{n+\frac{1}{2}})_{\hat{x}\hat{x}} + \left(\frac{3}{2}U_j^n - \frac{1}{2}U_j^{n-1}\right)\left[\frac{4}{3}(U_j^{n+\frac{1}{2}})_{\hat{x}} - \frac{1}{3}(U_j^{n+\frac{1}{2}})_{\ddot{x}}\right] = 0$$

$$j=1,2,\cdots,J-1;\ n=1,2,\cdots,N-1 \tag{5}$$

$$(\phi_j^n)_t+\frac{4}{3}\left(\frac{3}{2}U_j^n-\frac{1}{2}U_j^{n-1}\right)_{\hat{x}}-\frac{1}{3}\left(\frac{3}{2}U_j^n-\frac{1}{2}U_j^{n-1}\right)_{\ddot{x}}+\gamma\phi_j^{n+\frac{1}{2}}=0$$

$$j=1,2,\cdots,J-1;\ \ n=1,2,\cdots,N-1 \tag{6}$$

$$U_j^1-\frac{4}{3}(U_j^1)_{x\bar{x}}+\frac{1}{3}(U_j^1)_{\hat{x}\hat{x}}=u_0(x_j)-\frac{\partial^2 u_0}{\partial x^2}(x_j)+\tau\upsilon\frac{\partial^2 u_0}{\partial x^2}(x_j)-$$

$$-\tau\frac{\partial\rho_0}{\partial x}(x_j)-\tau u_0(x_j)\frac{\partial u_0}{\partial x}(x_j),\quad j=1,2,\cdots,J-1 \tag{7}$$

$$\phi_j^1=\rho_0(x_j)-\tau\frac{\partial u_0}{\partial x}(x_j)-\gamma\tau\rho_0(x_j),\ \ j=1,2,\cdots,J-1 \tag{8}$$

$$U_j^0=u_0(x_j),\ \phi_j^0=\rho_0(x_j),\ \ j=1,2,\cdots,J-1 \tag{9}$$

$$U^n\in Z_h^0,\quad \phi^n\in Z_h^0,\ n=0,1,2,\cdots,N \tag{10}$$

The truncation error of the difference scheme (5)-(10) is defined:

$$r_j^n=(u_j^n)_t-\frac{4}{3}(u_j^n)_{x\bar{x}t}+\frac{1}{3}(u_j^n)_{\hat{x}\hat{x}t}+\frac{4}{3}\left(\frac{3}{2}\rho_j^n-\frac{1}{2}\rho_j^{n-1}\right)_{\hat{x}}-\frac{1}{3}\left(\frac{3}{2}\rho_j^n-\frac{1}{2}\rho_j^{n-1}\right)_{\ddot{x}}-$$

$$-\frac{4}{3}\upsilon(u_j^{n+\frac{1}{2}})_{x\bar{x}}+\frac{1}{3}\upsilon(u_j^{n+\frac{1}{2}})_{\hat{x}\hat{x}}+\left(\frac{3}{2}u_j^n-\frac{1}{2}u_j^{n-1}\right)\left[\frac{4}{3}(u_j^{n+\frac{1}{2}})_{\hat{x}}-\frac{1}{3}(u_j^{n+\frac{1}{2}})_{\ddot{x}}\right]=0 \tag{11}$$

$$s_j^n=(\rho_j^n)_t+\frac{4}{3}\left(\frac{3}{2}u_j^n-\frac{1}{2}u_j^{n-1}\right)_{\hat{x}}-\frac{1}{3}\left(\frac{3}{2}u_j^n-\frac{1}{2}u_j^{n-1}\right)_{\ddot{x}}+\gamma\rho_j^{n+\frac{1}{2}}=0 \tag{12}$$

$$u_j^1-\frac{4}{3}(u_j^1)_{x\bar{x}}+\frac{1}{3}(u_j^1)_{\hat{x}\hat{x}}=u_0(x_j)-\frac{\partial^2 u_0}{\partial x^2}(x_j)+\tau\upsilon\frac{\partial^2 u_0}{\partial x^2}(x_j)-$$

$$-\tau\frac{\partial\rho_0}{\partial x}(x_j)-\tau u_0(x_j)\frac{\partial u_0}{\partial x}(x_j)+r_j^0 \tag{13}$$

$$\rho_j^1=\rho_0(x_j)-\tau\frac{\partial u_0}{\partial x}(x_j)-\gamma\tau\rho_0(x_j)+s_j^0 \tag{14}$$

According to Taylor expansion, when $h,\tau\to 0,|r_j^n|+|s_j^n|=O(\tau^2+h^4)$.

Convergence and stability of difference scheme

Lemma 1 [7] Assumes that $u_0\in H^1,\ \rho_0\in L_2,$ then there exists a constant $C_u,$ so that the solution of the initial boundary value problem (1)-(4) satisfies:

$$\|u\|_{L_2}\le C_u,\ \|u_x\|_{L_2}\le C_u,\ \|\rho\|_{L_2}\le C_u,\ \|u\|_{L_\infty}\le C_u \tag{15}$$

Lemma 2 [9] For $\forall U\in Z_h^0,$ there has always $\|U_{\ddot{x}}\|\le\|U_{\hat{x}}\|\le\|U_x\|.$

Theorem 1 Suppose $u_0 \in H^1$, and $\rho_0 \in L_2$, and the temporal step τ and spatial step h are sufficiently small, then the solutions U^n and ϕ^n of the difference scheme (5)-(10) converges to the solution of the initial boundary value problem (1)-(4) in terms of $\|\cdot\|_\infty$ and $\|\cdot\|$ respectively, and the order of convergence is $O(\tau^2 + h^4)$.

Proof Use mathematical induction. Denote $e_j^n = u_j^n - U_j^n$, $\eta_j^n = \rho_j^n - \phi_j^n$. Subtracting (5)-(8) from (11)-(14) , respectively, we get:

$$r_j^n = (e_j^n)_t - \frac{4}{3}(e_j^n)_{x\bar{x}t} + \frac{1}{3}(e_j^n)_{\hat{x}\hat{x}t} + \frac{4}{3}\left(\frac{3}{2}\eta_j^n - \frac{1}{2}\eta_j^{n-1}\right)_{\hat{x}} - \frac{1}{3}\left(\frac{3}{2}\eta_j^n - \frac{1}{2}\eta_j^{n-1}\right)_{\ddot{x}} -$$

$$-\frac{4}{3}\upsilon(e_j^{n+\frac{1}{2}})_{x\bar{x}} + \frac{1}{3}\upsilon(e_j^{n+\frac{1}{2}})_{\hat{x}\hat{x}} + P_{1,j} + P_{2,j} \tag{16}$$

$$s_j^n = (\eta_j^n)_t + \frac{4}{3}\left(\frac{3}{2}e_j^n - \frac{1}{2}e_j^{n-1}\right)_{\hat{x}} - \frac{1}{3}\left(\frac{3}{2}e_j^n - \frac{1}{2}e_j^{n-1}\right)_{\ddot{x}} + \gamma\eta_j^{n+\frac{1}{2}} = 0 \tag{17}$$

$$e_j^1 - \frac{4}{3}(e_j^1)_{x\bar{x}} + \frac{1}{3}(e_j^1)_{\hat{x}\hat{x}} = r_j^0 \tag{18}$$

$$\eta_j^1 = s_j^0 \tag{19}$$

where

$$P_{1,j} = \frac{4}{3}(u_j^{n+\frac{1}{2}})_{\hat{x}}\left(\frac{3}{2}u_j^n - \frac{1}{2}u_j^{n-1}\right) - \frac{4}{3}(U_j^{n+\frac{1}{2}})_{\hat{x}}\left(\frac{3}{2}U_j^n - \frac{1}{2}U_j^{n-1}\right)$$

$$P_{2,j} = -\frac{1}{3}(u_j^{n+\frac{1}{2}})_{\ddot{x}}\left(\frac{3}{2}u_j^n - \frac{1}{2}u_j^{n-1}\right) + \frac{1}{3}(U_j^{n+\frac{1}{2}})_{\ddot{x}}\left(\frac{3}{2}U_j^n - \frac{1}{2}U_j^{n-1}\right)$$

According to *Lemma 1* and (15), there are constants C_r and C_s, which are independent of τ and h, such that:

$$\left\|u^n\right\|_\infty \le C_u, \ \left\|r^n\right\|_\infty \le C_r(\tau^2 + h^4), \ \left\|s^n\right\|_\infty \le C_s(\tau^2 + h^4), \ \ n = 1, 2, , N \tag{20}$$

The following estimation can be obtained from the initial condition eq. (9):

$$\left\|e^0\right\| = 0, \ \left\|\eta^0\right\| = 0, \ \left\|U^0\right\|_\infty \le C_u \tag{21}$$

Taking the inner products of (18) and (19) with e^1 and η^1, respectively, and by (15) and the Cauchy-Schwarz inequality, we can obtain the following estimation:

$$\left\|e^1\right\| + \left\|e_x^1\right\| + \left\|\eta^l\right\|^2 \le \left\|e^1\right\|^2 + \frac{4}{3}\left\|e_x^1\right\|^2 - \frac{1}{3}\left\|e_x^1\right\|^2 + \left\|\eta^l\right\|^2 \le C_1(\tau^2 + h^4) \tag{22}$$

where C_1 is a constant independent of τ and h.

Suppose that:

$$\left\|e^l\right\| + \left\|e_x^l\right\| + \left\|\eta^l\right\| \le C_l(\tau^2 + h^4), \ \ l = 2, 3, \cdots, n, \ \ (n \le N - 1)$$

where $C_1(l=2,3,\cdots,n)$ are constants independent of τ and h. According to discrete Sobolev inequality [10] and Cauchy-Schwarz inequality, we get:

$$\left\|U^l\right\|_\infty \le \left\|u^l\right\|_\infty + \left\|e^l\right\|_\infty \le C_u + \frac{3}{2}C_0 C_t(\tau^2+h^4), \quad l=1,2,n \tag{23}$$

Then taking the inner products of (16) and (17) with $e^{n+\frac{1}{2}}$ and $\eta^{n+\frac{1}{2}}$, respectively, and using the summation by part, we get:

$$\frac{1}{2}\left\|e^n\right\|_t^2 + \frac{2}{3}\left\|e_x^n\right\|_t^2 - \frac{1}{6}\left\|e_{\hat{x}}^n\right\|_t^2 = \langle r^n, e^{n+\frac{1}{2}}\rangle - \frac{4}{3}\upsilon\left\|e_x^{n+\frac{1}{2}}\right\|^2 + \frac{1}{3}\upsilon\left\|e_{\hat{x}}^{n+\frac{1}{2}}\right\|^2 -$$

$$-\left\langle 2\eta_{\hat{x}}^n - \frac{2}{3}\eta_{\hat{x}}^{n-1}, e^{n+\frac{1}{2}}\right\rangle + \left\langle \frac{1}{2}\eta_{\ddot{x}}^n - \frac{1}{6}\eta_{\ddot{x}}^{n-1}, e^{n+\frac{1}{2}}\right\rangle - \left\langle P_1, e^{n+\frac{1}{2}}\right\rangle - \left\langle P_2, e^{n+\frac{1}{2}}\right\rangle \tag{24}$$

$$\frac{1}{2}\left\|\eta^n\right\|_t^2 = -\gamma\left\|\eta^{n+\frac{1}{2}}\right\|^2 + \left\langle s^n, \eta^{n+\frac{1}{2}}\right\rangle - \left\langle 2e_{\hat{x}}^n - \frac{2}{3}e_{\hat{x}}^{n-1}, \eta^{n+\frac{1}{2}}\right\rangle + \left\langle \frac{1}{2}e_{\ddot{x}}^n - \frac{1}{6}e_{\ddot{x}}^{n-1}, \eta^{n+\frac{1}{2}}\right\rangle \tag{25}$$

From *Lemma 1* and the mean value theorem, we get:

$$\left\|u_{\hat{x}}^{n+\frac{1}{2}}\right\|_\infty \le C_u, \left\|u_{\ddot{x}}^{n+\frac{1}{2}}\right\|_\infty \le C_u \tag{26}$$

Then, taking h and τ sufficiently small makes:

$$\frac{3}{2}C_0\left(\max_{1\le l\le n} C_l\right)(\tau^2+h^4) \le 1 \tag{27}$$

Thus, according to (23), (26), (27), *Lemma 2* and Cauchy-Schwarz inequality, we have:

$$-\left\langle P_1, e^{n+\frac{1}{2}}\right\rangle = -h\sum_{j=1}^{J-1}\left[(u_j^{n+\frac{1}{2}})_{\hat{x}}\left(2e_j^n - \frac{2}{3}e_j^{n-1}\right) + \left(2U_j^n - \frac{2}{3}U_j^{n-1}\right)(e_j^{n+\frac{1}{2}})_{\hat{x}}\right]e_j^{n+\frac{1}{2}} \le$$

$$\le \frac{1}{3}C_u\left(2\left\|e^{n+1}\right\|^2 + 5\left\|e^n\right\|^2 + \left\|e^{n-1}\right\|^2\right) + \frac{2}{3}(C_u+1)\left(\left\|e_x^{n+1}\right\|^2 + \left\|e_x^n\right\|^2 + \left\|e^{n+1}\right\|^2 + \left\|e^n\right\|^2\right) \tag{28}$$

$$-\left\langle 2\eta_{\hat{x}}^n - \frac{2}{3}\eta_{\hat{x}}^{n-1}, e^{n+\frac{1}{2}}\right\rangle \le \frac{1}{3}\left(3\left\|\eta^n\right\|^2 + \left\|\eta^{n-1}\right\|^2 + 2\left\|e_x^{n+1}\right\|^2 + 2\left\|e_x^n\right\|^2\right) \tag{29}$$

Similarly, we have:

$$-\left\langle P_2, e^{n+\frac{1}{2}}\right\rangle \le \frac{1}{12}C_u\left(2\left\|e^{n+1}\right\|^2 + 5\left\|e^n\right\|^2 + \left\|e^{n-1}\right\|^2\right) + \\ + \frac{1}{6}(C_u+1)\left(\left\|e_x^{n+1}\right\|^2 + \left\|e_x^n\right\|^2 + \left\|e^{n+1}\right\|^2 + \left\|e^n\right\|^2\right) \tag{30}$$

$$\left\langle \frac{1}{2}\eta_{\ddot{x}}^n - \frac{1}{6}\eta_{\ddot{x}}^{n-1}, e^{n+\frac{1}{2}}\right\rangle \le \frac{1}{12}\left(3\left\|\eta^n\right\|^2 + \left\|\eta^{n-1}\right\|^2 + 2\left\|e_x^{n+1}\right\|^2 + 2\left\|e_x^n\right\|^2\right) \tag{31}$$

$$-\left\langle 2e_{\hat{x}}^{n}-\frac{2}{3}e_{\hat{x}}^{n-1},\eta^{n+\frac{1}{2}}\right\rangle \le \frac{1}{3}\left(3\left\|e_x^n\right\|^2+\left\|e_x^{n-1}\right\|^2+2\left\|\eta^{n+1}\right\|^2+2\left\|\eta^n\right\|^2\right) \tag{32}$$

$$\left\langle \frac{1}{2}e_{\ddot{x}}^{n}-\frac{1}{6}e_{\ddot{x}}^{n-1},\eta^{n+\frac{1}{2}}\right\rangle \le \frac{1}{12}\left(3\left\|e_x^n\right\|^2+1\left\|e_x^{n-1}\right\|^2+2\left\|\eta^{n+1}\right\|^2+2\left\|\eta^n\right\|^2\right) \tag{33}$$

and

$$\left\langle s^n,\eta^{n+\frac{1}{2}}\right\rangle \le \frac{1}{2}\left\|s^n\right\|^2+\frac{1}{4}\left(\left\|\eta^{n+1}\right\|^2+\left\|\eta^n\right\|^2\right),\ \left\langle r^n,e^{n+\frac{1}{2}}\right\rangle = \frac{1}{2}\left\|r^n\right\|^2+\frac{1}{4}\left(\left\|e^{n+1}\right\|^2+\left\|e^n\right\|^2\right) \tag{34}$$

The combination of (24), (25), (28)-(35) yields:

$$\begin{aligned}
&\frac{1}{2}\left\|e^n\right\|_t^2+\frac{2}{3}\left\|e_x^n\right\|_t^2-\frac{1}{6}\left\|e_{\hat{x}}^n\right\|_t^2+\frac{1}{2}\left\|\eta^n\right\|_t^2 \le \frac{1}{2}\left\|r^n\right\|^2+\frac{1}{2}\left\|s^n\right\|^2+\\
&+\frac{1}{12}\left(3\left\|e^{n+1}\right\|^2+3\left\|e^n\right\|^2+10\left\|e_x^{n+1}\right\|^2+25\left\|e_x^n\right\|^2+5\left\|e_x^{n-1}\right\|^2+10\left\|\eta^{n+1}\right\|^2+25\left\|\eta^n\right\|^2+5\left\|\eta^{n-1}\right\|^2\right)+\\
&+\frac{5}{12}C_u\left(2\left\|e^{n+1}\right\|^2+5\left\|e^n\right\|^2+\left\|e^{n-1}\right\|^2\right)+\frac{5}{6}(C_u+1)\left(\left\|e_x^{n+1}\right\|^2+\left\|e_x^n\right\|^2+\left\|e^{n+1}\right\|^2+\left\|e^n\right\|^2\right)\le\\
&\le \frac{1}{2}\left\|r^n\right\|^2+\frac{1}{2}\left\|s^n\right\|^2+3(C_u+1)\left(\left\|e^{n+1}\right\|^2+\left\|e^n\right\|^2+\left\|e^{n-1}\right\|^2+\right.\\
&\left.+\left\|e_x^{n+1}\right\|^2+\left\|e_x^n\right\|^2+\left\|e_x^{n-1}\right\|^2+\left\|\eta^{n+1}\right\|^2+\left\|\eta^n\right\|^2+\left\|\eta^{n-1}\right\|^2\right)
\end{aligned} \tag{35}$$

Let $B^n=\|e^n\|^2+4/3\|e_x^n\|^2-1/3\|e_{\hat{x}}^n\|^2+\|\eta^n\|^2$. Multiplying both sides of (36) by 2τ, and then recursively summing from 1 to τ, we have:

$$B^{n+1}\le B^1+\tau\sum_{k=1}^{n}\left\|r^k\right\|^2+\tau\sum_{k=1}^{n}\left\|s^k\right\|+\tau\sum_{k=0}^{n+1}18(C_u+1)\left(\left\|e^k\right\|^2+\left\|e_x^k\right\|^2+\left\|\eta^k\right\|^2\right) \tag{36}$$

It follows from (20) that:

$$\tau\sum_{k=1}^{n}\|r^k\|^2\le n\tau\max_{1\le k\le n}\left\|r^k\right\|^2\le T(C_r)^2(\tau^2+h^4)^2 \tag{37}$$

$$\tau\sum_{k=1}^{n}\left\|s^k\right\|^2\le n\tau\max_{1\le k\le n}\left\|s^k\right\|^2\le T(C_s)^2(\tau^2+h^4)^2 \tag{38}$$

Suppose that the temporal step τ is sufficiently small to satisfy $\tau<1/[36(C_u+1)]$. Substituting (22), (37), (38) into (36), and using the discrete Gronwall inequality [10], we get:

$$\begin{aligned}
&\left\|e^{n+1}\right\|^2+\left\|e_x^{n+1}\right\|^2+\left\|\eta^{n+1}\right\|^2\le[T(C_r)^2+T(C_s)^2+C_1](\tau^2+h^4)e^{2T[18(C_u+1)]}\le\\
&\le(C_{n+1})^2(\tau^2+h^4)^2,\ n=1,2,\cdots,N-1
\end{aligned}$$

in which $C_{n+1} = [\sqrt{T}(C_r + C_s) + \sqrt{C_1}]e^{18T(C_u+1)}$. Thus by the induction hypothesis:

$$\|e^n\| \le O(\tau^2 + h^4),\ \|e_x^n\| \le O(\tau^2 + h^4),\ \|\eta^n\| \le O(\tau^2 + h^4),\ n = 1, 2, \cdots, N$$

Finally, according to the discrete Sobolev inequality [10], it can be obtained:

$$\|e^n\|_\infty \le O(\tau^2 + h^4),\ n = 1, 2, \cdots, N$$

Theorem 2 Suppose $u_0 \in H^1$, $\rho_0 \in L_2$. If the temporal step τ and spatial step h are sufficiently small, the solution of the difference scheme (5)-(10) satisfies:

$$\|U^n\|_\infty \le \tilde{C}_u,\ \|\phi^n\|_\infty \le \tilde{C}_\rho,\ n = 1, 2, \cdots, N$$

where $\tilde{C}_u$, $\tilde{C}_\rho$ are constant independent of τ and h.

Proof For sufficiently small τ and h, according to *Theorem 1*:

$$\|U^n\|_\infty \le \|u^n\|_\infty + \|e^n\|_\infty \le \tilde{C}_u,\ \|\phi^n\| \le \|\rho^n\| + \|\eta^n\| \le \tilde{C}_\rho$$

Numerical experiments

In the numerical experiment, the initial value function in problems (1)-(4) is taken as the initial value function of the SRLW equation, [7]:

$$u_0(x) = \frac{5}{2}\operatorname{sec}h^2 \frac{\sqrt{5}}{6}x,\ \rho_0(x) = \frac{5}{3}\operatorname{sec}h^2 \frac{\sqrt{5}}{6}x$$

We take $x_L = -20$, $x_R = 40$, and $T = 5.0$. Since the exact solutions of the eq. (1) and (2) are unknown, the error estimation method in [7] can be used to estimate the error using the numerical solution on the fine grid $(\tau = h = 1/160)$ as the exact solution. Regarding the different values of $\upsilon = \gamma = 0.5$, τ and h, the errors of the difference scheme (5)-(10) at several different times are shown in tab. 1, and the tests of theoretical accuracy are shown in tab. 2, respectively.

Table 1. The error of numerical solutions at various time (υ = γ = 0.5)

		τ = 0.2, h = 0.4		τ = 0.05, h = 0.2		τ = 0.0125, h = 0.1	
		$\|\cdot\|$	$\|\cdot\|_\infty$	$\|\cdot\|$	$\|\cdot\|_\infty$	$\|\cdot\|$	$\|\cdot\|_\infty$
u	t = 1	$7.18652 \cdot 10^{-3}$	$4.39422 \cdot 10^{-3}$	$5.55924 \cdot 10^{-4}$	$3.37368 \cdot 10^{-4}$	$2.76672 \cdot 10^{-5}$	$1.66711 \cdot 10^{-5}$
	t = 3	$2.14526 \cdot 10^{-2}$	$1.21955 \cdot 10^{-2}$	$1.35470 \cdot 10^{-3}$	$7.52078 \cdot 10^{-4}$	$6.45305 \cdot 10^{-5}$	$3.54269 \cdot 10^{-5}$
	t = 5	$2.07351 \cdot 10^{-2}$	$1.13921 \cdot 10^{-2}$	$1.29487 \cdot 10^{-3}$	$6.97498 \cdot 10^{-4}$	$6.15934 \cdot 10^{-5}$	$3.29078 \cdot 10^{-5}$
ρ	t = 1	$1.73411 \cdot 10^{-2}$	$1.08242 \cdot 10^{-2}$	$1.12492 \cdot 10^{-3}$	$7.10448 \cdot 10^{-4}$	$5.36563 \cdot 10^{-5}$	$3.36449 \cdot 10^{-5}$
	t = 3	$1.21412 \cdot 10^{-2}$	$6.83985 \cdot 10^{-3}$	$6.92738 \cdot 10^{-4}$	$3.85509 \cdot 10^{-4}$	$3.19317 \cdot 10^{-5}$	$1.75585 \cdot 10^{-5}$
	t = 5	$1.27553 \cdot 10^{-2}$	$6.40001 \cdot 10^{-3}$	$7.50992 \cdot 10^{-4}$	$3.62651 \cdot 10^{-4}$	$3.53210 \cdot 10^{-5}$	$1.65474 \cdot 10^{-5}$

Conclusion

In this paper, from the numerical results, the proposed difference schemes (5)-(10) for SRLW eqs. (1)-(4) is effective and has the accuracy of $O(\tau^2 + h^4)$, obviously. Meanwhile,

Table 2. Numerical verifications of the theoretical accuracy under $\|\cdot\|_\infty$ ($\upsilon = \gamma = 0.5$)

	$\|e^n(h,\tau)\|_\infty / \|e^{4n}(h/2,\tau/4)\|_\infty$			$\|\eta^n(h,\tau)\|_\infty / \|\eta^{4n}(h/2,\tau/4)\|_\infty$		
	$\tau = 0.2$ $h = 0.4$	$\tau = 0.05$ $h = 0.2$	$\tau = 0.0125$ $h = 0.1$	$\tau = 0.2$ $h = 0.4$	$\tau = 0.05$ $h = 0.2$	$\tau = 0.0125$ $h = 0.1$
$t = 1$	--	13.0250	20.2366	--	15.2357	21.1161
$t = 3$	--	16.2158	21.2290	--	17.7424	21.9557
$t = 5$	--	16.3329	21.1955	--	17.6479	21.9158

it is clear that scheme (6) is explicit about $\{\phi_j^{n+1}\}$, which can be solved directly, so the difference scheme (5)-(10) is essentially a semi-explicit linear difference scheme. Compared to other coupled second-order precision difference schemes, the numerical efficiency of the proposed scheme could be greatly improved.

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Nomenclature

u – fluid velocity, [ms^{-1}]
t – time, [s]
ρ – electron charge density, [Cs^{-1}]

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