A TWO-LEVEL HIGH ACCURACY LINEARIZED DIFFERENCE SCHEME FOR THE BENJAMIN-BONA-MAHONY EQUATION

by

Jin-Song HU*, Ju CHEN, Li-Jia YI, and Ming-Fang SU

School of Science, Xihua University, Chengdu, Sichuan, China

Original scientific paper
https://doi.org/10.2298/TSCI2202017H

In this article we propose the anomalous diffusion models with respect to monotone increasing functions. The Riesz-type fractional order derivatives operators with respect to power-law function are considered based on the extended work of Riesz. Two models for the anomalous diffusion processes are given to describe the special behaviors in the complex media.

Key words: *anomalous diffusion, Riesz fractional derivative, Riesz-type fractional derivative with respect to monotone increasing function, Riesz-type fractional derivative with respect to power-law function*

Introduction

To consider the dissipation principle of the non-linear wave propagation, the Benjamin-Bona-Mahony (BBM) equation [1] is proposed in the study of long waves in non-linear dispersion systems. Authors [2, 3] studied the attenuation of its solution, and the existence, uniqueness and convergence of the solution of BBM equation are also proved in [4-6]. Various numerical methods also have attracted the attention of many researchers [7-16]. In this paper, the following initial value condition and boundary value conditions of BBM equation are considered. Moreover, the following problem:

$$u_t - u_{xxt} + u_x - u_{xx} + uu_x = 0\,,\ (x,t) \in (x_L, x_R) \times (0,T] \tag{1}$$

$$u(x,0) = u_0(x)\,,\ x \in [x_L, x_R] \tag{2}$$

$$u(x_L,t) = u(x_R,t) = 0\,,\ t \in [0,T] \tag{3}$$

has the conserved quantities [15];

$$Q(t) = \int_{x_L}^{x_R} u(x,t)\,\mathrm{d}x = \int_{x_L}^{x_R} u(x)\,\mathrm{d}x = Q(0) \tag{4}$$

where $u_0(x)$ is a smooth functions and $Q(0)$ is a constant only related to the initial conditions.

In recent years, numerous numerical methods have been proposed for solving BBM eqs. (1)-(4). Huang *et al.* [15] proposes a two-layer non-linear Crank-Nicolson difference scheme with the theoretical accuracy of $O(\tau^2 + h^4)$ for problem (1)-(3) in which the non-linear iteration is required for numerical solutions. Zhang *et al.* [16] put forward a three-level

* Corresponding author, e-mail: hujs@mail.xhu.edu.cn

linear difference scheme with the theoretical accuracy of $O(\tau^2 + h^4)$, which can improve the efficiency of numerical solutions. However, this scheme is not self-starting.

In this paper, following the aforementioned work, the discretization of the non-linear term uu_x is used by a two-level linearization, and a two-level linearization difference scheme of the theoretical accuracy $O(\tau^2 + h^4)$ is obtained for problem (1)-(3), the conserved quantity (4) is reasonably simulated. Since only the data of the previous time layer need to be stored, the advantages of linear scheme without non-linear iteration are maintained in numerical solutions. That is, the proposed scheme has higher numerical efficiency, and the numerical results also show that the accuracy is obviously better than the other high-accuracy schemes in [15].

The finite difference scheme and conservation law

Firstly, for the domain $[x_R, x_L]\times[0, T]$, let $h = (x_R - x_L)/J$ be the step size for the spatial grid, and τ be the step size for the temporal direction such that $x_j = x_L + jh(0 \le j \le J)$, $t_n = n\tau$ $(n = 0, 1, 2,\ldots, N, N = [T/\tau])$.

Denote:

$$u_j^n = u(x_j, t_n), \quad U_j^n \approx u(x_j, t_n), \quad Z_h^0 = \{U = (U_j) \big| U_0 = U_J = 0, \quad j = 0, 1, \cdots, J-1, J\}$$

and

$$(U_j^n)_x = \frac{U_{j+1}^n - U_j^n}{h}, \; (U_j^n)_{\bar{x}} = \frac{U_j^n - U_{j-1}^n}{h}, \; (U_j^n)_{\hat{x}} = \frac{U_{j+1}^n - U_{j-1}^n}{2h}$$

$$(U_j^n)_t = \frac{U_j^{n+1} - U_j^n}{\tau}, \; U_j^{n+\frac{1}{2}} = \frac{U_j^{n+1} + U_j^n}{2}$$

$$\left\langle U^n, V^n \right\rangle = h\sum_{j=1}^{J-1} U_j^n V_j^n, \; \left\| U^n \right\|^2 = \left\langle U^n, U^n \right\rangle, \; \left\| U^n \right\|_\infty = \max_{1\le j\le J-1} \left| U_j^n \right|, \; (U_j^n)_{\ddot{x}} = \frac{U_{j+2}^n - U_{j-2}^n}{4h}$$

We propose a two-level linear finite difference scheme for the initial boundary value problem (1)-(3):

$$(U_j^n)_t - \frac{4}{3}(U_j^n)_{x\bar{x}t} + \frac{1}{3}(U_j^n)_{\hat{x}\hat{x}t} + \frac{4}{3}(U_j^{n+\frac{1}{2}})_{\hat{x}} - \frac{1}{3}(U_j^{n+\frac{1}{2}})_{\ddot{x}} -$$
$$-\frac{4}{3}(U_j^{n+\frac{1}{2}})_{x\bar{x}} + \frac{1}{3}(U_j^{n+\frac{1}{2}})_{\hat{x}\hat{x}} + P_1(U_j^n, U_j^{n+1}) - P_2(U_j^n, U_j^{n+1}) = 0$$
$$j = 1, 2, \cdots, J-1, \quad n = 1, 2, \cdots, N-1 \tag{5}$$

$$U_j^0 = u_0(x_j), \quad j = 0, 1, 2, \cdots, J \tag{6}$$

$$U^n \in Z_h^0, \quad n = 0, 1, 2, \cdots, N \tag{7}$$

where

$$P_1(U_j^n, U_j^{n+1}) = \frac{2}{3}[U_j^n (U_j^{n+1})_{\hat{x}} + U_j^{n+1}(U_j^n)_{\hat{x}}]$$

and

$$P_2(U_j^n, U_j^{n+1}) = \frac{1}{6}[U_j^n (U_j^{n+1})_{\ddot{x}} + U_j^{n+1}(U_j^n)_{\ddot{x}}]$$

Let $u(x,t)$ be the solution of problem (1)-(3), $u_j^n = u(x_j, t_n)$. The truncation error of (1)-(3) is in the following:

$$r_j^n = (u_j^n)_t - \frac{4}{3}(u_j^n)_{x\bar{x}t} + \frac{1}{3}(u_j^n)_{\hat{x}\hat{x}t} + \frac{4}{3}\left(u_j^{n+\frac{1}{2}}\right)_{\hat{x}} - \frac{1}{3}\left(u_j^{n+\frac{1}{2}}\right)_{\ddot{x}} - \frac{4}{3}\left(u_j^{n+\frac{1}{2}}\right)_{x\bar{x}} + \frac{1}{3}\left(u_j^{n+\frac{1}{2}}\right)_{\hat{x}\hat{x}} +$$

$$+P_1(u_j^n, u_j^{n+1}) - P_2(u_j^n, u_j^{n+1}), \quad j = 1, 2, \cdots, J-1, \quad n = 1, 2, \cdots, N-1 \tag{8}$$

$$u_j^0 = u_0(x_j), \quad j = 0, 1, 2, \cdots, J \tag{9}$$

$$u^n \in Z_h^0, \quad n = 0, 1, 2, \cdots, N \tag{10}$$

By the Taylor expansion, we know that:

$$\left|r_j^n\right| = O(\tau^2 + h^4) \tag{11}$$

holds if h, $\tau \to 0$.

Theorem 1 The difference scheme (5)-(7) is conserved with respect to the following discrete energy, *i.e.*:

$$Q^n = h\sum_{j=1}^{J-1} U_j^n = Q^{n-1} = \cdots = Q^0, \quad n = 1, 2, \cdots, N \tag{12}$$

Convergence and stability

Lemma 1 [16] For any $U \in Z_h^0$, there has always $\|U_{\ddot{x}}\| \le \|U_{\hat{x}}\| \le \|U_x\|$.

Lemma 2 [15] Suppose that $u_0 \in H^1$, then the solutions of the initial boundary value problem (1)-(3) satisfy $\|u\|_{L_2} \le C$, $\|u_x\|_{L_2} \le C$, $\|u\|_{L_\infty} \le C$.

Theorem 2 Suppose $u_0 \in H^1$. For sufficiently small temporal step τ and spatial step h, the solutions of scheme (5)-(7) converge to the solution of the initial boundary value problem (1)-(3) with the convergence order of $O(\tau^2 + h^4)$ by the norm $\|\cdot\|_\infty$ for U^n.

Proof. Subtracting (8)-(10) from (5)-(7) and letting $e_j^n = u_j^n - U_j^n$, we have:

$$r_j^n = (e_j^n)_t - \frac{4}{3}(e_j^n)_{x\bar{x}t} + \frac{1}{3}(e_j^n)_{\hat{x}\hat{x}t} + \frac{4}{3}\left(e_j^{n+\frac{1}{2}}\right)_{\hat{x}} - \frac{1}{3}\left(e_j^{n+\frac{1}{2}}\right)_{\ddot{x}} - \frac{4}{3}\left(e_j^{n+\frac{1}{2}}\right)_{x\bar{x}} + \frac{1}{3}\left(e_j^{n+\frac{1}{2}}\right)_{\hat{x}\hat{x}} +$$

$$+P_1(u_j^n, u_j^{n+1}) - P_1(U_j^n, U_j^{n+1}) - P_2(u_j^n, u_j^{n+1}) + P_2(U_j^n, U_j^{n+1}) \tag{13}$$

$$e_j^0 = 0, \quad j = 0, 1, 2, \cdots, J \tag{14}$$

$$e^n \in Z_h^0, \quad n = 0, 1, 2, \cdots, N \tag{15}$$

Next, we use the mathematical induction to prove the error estimates. From *Lemma 2* and eq. (12), there exist constants C_u, C_r, which are independent of τ and h, satisfy that:

$$\left\|u^n\right\|_\infty \le C_u, \quad \left\|r^n\right\|_\infty \le C_r(\tau^2+h^2), \quad n=1,2,\cdots,N \tag{16}$$

It follows from (14) and the initial conditions (6) that the following estimates:

$$\left\|e^0\right\| = 0, \quad \left\|U^0\right\|_\infty \le C_u \tag{17}$$

Suppose that:

$$\left\|e^l\right\| + \left\|e_x^l\right\| \le C_l(\tau^2+h^4), \quad l=2,3,\cdots,n, \quad n\le N-1 \tag{18}$$

where $C_l\,(l=2,3,\cdots,n)$ is also independent of τ and h. By the discrete Sobolev inequality [17] and the Cauchy-Schwarz inequality, we get:

$$\left\|e^l\right\|_\infty \le C_0\sqrt{\left\|e^l\right\|}\sqrt{\left\|e_x^l\right\|+\left\|e^l\right\|} \le \frac{1}{2}C_0\left(2\left\|e^l\right\|+\left\|e_x^l\right\|\right) \le \frac{3}{2}C_0C_l(\tau^2+h^2), \quad l=1,2,\cdots,n \tag{19}$$

$$\left\|U^l\right\|_\infty \le \left\|u^l\right\|_\infty + \left\|e^l\right\|_\infty \le C_u + \frac{3}{2}C_0C_l(\tau^2+h^2), \quad l=2,3,\cdots,n \tag{20}$$

Taking the inner product of (13) with $e^{n+1/2}$ and using the summation by part [17], we get:

$$\frac{1}{2}\left\|e^n\right\|_t^2 + \frac{2}{3}\left\|e_x^n\right\|_t^2 - \frac{1}{6}\left\|e_{\hat{x}}^n\right\|_t^2 = \langle r^n, e^{n+\frac{1}{2}}\rangle + \frac{4}{3}\left\|e_x^{n+\frac{1}{2}}\right\|^2 - \frac{1}{3}\left\|e_{\hat{x}}^{n+\frac{1}{2}}\right\|^2 -$$

$$-\left\langle P_1(u_j^n,u_j^{n+1}) - P_1(U_j^n,U_j^{n+1}), e^{n+\frac{1}{2}}\right\rangle + \left\langle P_2(u_j^n,u_j^{n+1}) - P_2(U_j^n,U_j^{n+1}), e^{n+\frac{1}{2}}\right\rangle \tag{21}$$

According to *Lemma 2* and the mean value theorem, the following result:

$$(u_j^{n+1})_{\hat{x}} = \frac{u(x_{j+1},t_{n+1}) - u(x_{j-1},t_{n+1})}{2h} = \frac{\partial}{\partial x}u(x_{\xi_j},t_{n+1}),\ (x_{j-1}\le\xi_j\le x_{j+1})$$

holds, that is:

$$\left\|u_{\hat{x}}^{n+1}\right\|_\infty \le C_u \tag{22}$$

Similarly, we have:

$$\left\|u_{\ddot{x}}^{n+1}\right\|_\infty \le C_u \tag{23}$$

If h and τ are sufficiently small and satisfy that:

$$\frac{3}{2}C_0\left(\max_{0\le l\le n} C_l\right)(\tau^2+h^2) \le 1 \tag{24}$$

then, it follows from (21)-(23) and the Cauchy-Schwarz inequality that:

$$-\left\langle P_1(u^n,u^{n+1})-P_1(u^n,u^{n+1}),e^{n+\frac{1}{2}}\right\rangle=$$

$$=-\frac{2}{3}h\sum_{j=1}^{J-1}[e_j^n(u_j^{n+1})_{\hat{x}}+U_j^n(e_j^{n+1})_{\hat{x}}]e_j^{n+\frac{1}{2}}-\frac{2}{3}h\sum_{j=1}^{J-1}[u_j^{n+1}(e_j^n)_{\hat{x}}+e_j^{n+1}(U_j^n)_{\hat{x}}]\,e_j^{n+\frac{1}{2}}=$$

$$=-\frac{2}{3}h\sum_{j=1}^{J-1}[e_j^n(u_j^{n+1})_{\hat{x}}+U_j^n(e_j^{n+1})_{\hat{x}}]e_j^{n+\frac{1}{2}}-\frac{2}{3}h\sum_{j=1}^{J-1}u_j^{n+1}(e_j^n)_{\hat{x}}e_j^{n+\frac{1}{2}}+$$

$$+\frac{2}{3}h\sum_{j=1}^{J-1}U_j^n\left[(e_j^{n+1})_{\hat{x}}e_j^{n+\frac{1}{2}}+e_j^{n+1}\left(e_j^{n+\frac{1}{2}}\right)_{\hat{x}}\right]\le\frac{1}{3}C_u\left(\left\|e^n\right\|^2+\left\|e_x^n\right\|^2+2\left\|e^{n+\frac{1}{2}}\right\|^2\right)+$$

$$+\frac{1}{3}\left[C_u+\frac{3}{2}C_0C_n(\tau^2+h^4)\right]\left(2\left\|e_x^{n+1}\right\|^2+2\left\|e^{n+\frac{1}{2}}\right\|^2+\left\|e^{n+1}\right\|^2+\left\|e_x^{n+\frac{1}{2}}\right\|^2\right)\le$$

$$\le\frac{1}{3}C_u\left(2\left\|e^n\right\|^2+\left\|e_x^n\right\|^2+\left\|e^{n+1}\right\|^2\right)+\frac{1}{6}(C_u+1)\left(5\left\|e_x^{n+1}\right\|^2+\left\|e_x^n\right\|^2+4\left\|e^{n+1}\right\|^2+2\left\|e^n\right\|^2\right)\tag{25}$$

$$\left\langle P_2(u^n,u^{n+1})-P_2(U^n,U^{n+1}),e^{n+\frac{1}{2}}\right\rangle=$$

$$=-\frac{1}{6}h\sum_{j=1}^{J-1}[e_j^n(u_j^{n+1})_{\ddot{x}}+U_j^n(e_j^{n+1})_{\ddot{x}}]e_j^{n+\frac{1}{2}}+\frac{1}{6}h\sum_{j=1}^{J-1}[u_j^{n+1}(e_j^{n+1})_{\ddot{x}}+e_j^{n+1}(U_j^{n+1})_{\ddot{x}}]\,e_j^{n+\frac{1}{2}}=$$

$$=-\frac{1}{6}h\sum_{j=1}^{J-1}[e_j^n(u_j^{n+1})_{\ddot{x}}+U_j^n(e_j^{n+1})_{\ddot{x}}]e_j^{n+\frac{1}{2}}+\frac{1}{6}h\sum_{j=1}^{J-1}u_j^{n+1}(e_j^n)_{\ddot{x}}e_j^{n+\frac{1}{2}}+$$

$$+\frac{1}{6}h\sum_{j=1}^{J-1}U_j^n[(e_j^{n+1})_{\ddot{x}}e_j^{n+\frac{1}{2}}+e_j^{n+1}(e_j^{n+\frac{1}{2}})_{\ddot{x}}]\le\frac{1}{12}C_u\left(\left\|e^n\right\|^2+\left\|e_x^n\right\|^2+2\left\|e^{n+\frac{1}{2}}\right\|^2\right)+$$

$$+\frac{1}{12}\left[C_u+\frac{3}{2}C_0C_n(\tau^2+h^4)\right]\left(2\left\|e_x^{n+1}\right\|^2+2\left\|e^{n+\frac{1}{2}}\right\|^2+\left\|e^{n+1}\right\|^2+\left\|e_x^{n+\frac{1}{2}}\right\|^2\right)\le$$

$$\le\frac{1}{12}C_u\left(2\left\|e^n\right\|^2+\left\|e_x^n\right\|^2+\left\|e^{n+1}\right\|^2\right)+\frac{1}{24}(C_u+1)\left(5\left\|e_x^{n+1}\right\|^2+\left\|e_x^n\right\|^2+4\left\|e^{n+1}\right\|^2+2\left\|e^n\right\|^2\right)\tag{26}$$

and

$$\langle r^n,e^{n+\frac{1}{2}}\rangle=\frac{1}{2}\langle r^n,e^{n+1}+e^n\rangle\le\frac{1}{2}\left\|r^n\right\|^2+\frac{1}{4}\left[\left\|e^{n+1}\right\|^2+\left\|e^n\right\|^2\right]\tag{27}$$

Substituting (25)-(27) into (21), we get:

$$\left(\|e^{n+1}\|^2-\|e^n\|^2\right)+\frac{4}{3}\left(\|e_x^{n+1}\|^2-\|e_x^n\|^2\right)-\frac{1}{3}\left(\|e_{\hat{x}}^{n+1}\|^2-\|e_{\hat{x}}^n\|^2\right)\le$$

$$\le\tau\|r^n\|^2+\frac{1}{2}\tau\left(\|e^{n+1}\|^2+\|e^n\|^2\right)+2\tau\left(\|e_x^{n+1}\|^2+\|e_x^n\|^2\right)+\frac{5}{12}\tau C_u\left(2\|e^n\|^2+\|e_x^n\|^2+\|e^{n+1}\|^2\right)+$$

$$+\frac{5}{24}\tau(C_u+1)\left(5\|e_x^{n+1}\|^2+\|e_x^n\|^2+4\|e^{n+1}\|^2+2\|e^n\|^2\right)\le$$

$$\le\tau\|r^n\|^2+3\tau(C_u+1)\left(\|e^{n+1}\|^2+\|e^n\|^2+\|e_x^{n+1}\|^2+\|e_x^n\|^2\right) \tag{28}$$

Setting

$$B^n=\|e^n\|^2+\frac{4}{3}\|e_x^n\|^2-\frac{1}{3}\|e_{\hat{x}}^n\|^2$$

and summing up (28) from 1 to *n*, we get:

$$B^{n+1}\le B^1+\tau\sum_{k=1}^{n}\|r^k\|^2+6(C_u+1)\tau\sum_{k=1}^{n+1}\left(\|e^k\|^2+\|e_x^k\|^2\right) \tag{29}$$

It follows from (16) and (28) that:

$$\tau\sum_{k=1}^{n}\|r^k\|^2\le n\tau\max_{1\le k\le n}\|r^k\|^2\le T(C_r)^2(\tau^2+h^4)^2 \tag{30}$$

$$B^1=C_1^2(\tau^2+h^4)^2 \tag{31}$$

Then, substituting (30) and (31) into (29), and applying the discrete Gronwall inequality [17], if *h* and *τ* are sufficiently small and satisfy that $\tau<1/[12(C_u+1)]$, we get:

$$\|e^{n+1}\|^2+\|e_x^{n+1}\|^2\le B^{n+1}\le[T(C_r)^2+C_1^2](\tau^2+h^4)^2e^{2T[6(C_u+1)]}\le$$

$$\le(C_{n+1})^2(\tau^2+h^4)^2,\quad n=1,2,\cdots,N-1$$

where

$$C_{n+1}=(\sqrt{T}C_r+C_1)e^{6T(C_u+1)}$$

Obviously, C_{n+1} is a constant independent of *n*. Therefore, by the mathematical induction, we get:

$$\|e^n\|\le O(\tau^2+h^4),\quad \|e_x^n\|\le O(\tau^2+h^4),\quad n=1,2,\cdots,N$$

Finally, it follows from the discrete Sobolev inequality [18] that:

$$\|e^n\|_\infty\le O(\tau^2+h^4),\quad n=1,2,\cdots,N$$

Theorem 3 Suppose that $u_0\in H^2$. If *h* and *τ* are sufficiently small, then the solutions of difference scheme (5)-(7) satisfy $\|U^n\|_\infty\le\tilde{C}_0,\quad n=1,2,\cdots,N,$ where $\tilde{C}_0$ is also independent of *τ* and *h*.

Proof For sufficiently small h and τ, by *Theorem 2*, we get:

$$\left\|U^n\right\|_\infty \le \left\|u^n\right\|_\infty + \left\|e^n\right\|_\infty \le \tilde{C}_0$$

Remark 1. Theorem 3 shows that the numerical solution of the difference scheme (5)-(7) is unconditionally stable.

Numerical experiments

In the following experiments, the initial function of problem (1)-(3) can be set in the following form [15]: $u(x,0) = \sec h^2(x/4)$.

We take x_L = 20, x_R = 40, and T = 10. Since the exact solutions of the BBM eqs. (1)-(3) is unknown, we set the numerical solutions on the mesh $\tau = h = 1/160$ as the reference solution. For comparison in this paper, the two-layers linear scheme (5)-(7) is named as Scheme 1, and the two-level non-linear scheme in [15] is named as Scheme 2, the three layer linear scheme in [16] as Scheme 3. For different values of τ and h, l_∞ error at several different times are shown in tab. 1. Finally, the conserved quantity (4) of numerical solutions is shown in tab. 2.

Table 1. The l_∞ error comparison of the three schemes at several different times

	τ = 0.4, h = 0.2			τ = 0.1, h = 0.1			τ = 0.025, h = 0.05		
	Scheme1	Scheme 2	Scheme 3	Scheme 1	Scheme 2	Scheme 3	Scheme 1	Scheme 2	Scheme 3
t = 2	$2.2303\cdot10^{-3}$	$2.5132\cdot10^{-3}$	$2.7454\cdot10^{-3}$	$1.3951\cdot10^{-4}$	$1.5724\cdot10^{-4}$	$1.9144\cdot10^{-4}$	$8.2092\cdot10^{-6}$	$9.2532\cdot10^{-6}$	$1.1577\cdot10^{-5}$
t = 4	$2.7610\cdot10^{-3}$	$2.9814\cdot10^{-3}$	$3.6096\cdot10^{-3}$	$1.7223\cdot10^{-4}$	$1.8561\cdot10^{-4}$	$2.4630\cdot10^{-4}$	$1.0132\cdot10^{-5}$	$1.0919\cdot10^{-5}$	$1.4812\cdot10^{-5}$
t = 6	$2.7384\cdot10^{-3}$	$2.8743\cdot10^{-3}$	$3.7131\cdot10^{-3}$	$1.7042\cdot10^{-4}$	$1.7857\cdot10^{-4}$	$2.5183\cdot10^{-4}$	$1.0025\cdot10^{-5}$	$1.0506\cdot10^{-5}$	$1.5127\cdot10^{-5}$
t = 8	$2.5592\cdot10^{-3}$	$2.6313\cdot10^{-3}$	$3.5723\cdot10^{-3}$	$1.5900\cdot10^{-4}$	$1.6345\cdot10^{-4}$	$2.4195\cdot10^{-4}$	$9.3522\cdot10^{-6}$	$9.6135\cdot10^{-6}$	$1.4531\cdot10^{-5}$
t = 10	$2.3464\cdot10^{-3}$	$2.6313\cdot10^{-3}$	$3.3579\cdot10^{-3}$	$1.4566\cdot10^{-4}$	$1.4758\cdot10^{-4}$	$2.2774\cdot10^{-4}$	$8.5670\cdot10^{-6}$	$8.6801\cdot10^{-6}$	$1.3683\cdot10^{-5}$

Table 2. Numerical simulations of the conservation invariant (4)

	τ = 0.4, h = 0.2	τ = 0.1, h = 0.1	τ = 0.025, h = 0.05
t = 2	7.999477502867281	7.999450190471835	7.999443302979656
t = 4	7.999468843609141	7.999449092562881	7.999442201778074
t = 6	7.999415162464388	7.999440961334592	7.999434116090026
t = 8	7.999135825831214	7.999390383543243	7.999383814257381
t = 10	7.999135825831214	7.999124287260258	7.999118965471003

It can be seen from the numerical experiments that Scheme 1 has the theoretical accuracy of the second order in time and the fourth order in space, and is obviously better than the two-level non-linear Scheme 2 and the three-level linear Scheme 3. Table 2 also shows the reasonably simulations of the conserved quantity (4).

Conclusion

A novel numerical scheme for the initial-boundary value problem of BBM equation with a homogeneous boundary is considered. A two-level linearized difference scheme is proposed with theoretical accuracy $O(\tau^2 + h^4)$. Also, the conservation property of the problem is verified. Therefore, the two-level linear difference scheme proposed in this paper for the initial boundary value problem (1)-(3) is more effective.

Acknowledgment

This work is partially supported by the Applied Basic Research Project of Sichuan Province (No. 2019YJ0387) and the National Natural Science Foundation of China (No. 11701481).

Nomenclature

t – time, [s]
u – velocity, [ms^{-1}]
x – co-ordinates, [m]

References

[1] Benjamin, T. B., *et al*., Model Equations For Long Waves Non-Linear Dispersive System, *Philosophical Transactions of the Royal Society London*, *272* (1972), 1220, pp. 47-78
[2] Mei, M., Large-Time Behavior of Solution for Generalized Benjamin-Bona-Mahony-Burgers Equation, *Non-linear Analysis: Theory Methods & Applications*, *33* (1998), 7, pp. 699-714
[3] Mei, M., Decay Rates of Solutions for Generalized Benjamin-Bona-Mahony-Burgers Equation, *Journal of Differential Equations*, *158* (1999), 2, pp. 314-340
[4] Amick, C. J., *et al*., Decay of Solution of Some Non-linear Wave Equation, *Journal of Differential Equations*, *81* (1989), 1, pp. 1-49
[5] Bona, J. L., Dougalis, V. A., An iNitial and Boundary Value Problem for a Model Equation for Propagation Long Waves, *Journal of Mathematical Analysis & Applications*, *75* (1980), 2, pp. 503-522
[6] Wang, B. X., Attractors and Approximate Inertial Manifolds for the Generalized Benjamin-Bona- Mahony Equation, *Mathematical Methods in the Applied Sciences*, *20* (1997), 3, pp. 189-203
[7] Achouri, T., *et al*., On the Convergence of Difference Schemes for the Benjamin- Bona-Mahanoy (BBM) Equation, *Applied Mathematics and Computation*, *182* (2006), 2, pp. 999-1005
[8] Omrani, K., The Convergence of Fully Discrete Galerkin Approximations for the Benjamin- Bona-Mahony (BBM) Equation, *Applied Mathematics and Computation*, *180* (2006), 2, pp. 614-621
[9] Hu, J. S., Wang, Y. L., Quasi-Compact Difference Algorithm for Benjamin-Bona-Mahony Equations, (in Chinese), *Journal of Southwest Normal University (Natural Science Edition)*, (2010), 2, pp. 35-64
[10] Zhang, Y., *et al.*, Average Implicit Difference Schemes for Benjamin-Bona-Mahony Equations, (in Chinese), *Journal of Sichuan University (Natural Science), 56* (2012), 3, pp. 955-959
[11] Che, H., *et al.*, Numerical Analysis of A Linear-Implicit Average Scheme for Generalized Benjamin-Bona-Mahony-Burgers Equation, *Journal of Applied Mathematics, 2012* (2012), ID 308410
[12] Xu, Y., *et al.*, Mixed Finite Element Analysis for Dissipative SRLW Equations with Damping Term, *Applied Mathematics, 38* (2015), 4, pp. 597-610
[13] Yu, X., *et al.*, Numerical Method for the Time Optimal Control Problem Governed by the Benjamin-Bona-Mahony Equation, *International Journal of Computational Science & Engineering*, *13* (2016), 3, pp. 296-302
[14] Li, C., Linearized Difference Schemes for a BBM Equation with a Fractional Nonlocal Viscous Term, *Applied Mathematics and Computation*, *311* (2017), Oct., pp. 240-250
[15] Huang, J. T., *et al.*, A High-Precision Non-Linear CN Difference Scheme for Solving BBM Equations, (in Chinese), *Journal of Sichuan University (Natural Science), 56* (2019), 3, pp. 387-391
[16] Zhang, H., *et al.*, A High-Precision Linear Difference Scheme for the Benjamin-Bona-Mahony Equation, (in Chinese), *Journal of Sichuan University (Natural Science), 56* (2019), 5, pp. 813-818
[17] Zhou, Y., *Application of Discrete Functional Analysis to the Finite Difference Method*, International Academic Publishers, Beijing, China, 1990
[18] Zhou, Y. L., *Application of Discrete Functional Analysis to the Finite Difference Methods,* International Academic Publishers, Beijing, 1990

Paper submitted: August 11, 2021
Paper revised: August 27, 2021
Paper accepted: September 1, 2021