A NEW INSIGHT ON ANALYTICAL THEORY OF THE SCALING LAW HEAT CONDUCTION ASSOCIATED WITH THE RICHARDSON SCALING LAW

by

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In this article, we propose a new model for the scaling law heat conduction equation associated with the Richardson scaling law. To find the analytical solution for it, we present a scaling law series associated with the Kohlrausch-Williams-Watts function analogous to the Fourier series. The proposed technology is efficient to handle the Richardson scaling law problems in mathematical physics.

Key words: *scaling law heat conduction equation, scaling law series, Kohlrausch-Williams-Watts function, Richardson scaling law*

Introduction

Jean Baptiste Joseph Fourier [1] proposed the theory of the well-known Fourier heat conduction equation in the 3-D case by:

$$\frac{\partial \Xi(x,y,z,t)}{\partial t} = \gamma \nabla^2 \Xi(x,y,z,t) \tag{1}$$

where ∇^2 is the Laplace operator and γ is the thermal diffusivity of the medium. The 1-D heat conduction equation reads:

$$\frac{\partial \Xi(x,t)}{\partial t} = \gamma \frac{\partial^2 \Xi(x,t)}{\partial u^2} \tag{2}$$

Richardson [2] suggested the well-known scaling law:

$$\beta(x) = \lambda x^D \tag{3}$$

where $\lambda \in (0,+\infty)$ is the normalization constant, $x \in (-\infty,+\infty)$ and $D \in (0,+\infty)$ is the scaling exponent. The scaling law derivative of the function $\Im_\beta(x)$ associated with the Richardson scaling law (3) is defined [3]:

$${}^{\mathrm{RSL}}\mathrm{D}_x^{(1)} \Im_\beta(x) = \frac{x^{1-D}}{\lambda D} \frac{\mathrm{d}\Im_\beta(x)}{\mathrm{d}x} \tag{4}$$

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The differential of the function $\Im_\beta(x)$ associated with the Richardson scaling law (5), denoted by $\mathrm{d}\Im_\beta(x)$, is [3]:

$$\mathrm{d}\Im_\beta(x) = (\mathrm{D}\lambda x^{D-1})^n \ {}^{\mathrm{RSL}}\mathrm{D}_t^{(n)}\Im_\beta(x)\mathrm{d}x \tag{5}$$

The scaling law integral of the function $\varpi_\beta(x)$ associated with the Richardson scaling law (3) is defined [3]:

$$ {}^{\mathrm{RSL}}_{\ a}I_x^{(1)}\varpi_\beta(x) = (\lambda D)\int_a^x \varpi_\beta(x)x^{D-1}\mathrm{d}x \tag{6}$$

The scaling law indefinite integral of the function $\varpi_\beta(x)$ associated with the Richardson scaling law (3) is defined [3]:

$$ {}^{\mathrm{RSL}}I_x^{(1)}\varpi_\beta(x) = (\lambda D)\int \varpi_\beta(x)x^{D-1}\mathrm{d}x \tag{7}$$

The improper scaling law integrals of the function $\varpi_\beta(x)$ associated with the Richardson scaling law (3) are defined [3]:

$$\lim_{a\to-\infty} {}^{\mathrm{RSL}}_{\ a}I_x^{(1)}\varpi_\beta(x) = \lim_{a\to-\infty}\left[(\lambda D)\int_a^x \varpi_\beta(x)x^{D-1}\mathrm{d}x\right] = (\lambda D)\int_{-\infty}^x \varpi_\beta(x)x^{D-1}\mathrm{d}x \tag{8}$$

$$\lim_{b\to\infty} {}^{\mathrm{RSL}}_{\ x}I_b^{(1)}\varpi_\beta(x) = \lim_{b\to\infty}\left[(\lambda D)\int_x^b \varpi_\beta(x)x^{D-1}\mathrm{d}x\right] = (\lambda D)\int_x^\infty \varpi_\beta(x)x^{D-1}\mathrm{d}x \tag{9}$$

and

$$\lim_{\substack{b\to\infty\\ a\to-\infty}} {}^{\mathrm{RSL}}_{\ a}I_b^{(1)}\varpi_\beta(x) = \lim_{\substack{b\to\infty\\ a\to-\infty}}\left[(\lambda D)\int_a^b \varpi_\beta(x)x^{D-1}\mathrm{d}x\right] = (\lambda D)\int_{-\infty}^\infty \varpi_\beta(x)x^{D-1}\mathrm{d}x \tag{10}$$

The net change theorem for the scaling law integral associated with the Richardson scaling law (3) reads [3]:

$$\Im_\beta(b) - \Im_\beta(a) = {}^{\mathrm{RSL}}_{\ a}I_b^{(1)}[{}^{\mathrm{RSL}}\mathrm{D}_x^{(1)}\Im_\beta(x)] \tag{11}$$

The properties of the scaling calculus associated with the Richardson scaling law (2) are [3]:

$$ {}^{\mathrm{RSL}}\mathrm{D}_x^{(1)}1 = 0,\ {}^{\mathrm{RSL}}\mathrm{D}_x^{(1)}(\lambda x^D) = 1,\ {}^{\mathrm{RSL}}\mathrm{D}_x^{(1)}(\lambda x^D)^n = n(\lambda x^D)^{n-1} \tag{12a,b,c}$$

$$ {}^{\mathrm{RSL}}\mathrm{D}_x^{(1)}e^{s\lambda x^D} = se^{\lambda x^D},\ {}^{\mathrm{RSL}}\mathrm{D}_x^{(1)}\ln(\lambda x^D) = \frac{1}{\lambda x^D},\ {}^{\mathrm{RSL}}\mathrm{D}_x^{(1)}s^{\lambda x^D} = (\ln s)s^{\lambda x^D} \tag{13a,b,c}$$

$$ {}^{\mathrm{RSL}}_{\ a}I_b^{(1)}1 = \lambda b^D - \lambda a^D,\ {}^{\mathrm{RSL}}_{\ a}I_b^{(1)}[n(\lambda x^D)^{n-1}] = (\lambda b^D)^n - (\lambda a^D)^n \tag{14a,b}$$

$$ {}^{\mathrm{RSL}}_{\ a}I_b^{(1)}\left(\frac{1}{\lambda x^D}\right) = \ln(\lambda b^D) - \ln(\lambda a^D),\ {}^{\mathrm{RSL}}_{\ a}I_b^{(1)}[(\ln s)s^{\lambda x^D}] = s^{\lambda b^D} - s^{\lambda a^D} \tag{15a,b}$$

$$ {}^{\mathrm{RSL}}_{\ a}I_b^{(1)}(e^{\lambda x^D}) = e^{\lambda b^D} - e^{\lambda a^D},\ {}^{\mathrm{RSL}}_{\ a}I_b^{(1)}(se^{\lambda x^D}) = e^{s\lambda b^D} - e^{s\lambda a^D} \tag{16a,b}$$

where s is a constant, and $e^{\lambda x^D} = \Sigma_{n=0}^{\infty}\lambda^n x^{nD}/n!$ is the Kohlrausch-Williams-Watts function [4, 5]. The scaling law heat conduction equation may be used to describe the heat transportations in the sheared granular materials [6], carbon nanotube materials [7], and carbon nanotubes [8].

To solve the problems, the non-Fourier law [3]:

$$-\kappa\, {}^{\mathrm{RSL}}\nabla_D \Xi(x,y,z,t) = Q(x,y,z,t) \tag{17}$$

was suggested, where κ is the thermal conductivity of the scaling law materials, $\Xi(x,y,z,t)$ is the temperature distribution, and the scaling law gradient associated with the Richardson scaling law (3) is [3]:

$${}^{\mathrm{RSL}}\nabla_D = l(\lambda D x^{D-1})\frac{\partial}{\partial x} + j(\lambda D y^{D-1})\frac{\partial}{\partial y} + k(\lambda D z^{D-1})\frac{\partial}{\partial z} \tag{18}$$

and the scaling law Laplace-type associated with the Richardson scaling law (3) is [3]:

$${}^{\mathrm{RSL}}\Delta_D = {}^{\mathrm{RSL}}\nabla_D^2 = {}^{\mathrm{RSL}}\nabla_D\, {}^{\mathrm{RSL}}\nabla_D = \left(\lambda D x^{D-1}\frac{\partial}{\partial x}\right)^2 + \left(\lambda D y^{D-1}\frac{\partial}{\partial y}\right)^2 + \left(\lambda D z^{D-1}\frac{\partial}{\partial z}\right)^2 =$$

$$= [\lambda^2 D^2 x^{2(D-1)}]\frac{\partial^2}{\partial x^2} + [\lambda^2 D^2 y^{2(D-1)}]\frac{\partial^2}{\partial y^2} + [\lambda^2 D^2 z^{2(D-1)}]\frac{\partial^2}{\partial z^2} \tag{19}$$

The main target of the paper is to set up the theory of the scaling law heat conduction associated with the Richardson scaling law (3), and to propose the theory of an scaling law series associated with the scaling law subtrigonometric series and Kohlrausch-Williams-Watts function to solve the scaling law heat conduction equation in the 1-D case.

Theory of a scaling law series associated with the Kohlrausch-Williams-Watts function

In this section, we suggest analytical theory of a scaling law series associated with the Kohlrausch-Williams-Watts function analogous to the Fourier series.

The Kohlrausch-Williams-Watts function is represented [9, 10]:

$$e^{i\lambda x^D} = \cos(\lambda x^D) + i\sin(\lambda x^D) \tag{20}$$

where $\cos(\lambda x^D) = \Sigma_{n=0}^{\infty}\lambda^{2n}x^{2nD}/(2n)!$ and $\sin(\lambda x^D) = \Sigma_{n=0}^{\infty}\lambda^{2n+1}x^{(2n+1)D}/(2n+1)!$.

The scaling law series associated with the scaling law subtrigonometric series is:

$$\hat{\aleph}(x) = \frac{1}{2}\hat{a}_0 + \sum_{n=1}^{\infty}\hat{b}_n\cos(n\lambda x^D) + \sum_{n=1}^{\infty}\hat{c}_n\sin(n\lambda x^D) \tag{21}$$

where $\hat{a}_0$, $\hat{b}_n$, and $\hat{c}_n$ are the coefficients for the scaling law series of the function $\aleph(x)$.

It is observed that $\aleph(\lambda x^D) = \hat{\aleph}(x)$, where $\aleph(x)$ has the period 2π, and that $\hat{\aleph}(x)$ can be expressed by the scaling law subtrigonometric series.

This implies that:

$$\int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} \sin(n_1\lambda x^D)\sin(n_2\lambda x^D)x^{D-1}\mathrm{d}x = \pi/\lambda D \quad (n_1 = n_2) \tag{22}$$

$$\int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} \sin(n_1\lambda x^D)\sin(n_2\lambda x^D)x^{D-1}\mathrm{d}x = 0 \quad (n_1 \neq n_2) \tag{23}$$

$$\int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} \cos(n_1\lambda x^D)\cos(n_2\lambda x^D)x^{D-1}\mathrm{d}x = \pi/\lambda D \quad (n_1 = n_2) \tag{24}$$

$$\int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} \cos(n_1\lambda x^D)\cos(n_2\lambda x^D)x^{D-1}\mathrm{d}x = 0 \quad (n_1 \neq n_2) \tag{25}$$

$$\int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} \sin(n_1\lambda x^D)\cos(n_2\lambda x^D)x^{D-1}\mathrm{d}x = 0 \tag{26}$$

$$\int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} \sin(n_1\lambda x^D)x^{D-1}\mathrm{d}x = 0 \tag{27}$$

and

$$\int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} \cos(n_1\lambda x^D)x^{D-1}\mathrm{d}x = 0 \tag{28}$$

It is easy to find that:

$$\int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} e^{in_1\lambda x^D} e^{-in_2\lambda x^D} x^{D-1}\mathrm{d}x = \pi/\lambda D \quad (n_1 = n_2) \tag{29}$$

$$\int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} e^{in_1\lambda x^D} e^{-in_2\lambda x^D} x^{D-1}\mathrm{d}x = 0 \quad (n_1 \neq n_2) \tag{30}$$

$$\int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} e^{in_1\lambda x^D} x^{D-1}\mathrm{d}x = 0 \quad (n_1 \neq 0) \tag{31}$$

and

$$\int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} x^{D-1}\mathrm{d}x = \pi/\lambda D \tag{32}$$

Let us define the scaling law product of two functions $\phi_{n_1}(x)$ and $\psi_{n_2}(x)$ with weighted function $\alpha(x) = x^{D-1}$ by:

$$\left\langle \phi_{n_1}(x), \psi_{n_2}(x) \right\rangle_{\alpha(x)} = \int_a^b \phi_{n_1}(x)\psi_{n_2}(x)x^{D-1}\mathrm{d}x \tag{33}$$

If:

$$\left\langle \phi_{n_1}(x), \psi_{n_2}(x) \right\rangle_{\alpha(x)} = \int_a^b \phi_{n_1}(x)\psi_{n_2}(x)x^{D-1}\mathrm{d}x = 0 \quad (n_1 \neq n_2) \tag{34}$$

and

$$\left\langle \phi_{n_1}(x), \psi_{n_2}(x) \right\rangle_{\alpha(x)} = \int_a^b \phi_{n_1}(x)\psi_{n_2}(x)x^{D-1}\mathrm{d}x = \Lambda \quad (n_1 = n_2) \tag{35}$$

where Λ is a constant, then two functions $\phi_{n_1}(x)$ and $\psi_{n_2}(x)$ are the weighted orthonormal functions with weighted function $\alpha(x) = x^{D-1}$.

Thus:

$$\left\langle \sin(n_1\lambda x^D), \sin(n_2\lambda x^D) \right\rangle_{\alpha(x)=x^{D-1}} = \pi/\lambda D \quad (n_1 = n_2) \tag{36}$$

$$\left\langle \sin(n_1\lambda x^D), \sin(n_2\lambda x^D) \right\rangle_{\alpha(x)=x^{D-1}} = 0 \quad (n_1 \neq n_2) \tag{37}$$

$$\left\langle \cos(n_1\lambda x^D), \cos(n_2\lambda x^D) \right\rangle_{\alpha(x)=x^{D-1}} = \pi/\lambda D \quad (n_1 = n_2) \tag{38}$$

$$\left\langle \cos(n_1\lambda x^D), \cos(n_2\lambda x^D) \right\rangle_{\alpha(x)=x^{D-1}} = 0 \quad (n_1 \neq n_2) \tag{39}$$

$$\left\langle e^{in_1\lambda x^D}, e^{in_2\lambda x^D} \right\rangle_{\alpha(x)=x^{D-1}} = \pi/\lambda D \quad (n_1 = n_2) \tag{40}$$

$$\left\langle e^{in_1\lambda x^D}, e^{in_2\lambda x^D} \right\rangle_{\alpha(x)=x^{D-1}} = 0 \quad (n_1 \neq n_2) \tag{41}$$

where $a = -\sqrt[D]{\pi/\lambda}$ and $b = \sqrt[D]{\pi/\lambda}$.

Now, by (14 a, b), we find

$${}_{a}^{\mathrm{RSL}}I_b^{(1)}1 = (\lambda D)\int_a^b x^{D-1}\mathrm{d}x = \lambda b^D - \lambda a^D \tag{42}$$

such that:

$$\int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} \hat{\aleph}(x)x^{D-1}\mathrm{d}x = \int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} \left[\frac{1}{2}\hat{a}_0 + \sum_{n=1}^{\infty}\hat{b}_n\cos(n\lambda x^D) + \sum_{n=1}^{\infty}\hat{c}_n\sin(n\lambda x^D)\right]x^{D-1}\mathrm{d}x =$$

$$= \int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} \frac{1}{2}\hat{a}_0 x^{D-1}\mathrm{d}x = \frac{\hat{a}_0}{2}\int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} x^{D-1}\mathrm{d}x = \hat{a}_0\pi/\lambda D \tag{43}$$

$$\int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} \hat{\aleph}(x)\cos(n\lambda x^D)x^{D-1}\mathrm{d}x =$$

$$= \int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} \left[\frac{1}{2}\hat{a}_0 + \sum_{n=1}^{\infty}\hat{b}_n \cos(n\lambda x^D) + \sum_{n=1}^{\infty}\hat{c}_n \sin(n\lambda x^D)\right]\cos(n\lambda x^D)x^{D-1}\mathrm{d}x =$$

$$= \int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} \left[\sum_{n=1}^{\infty}\hat{b}_n \cos(n\lambda x^D)\right]\cos(n\lambda x^D)x^{D-1}dx =$$

$$= \hat{b}_n \int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} \cos(n\lambda x^D)\cos(n\lambda x^D)x^{D-1}\mathrm{d}x =$$

$$= \hat{b}_n \pi/\lambda D \tag{44}$$

and

$$\int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} \hat{\aleph}(x)\sin(n\lambda x^D)x^{D-1}\mathrm{d}x =$$

$$= \int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} \left[\frac{1}{2}\hat{a}_0 + \sum_{n=1}^{\infty}\hat{b}_n \cos(n\lambda x^D) + \sum_{n=1}^{\infty}\hat{c}_n \sin(n\lambda x^D)\right]\sin(n\lambda x^D)x^{D-1}\mathrm{d}x =$$

$$= \int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} \left[\sum_{n=1}^{\infty}\hat{c}_n \sin(n\lambda x^D)\right]\sin(n\lambda x^D)x^{D-1}\mathrm{d}x =$$

$$= \hat{c}_n \int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} \sin(n\lambda x^D)\sin(n\lambda x^D)x^{D-1}\mathrm{d}x =$$

$$= \hat{c}_n \pi/\lambda D \tag{45}$$

Thus, with the aid of (43)-(45), we have:

$$\hat{a}_0 = \lambda D/\pi \int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} \hat{\aleph}(x)x^{D-1}\mathrm{d}x \tag{46}$$

$$\hat{b}_n = \lambda D/\pi \int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} \hat{\aleph}(x)\cos(n\lambda x^D)x^{D-1}\mathrm{d}x \tag{47}$$

and

$$\hat{c}_n = \lambda D/\pi \int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} \hat{\aleph}(x)\sin(n\lambda x^D)x^{D-1}\mathrm{d}x \tag{48}$$

such that we determine the scaling law subtrigonometric series (21).

Let us define a scaling law series associated with the Kohlrausch-Williams-Watts function:

$$\hat{\aleph}(x) = \sum_{n=-\infty}^{\infty} \hat{g}_n e^{in\lambda x^D} \tag{49}$$

where $\hat{g}_n$ are the coefficient for the scaling law series of the function $\hat{\aleph}(x)$.

We now compute:

$$\int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} \hat{\aleph}(x) e^{-in\lambda x^D} x^{D-1} \mathrm{d}x = \hat{g}_n \int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} e^{in\lambda x^D} e^{-in\lambda x^D} x^{D-1} \mathrm{d}x = \hat{g}_n \pi\lambda/D \tag{50}$$

Thus, with use of (50), we obtain:

$$\hat{g}_n = \lambda D/\pi \int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} \lambda \hat{\aleph}(x) e^{-in\lambda x^D} x^{D-1} \mathrm{d}x \tag{51}$$

Let $\hat{\aleph}(\lambda x^D) = \hat{\aleph}[\lambda(-x)^D]$ with the period 2π. Then we obtain:

$$\hat{\aleph}(x) = \frac{1}{2}\hat{a}_0 + \sum_{n=1}^{\infty} \hat{b}_n \cos(n\lambda x^D) \tag{52}$$

where

$$\hat{a}_0 = \lambda D/\pi \int_{\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} \hat{\aleph}(x) x^{D-1} \mathrm{d}x = 2\lambda D/\pi \int_{0}^{\sqrt[D]{\pi/\lambda}} \hat{\aleph}(x) x^{D-1} \mathrm{d}x \tag{53}$$

and

$$\hat{b}_n = \lambda D/\pi \int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} \hat{\aleph}(x) \cos(n\lambda x^D) x^{D-1} \mathrm{d}x = 2\lambda D/\pi \int_{0}^{\sqrt[D]{\pi/\lambda}} \hat{\aleph}(x) \cos(n\lambda x^D) x^{D-1} \mathrm{d}x \tag{54}$$

Let $\hat{\aleph}(\lambda x^D) = -\hat{\aleph}[\lambda(-x)^D]$ with the period 2π. Then we obtain:

$$\hat{\aleph}(x) = \sum_{n=1}^{\infty} \hat{c}_n \sin(n\lambda x^D) \tag{55}$$

where

$$\hat{c}_n = \lambda D/\pi \int_{-\sqrt[D]{\pi/\lambda}}^{\sqrt[D]{\pi/\lambda}} \hat{\aleph}(x) \sin(n\lambda x^D) x^{D-1} \mathrm{d}x = 2\lambda D/\pi \int_{0}^{\sqrt[D]{\pi/\lambda}} \hat{\aleph}(x) \sin(n\lambda x^D) x^{D-1} \mathrm{d}x \tag{56}$$

The general theory of the scaling law series associated with the Kohlrausch-Williams-Watts function [10].

Theory of the scaling law heat conduction

Let us recall that the non-Fourier law [3]:

$$-\kappa\,{}^{\text{RSL}}\nabla_D \Xi(x,y,z,t) = \mathbf{Q}(x,y,z,t) \tag{57}$$

where κ is the constant, and:

$$-{}^{\text{RSL}}\nabla_D \mathbf{Q}(x,y,z,t) = \upsilon\rho(\lambda D t^{D-1})\frac{\partial \Xi(x,y,z,t)}{\partial t} \tag{58}$$

where υ is the specific heat capacity and ρ is the density (mass per unit scaling law volume) of the scaling law material.

Connected with (57) and (58), we obtain the scaling law heat conduction equation:

$$-{}^{\text{RSL}}\nabla_D[-\kappa\,{}^{\text{RSL}}\nabla_D \Xi(x,y,z,t)] = \upsilon\rho(\lambda D t^{D-1})\frac{\partial \Xi(x,y,z,t)}{\partial t} \tag{59}$$

This implies that the scaling law heat conduction equation reads:

$${}^{\text{RSL}}\nabla_D[\kappa\,{}^{\text{RSL}}\nabla_D \Xi(x,y,z,t)] = \upsilon\rho(\lambda \text{D} t^{D-1})\frac{\partial \Xi(x,y,z,t)}{\partial t} \tag{60}$$

If κ is a constant, then the scaling law heat conduction equation:

$$(\lambda D t^{D-1})\frac{\partial \Xi(x,y,z,t)}{\partial t} = \gamma\,{}^{\text{RSL}}\nabla_D^2 \Xi(x,y,z,t) \tag{61}$$

where $\gamma = \kappa/(\upsilon\rho)$ is the thermal diffusivity of the medium.

The 1-D scaling law heat conduction equation can be expressed:

$$(\lambda D t^{D-1})\frac{\partial \Xi(x,t)}{\partial t} = \gamma[\lambda^2 D^2 x^{2(D-1)}]\frac{\partial^2 \Xi(x,t)}{\partial x^2} \tag{62}$$

subject to the initial and boundary conditions:

$$\Xi(x,0) = v(x) \tag{63}$$

and

$$\Xi(0,t) = \Xi(L,t) = 0 \tag{64}$$

where $x \in [0, L]$ and $t > 0$.

Analytical solution for the scaling law heat conduction equation in the 1-D case

Suppose that:

$$\Xi(x,t) = \Sigma(t)\Pi(x) \tag{65}$$

such that:

$$(\lambda D t^{D-1})\frac{\partial[\Sigma(t)\Pi(x)]}{\partial t}=\gamma[\lambda^2 D^2 x^{2(D-1)}]\frac{\partial^2[\Sigma(t)\Pi(x)]}{\partial x^2} \tag{66}$$

which leads to:

$$\Pi(x)(\lambda D t^{D-1})\frac{\partial\Sigma(t)}{\partial t}=\gamma\Sigma(t)[\lambda^2 D^2 x^{2(D-1)}]\frac{\partial^2\Pi(x)}{\partial x^2} \tag{67}$$

This implies that:

$$\frac{1}{\Sigma(t)}(\lambda D t^{D-1})\frac{\partial\Sigma(t)}{\partial t}=\frac{\gamma}{\Pi(x)}[\lambda^2 D^2 x^{2(D-1)}]\frac{\partial^2\Pi(x)}{\partial x^2} \tag{68}$$

Let:

$$\frac{1}{\Sigma(t)}(\lambda D t^{D-1})\frac{\partial\Sigma(t)}{\partial t}=-\Re \tag{69}$$

such that:

$$\frac{\gamma}{\Pi(x)}[\lambda^2 D^2 x^{2(D-1)}]\frac{\partial^2\Pi(x)}{\partial x^2}=-\Re \tag{70}$$

where $\Re$ is a constant.

Thus:

$$(\lambda D t^{D-1})\frac{\partial\Sigma(t)}{\partial t}=-\Re\Sigma(t) \tag{71}$$

and

$$[\lambda^2 D^2 x^{2(D-1)}]\frac{\partial^2\Pi(x)}{\partial x^2}=-\xi\Pi(x) \tag{72}$$

where $\xi=\Re/\gamma$.

According to [9, 10], we find that:

$$\Sigma(t)=\mathrm{T}e^{-\Re\lambda t^D} \tag{73}$$

and

$$\Pi(x)=\Pi_1\cos(\sqrt{\xi}\lambda x^D)+\Pi_2\sin(\sqrt{\xi}\lambda x^D) \tag{74}$$

where T, Π_1, and Π_2 are the constants.

Since:

$$\Xi(0,t)=\Xi(L,t)=0 \tag{75}$$

then $\Pi_1=0$ and:

$$\Pi(x)=\Pi_2\sin(\sqrt{\xi}\lambda L^D)=0 \tag{76}$$

Here, we have $\sqrt{\xi}\lambda L^D = n\pi,$ which leads to:

$$\Re = \gamma\left(\frac{n\pi}{\lambda L^D}\right)^2 \text{ and } \xi = \left(\frac{n\pi}{\lambda L^D}\right)^2$$

Thus:

$$\Pi(x) = \sum_{n=1}^{\infty} \vartheta(n)\sin\left(\frac{n\pi}{L^D}x^D\right) = v(x) \tag{77}$$

which yields that

$$\vartheta(n) = \frac{D}{L^D}\int_{-L}^{L} v(x)\sin\left(n\frac{\pi}{L^D}x^D\right)x^{D-1}\mathrm{d}x = 2\frac{D}{L^D}\int_{0}^{L} v(x)\sin\left(n\frac{\pi}{L^D}x^D\right)x^{D-1}\mathrm{d}x \tag{78}$$

From (73) and (77) we have $\mathrm{T} = 1$ such that:

$$\Xi(x,t) = \sum_{n=1}^{\infty} \vartheta(n)\sin\left(\frac{n\pi}{L^D}x^D\right)e^{-\gamma\left(\frac{n\pi}{\lambda L^D}\right)^2 \lambda t^D} \tag{79}$$

Conclusion

This work studied the scaling law heat conduction process associated with the Richardson scaling law and the analytical theory of the scaling law heat conduction equation associated with the Richardson scaling law. We also proposed the theory of the scaling law series associated with the Kohlrausch-Williams-Watts function. The result is important for us to find the scaling law series solutions for the scaling law PDE in mathematical physics.

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Nomenclature

x, *y*, *z* – space co-ordinates, [m]
t – time, [s]

Greek symbols

γ – heat conductivity, [Wm^{-1}K^{-1}]

$\Xi(x, y, z, t)$ – temperature, [K]
ρ – density, [kgm^{-3}]
υ – specific heat capacity, [Jkg^{-1}K^{-1}]

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