ANOMALOUS DIFFUSION MODELS WITH RESPECT TO MONOTONE INCREASING FUNCTIONS

by

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In this article we propose the anomalous diffusion models with respect to monotone increasing functions. The Riesz-type fractional order derivatives operators with respect to power-law function are considered based on the extended work of Riesz. Two models for the anomalous diffusion processes are given to describe the special behaviors in the complex media.

Key words: *anomalous diffusion, Riesz fractional derivative, Riesz-type fractional derivative with respect to monotone increasing function, Riesz-type fractional derivative with respect to power-law function*

Introduction

Fick's laws of diffusion, due to Fick [1, 2], has been used to describe the law governing the transport of mass by using the diffusive means. However, there exist a number of frameworks of anomalous diffusion models studied by many scientists, Richardson [3], Bouchaud and Georges [4], Alcazar-Cano and Delgado-Buscalioni [5], Metzler *et al.* [6], Knackstedt *et al.* [7], *etc.*

Fractional diffusion, as one of interesting anomalous diffusion models, has researched by many researchers. For example, Jeon studied the anomalous diffusion applied to describe the complex behaviors of the phospholipids and cholesterols in the lipid bilayer [8]. The anomalous diffusion was proposed in the sense of the local fractional calculus [9]. The anomalous diffusion was suggested with the aid of the fractal derivative [10].

As is well known, the Riesz fractional derivative (RFD) in the 1-D case, proposed by Riesz [11], is connected with the Marchaud fractional derivative (MFD) [12, 13] and Liouville-Weyl fractional derivative (LWFD) [12, 13]. The RFD with respect to monotone increasing functions (MIF) have been proposed based on the LWFD with respect to MIF [12, 14]. The MFD with respect to power-law function (PLF) was proposed in [14]. The main target of the present paper is to propose the Riesz type fractional derivative (RTFD) with respect to PLF with the aid of the MFD with respect to PLF, and to suggest the anomalous diffusion models with respect to MIF.

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The Riesz fractional derivate

The left-sided MFD is defined as [12]:

$$ {}_{M}\mathrm{D}_{+}^{\alpha}\mathrm{H}(t)=\frac{\alpha}{\Gamma(1-\alpha)}\int_{-\infty}^{t}\frac{\mathrm{H}(t)-\mathrm{H}(\tau)}{(t-\tau)^{\alpha+1}}\mathrm{d}\tau \tag{1} $$

and the right-sided MFD as [12]:

$$ {}_{M}\mathrm{D}_{-}^{\alpha}\mathrm{H}(t)=\frac{\alpha}{\Gamma(1-\alpha)}\int_{t}^{\infty}\frac{\mathrm{H}(t)-\mathrm{H}(\tau)}{(\tau-t)^{\alpha+1}}\mathrm{d}\tau \tag{2} $$

It has been proved by Samko [12, 13] that:

$$ {}_{M}\mathrm{D}_{+}^{\alpha}\mathrm{H}(t)=\frac{\alpha}{\Gamma(1-\alpha)}\int_{0}^{\infty}\frac{\mathrm{H}(t)-\mathrm{H}(t-\tau)}{\tau^{\alpha+1}}\mathrm{d}\tau=\frac{\alpha}{\Gamma(1-\alpha)}\int_{-\infty}^{t}\frac{\mathrm{H}(t)-\mathrm{H}(\tau)}{(t-\tau)^{\alpha+1}}\mathrm{d}\tau \tag{3} $$

and

$$ {}_{M}\mathrm{D}_{-}^{\alpha}\mathrm{H}(t)=\frac{\alpha}{\Gamma(1-\alpha)}\int_{0}^{\infty}\frac{\mathrm{H}(t)-\mathrm{H}(t+\tau)}{\tau^{\alpha+1}}\mathrm{d}\tau=\frac{\alpha}{\Gamma(1-\alpha)}\int_{t}^{\infty}\frac{\mathrm{H}(t)-\mathrm{H}(\tau)}{(\tau-t)^{\alpha+1}}\mathrm{d}\tau \tag{4} $$

Let:

$$ c_1=-\left[2\cos\left(\frac{\pi\alpha}{2}\right)\right]^{-1} \tag{5} $$

The RFD of the function H(t) is defined [12, 15]:

$$ (-\Delta)^{\frac{\alpha}{2}}\mathrm{H}(t)=c_1[{}_{M}\mathrm{D}_{+}^{\alpha}\mathrm{H}(t)+{}_{M}\mathrm{D}_{-}^{\alpha}\mathrm{H}(t)] \tag{6} $$

It has been pointed by Richard [12, 13, 15] that for $0<\alpha<2$ [12, 15]:

$$ \begin{aligned} {}_{L}D_{+}^{\alpha}\mathrm{H}(t)&={}_{L}I_{+}^{-\alpha}\mathrm{H}(t)=\frac{\mathrm{d}}{\mathrm{d}t}{}_{L}I_{+}^{1-\alpha}\mathrm{H}(t)=\\ &=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{t}\frac{\mathrm{H}(\tau)}{(t-\tau)^{\alpha}}\mathrm{d}\tau=\\ &=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{0}^{\infty}\frac{\mathrm{H}(t-\tau)}{\tau^{\alpha}}\mathrm{d}\tau=\\ &=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{0}^{\infty}\frac{1}{\tau^{\alpha}}\left[-\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{H}(t-\tau)\right]\mathrm{d}\tau=\\ &=\frac{\alpha}{\Gamma(1-\alpha)}\int_{0}^{\infty}\frac{\mathrm{H}(t)-\mathrm{H}(t-\tau)}{\tau^{\alpha+1}}\mathrm{d}\tau \end{aligned} \tag{7} $$

where the left-sided LWFD is [12, 15]:

$$ {}_{L}\mathrm{D}_{+}^{\alpha}\mathrm{H}(t)={}_{L}I_{+}^{-\alpha}\mathrm{H}(t)=\frac{\mathrm{d}}{\mathrm{d}t}{}_{L}I_{+}^{1-\alpha}\mathrm{H}(t)=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{t}\frac{\mathrm{H}(\tau)}{(t-\tau)^{\alpha}}\mathrm{d}\tau \tag{8} $$

and [12, 15]:

$$ {}_{L}\mathrm{D}_{-}^{\alpha}\mathrm{H}(t) = {}_{L}I_{-}^{-\alpha}\mathrm{H}(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\,{}_{L}I_{-}^{1-\alpha}\mathrm{H}(t) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{t}^{\infty}\frac{\mathrm{H}(\tau)}{(\tau-t)^{\alpha}}\mathrm{d}\tau = $$

$$ = -\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{0}^{\infty}\frac{\mathrm{H}(t+\tau)}{\tau^{\alpha}}\mathrm{d}\tau = -\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{0}^{\infty}\frac{1}{\tau^{\alpha}}\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{H}(t-\tau)\right]\mathrm{d}\tau = $$

$$ = \frac{\alpha}{\Gamma(1-\alpha)}\int_{0}^{\infty}\frac{\Phi(t)-\mathrm{H}(t+\tau)}{\tau^{\alpha+1}}\mathrm{d}\tau \tag{9} $$

and the right-sided LWFD is [12, 15]:

$$ {}_{L}\mathrm{D}_{-}^{\alpha}\mathrm{H}(t) = {}_{L}I_{-}^{-\alpha}\mathrm{H}(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\,{}_{L}I_{-}^{1-\alpha}\mathrm{H}(t) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{t}^{\infty}\frac{\mathrm{H}(\tau)}{(\tau-t)^{\alpha}}\mathrm{d}\tau \tag{10} $$

For $0<\alpha<2$, by using eqs. (3), (4), (7), and (8), we have [12, 15]:

$$ {}_{M}\mathrm{D}_{+}^{\alpha}\mathrm{H}(t) = {}_{L}\mathrm{D}_{+}^{\alpha}\mathrm{H}(t) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{t}\frac{\mathrm{H}(\tau)}{(t-\tau)^{\alpha}}\mathrm{d}\tau \tag{11} $$

and

$$ {}_{M}\mathrm{D}_{-}^{\alpha}\mathrm{H}(t) = {}_{L}\mathrm{D}_{-}^{\alpha}\mathrm{H}(t) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{t}^{\infty}\frac{\mathrm{H}(\tau)}{(\tau-t)^{\alpha}}\mathrm{d}\tau \tag{12} $$

such that (6) can be rewritten as [12, 15]:

$$ (-\Delta)^{\frac{\alpha}{2}}\mathrm{H}(t) = c_1[{}_{M}\mathrm{D}_{+}^{\alpha}\mathrm{H}(t) + {}_{M}\mathrm{D}_{-}^{\alpha}\mathrm{H}(t)] = $$

$$ = c_1[{}_{L}\mathrm{D}_{+}^{\alpha}\mathrm{H}(t) + {}_{L}\mathrm{D}_{-}^{\alpha}\mathrm{H}(t)] = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}\frac{\mathrm{H}(\tau)}{|\tau-t|^{\alpha}}\mathrm{d}\tau \tag{13} $$

The Riesz fractional integral of the function H(*t*) is obtained when $\alpha<0$ [12, 15].

The RTFD with respect to MIF

Let $\mathrm{H}_{\theta}(t) = (\mathrm{H}\circ\theta)(t) = \mathrm{H}[\theta(t)]$, where $\theta^{(1)}(t)>0$ for $t\in\mathbb{R}$ and let $\lim_{t\to\pm\infty}\theta(t)=\pm\infty$ and $\lim_{t\to 0}\theta(t)=0$.

Let us recall the Marchaud-type fractional derivative (MTFD) with respect to MIF defined by author and coauthors [14].

The left-sided MTFD with respect to MIF $\theta(t)$ is defined [14]:

$$ {}_{M}\mathrm{D}_{+,\theta}^{\alpha}\mathrm{H}_{\theta}(t) = \frac{\alpha}{\Gamma(1-\alpha)}\int_{-\infty}^{t}\frac{\mathrm{H}_{\theta}(t)-\mathrm{H}_{\theta}(\tau)}{[\theta(t)-\theta(\tau)]^{\alpha+1}}\theta^{(1)}(\tau)\mathrm{d}\tau \tag{14} $$

and the right-sided MTFD with respect to MIF $\theta(t)$ [14]:

$$ {}_{M}\mathrm{D}_{-,\theta}^{\alpha}\mathrm{H}_{\theta}(t)=\frac{\alpha}{\Gamma(1-\alpha)}\int_{t}^{\infty}\frac{\mathrm{H}_{\theta}(t)-\mathrm{H}_{\theta}(\tau)}{[\theta(\tau)-\theta(t)]^{\alpha+1}}\theta^{(1)}(\tau)\mathrm{d}\tau \tag{15}$$

For $0<\alpha<2$, it is not difficult to find that [12, 14]:

$$ {}_{L}\mathrm{D}_{+,\theta}^{\alpha}\mathrm{H}_{\theta}(t)={}_{L}I_{+,\theta}^{-\alpha}\mathrm{H}_{\theta}(t)=\left[\frac{1}{\theta^{(1)}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right]{}_{L}I_{+,\theta}^{1-\alpha}\mathrm{H}_{\theta}(t)= $$

$$ =\frac{1}{\Gamma(1-\alpha)}\left[\frac{1}{\theta^{(1)}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right]\int_{-\infty}^{t}\frac{\mathrm{H}_{\theta}(\tau)}{[\theta(t)-\theta(\tau)]^{\alpha}}\theta^{(1)}(\tau)\mathrm{d}\tau= $$

$$ =\frac{\alpha}{\Gamma(1-\alpha)}\int_{0}^{\infty}\frac{\mathrm{H}_{\theta}(t)-\mathrm{H}[\theta(t)-\theta(\tau)]}{[\theta(\tau)]^{\alpha+1}}\theta^{(1)}(\tau)\mathrm{d}\tau \tag{16}$$

where the left-sided Kilbas-Srivastava-Trujillo fractional derivative (KSTFD) is [12, 14]:

$$ {}_{L}\mathrm{D}_{+,\theta}^{\alpha}\mathrm{H}_{\theta}(t)={}_{L}I_{+,\theta}^{-\alpha}\mathrm{H}_{\theta}(t)=\frac{1}{\Gamma(1-\alpha)}\left[\frac{1}{\theta^{(1)}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right]\int_{-\infty}^{t}\frac{\mathrm{H}_{\theta}(\tau)}{[\theta(t)-\theta(\tau)]^{\alpha}}\theta^{(1)}(\tau)\mathrm{d}\tau \tag{17}$$

and [12, 14]:

$$ {}_{L}\mathrm{D}_{-,\theta}^{\alpha}\mathrm{H}_{\theta}(t)={}_{L}I_{-,\theta}^{-\alpha}\mathrm{H}_{\theta}(t)=\left[-\frac{1}{\theta^{(1)}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right]{}_{L}I_{-,\theta}^{1-\alpha}\mathrm{H}_{\theta}(t)= $$

$$ =\frac{1}{\Gamma(1-\alpha)}\left[-\frac{1}{\theta^{(1)}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right]\int_{t}^{\infty}\frac{\mathrm{H}_{\theta}(\tau)}{[\theta(\tau)-\theta(t)]^{\alpha}}\theta^{(1)}(\tau)\mathrm{d}\tau= $$

$$ =\frac{\alpha}{\Gamma(1-\alpha)}\int_{0}^{\infty}\frac{\mathrm{H}_{\theta}(t)-\mathrm{H}[\theta(t)+\theta(\tau)]}{[\theta(\tau)]^{\alpha+1}}\theta^{(1)}(\tau)\mathrm{d}\tau \tag{18}$$

and the right-sided KSTFD is [12, 14]:

$$ {}_{L}\mathrm{D}_{-,\theta}^{\alpha}\mathrm{H}_{\theta}(t)={}_{L}I_{-,\theta}^{-\alpha}\mathrm{H}_{\theta}(t)=\frac{1}{\Gamma(1-\alpha)}\left[\frac{1}{\theta^{(1)}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right]\int_{t}^{\infty}\frac{\mathrm{H}_{\theta}(\tau)}{[\theta(\tau)-\theta(t)]^{\alpha}}\theta^{(1)}(\tau)\mathrm{d}\tau \tag{19}$$

There exist [12, 14]:

$$ {}_{M}\mathrm{D}_{+,\theta}^{\alpha}\mathrm{H}_{\theta}(t)=\frac{\alpha}{\Gamma(1-\alpha)}\int_{0}^{\infty}\frac{\mathrm{H}_{\theta}(t)-\mathrm{H}[\theta(t)-\theta(\tau)]}{[\theta(\tau)]^{\alpha+1}}\theta^{(1)}(\tau)\mathrm{d}\tau= $$

$$ =\frac{\alpha}{\Gamma(1-\alpha)}\int_{-\infty}^{t}\frac{\mathrm{H}_{\theta}(t)-\mathrm{H}_{\theta}(\tau)}{[\theta(t)-\theta(\tau)]^{\alpha+1}}\theta^{(1)}(\tau)\mathrm{d}\tau \tag{20}$$

and

$$
{}_{M}\mathrm{D}_{-,\theta}^{\alpha}\mathrm{H}_{\theta}(t)=\frac{\alpha}{\Gamma(1-\alpha)}\int_{0}^{\infty}\frac{\mathrm{H}_{\theta}(t)-\mathrm{H}[\theta(t)+\theta(\tau)]}{[\theta(\tau)]^{\alpha+1}}\theta^{(1)}(\tau)\mathrm{d}\tau=
$$

$$
=\frac{\alpha}{\Gamma(1-\alpha)}\int_{-\infty}^{t}\frac{\mathrm{H}_{\theta}(t)-\mathrm{H}_{\theta}(\tau)}{[\theta(\tau)-\theta(t)]^{\alpha+1}}\theta^{(1)}(\tau)\mathrm{d}\tau \tag{21}
$$

For $0<\alpha<2$, it is easy to see that [12, 14]:

$$
{}_{M}\mathrm{D}_{+,\theta}^{\alpha}\mathrm{H}_{\theta}(t)={}_{L}\mathrm{D}_{+,\theta}^{\alpha}\mathrm{H}_{\theta}(t)={}_{L}I_{+,\theta}^{-\alpha}\mathrm{H}_{\theta}(t)=\left[\frac{1}{\theta^{(1)}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right]{}_{L}I_{+,\theta}^{1-\alpha}\mathrm{H}_{\theta}(t)=
$$

$$
=\frac{1}{\Gamma(1-\alpha)}\left[\frac{1}{\theta^{(1)}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right]\int_{-\infty}^{t}\frac{\mathrm{H}_{\theta}(\tau)}{[\theta(t)-\theta(\tau)]^{\alpha}}\theta^{(1)}(\tau)\mathrm{d}\tau \tag{22}
$$

and [12, 14]:

$$
{}_{M}\mathrm{D}_{-,\theta}^{\alpha}\mathrm{H}_{\theta}(t)={}_{L}\mathrm{D}_{-,\theta}^{\alpha}\mathrm{H}_{\theta}(t)={}_{L}I_{-,\theta}^{-\alpha}\mathrm{H}_{\theta}(t)=\left[-\frac{1}{\theta^{(1)}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right]{}_{L}I_{-,\theta}^{1-\alpha}\mathrm{H}_{\theta}(t)=
$$

$$
=\frac{1}{\Gamma(1-\alpha)}\left[-\frac{1}{\theta^{(1)}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right]\int_{t}^{\infty}\frac{\mathrm{H}_{\theta}(\tau)}{[\theta(\tau)-\theta(t)]^{\alpha}}\theta^{(1)}(\tau)\mathrm{d}\tau \tag{23}
$$

The RTFD with respect to MIF $\theta(t)$ is defined [12, 14]:

$$
(-\Delta)_{\theta}^{\frac{\alpha}{2}}\mathrm{H}_{\theta}(t)=c_1[{}_{L}\mathrm{D}_{+,\theta}^{\alpha}\mathrm{H}_{\theta}(t)+{}_{L}\mathrm{D}_{-,\theta}^{\alpha}\mathrm{H}_{\theta}(t)]=c_1[{}_{M}\mathrm{D}_{+,\theta}^{\alpha}\mathrm{H}_{\theta}(t)+{}_{M}\mathrm{D}_{-,\theta}^{\alpha}\mathrm{H}_{\theta}(t)] \tag{24}
$$

Thus, by eqs. (22)-(24), we obtain:

$$
(-\Delta)_{\theta}^{\frac{\alpha}{2}}\mathrm{H}_{\theta}(t)=\frac{c_1}{\Gamma(1-\alpha)}\left[\frac{1}{\theta^{(1)}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right]\int_{-\infty}^{t}\frac{\mathrm{H}_{\theta}(\tau)}{[\theta(t)-\theta(\tau)]^{\alpha}}\theta^{(1)}(\tau)\mathrm{d}\tau+
$$

$$
+\frac{1}{\Gamma(1-\alpha)}\left[-\frac{1}{\theta^{(1)}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right]\int_{t}^{\infty}\frac{\mathrm{H}_{\theta}(\tau)}{[\theta(\tau)-\theta(t)]^{\alpha}}\theta^{(1)}(\tau)\mathrm{d}\tau=
$$

$$
=\frac{c_1}{\Gamma(1-\alpha)}\left[\frac{1}{\theta^{(1)}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right]\int_{-\infty}^{\infty}\frac{\mathrm{H}_{\theta}(\tau)}{|\theta(t)-\theta(\tau)|^{\alpha}}\theta^{(1)}(\tau)\mathrm{d}\tau
$$

It is noted that:

$$
(-\Delta)_{\theta}^{\frac{\alpha}{2}}\mathrm{H}_{\theta}(t)=c_1[{}_{L}\mathrm{D}_{+,\theta}^{\alpha}\mathrm{H}_{\theta}(t)+{}_{L}\mathrm{D}_{-,\theta}^{\alpha}\mathrm{H}_{\theta}(t)] \tag{25}
$$

was proposed in [12, 14], and:

$$
(-\Delta)_{\theta}^{\frac{\alpha}{2}}\mathrm{H}_{\theta}(t)=c_1[{}_{M}\mathrm{D}_{+,\theta}^{\alpha}\mathrm{H}_{\theta}(t)+{}_{M}\mathrm{D}_{-,\theta}^{\alpha}\mathrm{H}_{\theta}(t)] \tag{26}
$$

was suggested in [16].

The Riesz-type fractional integral (RTFI) with respect to MIF $\theta(t)$ is obtained when $\alpha<0$ [12, 14].

The RTFD with respect to PLF

The left-sided MTFD with respect to PLF is defined as [14]:

$$ {}_{M}\mathrm{D}_{+}^{\alpha}\mathrm{H}(t^{\beta}) = \frac{\alpha\beta}{\Gamma(1-\alpha)}\int_{-\infty}^{t}\frac{\mathrm{H}(t^{\beta})-\mathrm{H}(\tau^{\beta})}{(t^{\beta}-\tau^{\beta})^{\alpha+1}}\tau^{\beta-1}\mathrm{d}\tau \tag{27} $$

and the right-sided MTFD with respect to PLF as [14]:

$$ {}_{M}\mathrm{D}_{-}^{\alpha}\mathrm{H}(t^{\beta}) = \frac{\alpha\beta}{\Gamma(1-\alpha)}\int_{-\infty}^{t}\frac{\mathrm{H}(t^{\beta})-\mathrm{H}(\tau^{\beta})}{(\tau^{\beta}-t^{\beta})^{\alpha+1}}\tau^{\beta-1}\mathrm{d}\tau \tag{28} $$

It is seen that:

$$ {}_{M}\mathrm{D}_{+}^{\alpha}\mathrm{H}(t^{\beta}) = {}_{L}\mathrm{D}_{+}^{\alpha}\mathrm{H}(t^{\beta}) = {}_{L}\tilde{I}_{+}^{-\alpha}\mathrm{H}(t^{\beta}) = \left(\frac{t^{1-\beta}}{\beta}\frac{\mathrm{d}}{\mathrm{d}t}\right){}_{L}I_{+}^{1-\alpha}\mathrm{H}(t^{\beta}) = $$
$$ = \frac{\beta}{\Gamma(1-\alpha)}\left(\frac{t^{1-\beta}}{\beta}\frac{\mathrm{d}}{\mathrm{d}t}\right)\int_{-\infty}^{t}\frac{\tau^{\beta-1}}{(t^{\beta}-\tau^{\beta})^{\alpha}}\mathrm{H}(\tau^{\beta})\mathrm{d}\tau = $$
$$ = \frac{1}{\Gamma(1-\alpha)}\left(t^{1-\beta}\frac{\mathrm{d}}{\mathrm{d}t}\right)\int_{-\infty}^{t}\frac{\tau^{\beta-1}}{(t^{\beta}-\tau^{\beta})^{\alpha}}\mathrm{H}(\tau^{\beta})\mathrm{d}\tau \tag{29} $$

and

$$ {}_{M}\mathrm{D}_{-}^{\alpha}\mathrm{H}(t^{\beta}) = {}_{L}\mathrm{D}_{-}^{\alpha}\mathrm{H}(t^{\beta}) = {}_{L}\tilde{I}_{-}^{-\alpha}\mathrm{H}(t^{\beta}) = \left(-\frac{t^{1-\beta}}{\beta}\frac{\mathrm{d}}{\mathrm{d}t}\right){}_{L}I_{-}^{1-\alpha}\mathrm{H}(t^{\beta}) = $$
$$ = \frac{\beta}{\Gamma(1-\alpha)}\left(-\frac{t^{1-\beta}}{\beta}\frac{\mathrm{d}}{\mathrm{d}t}\right)\int_{-\infty}^{t}\frac{\tau^{\beta-1}}{(\tau^{\beta}-t^{\beta})^{\alpha}}\mathrm{H}(\tau^{\beta})\mathrm{d}\tau = $$
$$ = \frac{1}{\Gamma(1-\alpha)}\left(-t^{1-\beta}\frac{\mathrm{d}}{\mathrm{d}t}\right)\int_{-\infty}^{t}\frac{\tau^{\beta-1}}{(\tau^{\beta}-t^{\beta})^{\alpha}}\mathrm{H}(\tau^{\beta})\mathrm{d}\tau \tag{30} $$

The RTFD with respect to PLF is defined:

$$ (-\Delta)^{\frac{\alpha}{2}}\Phi(t^{\beta}) = c_1[{}_{L}\mathrm{D}_{+}^{\alpha}\Phi(t^{\beta}) + {}_{L}\mathrm{D}_{-}^{\alpha}\Phi(t^{\beta})] = c_1[{}_{M}\mathrm{D}_{+}^{\alpha}\Phi(t^{\beta}) + {}_{M}\mathrm{D}_{-}^{\alpha}\Phi(t^{\beta})] = $$
$$ = \frac{1}{\Gamma(1-\alpha)}\left(t^{1-\beta}\frac{\mathrm{d}}{\mathrm{d}t}\right)\int_{-\infty}^{\infty}\frac{\tau^{\beta-1}}{\left|t^{\beta}-\tau^{\beta}\right|^{\alpha}}\Phi(\tau^{\beta})\mathrm{d}\tau \tag{31} $$

The RTFI with respect to PLF is defined:

$$ (-\Delta)^{\frac{\alpha}{2}}\mathrm{H}(t^{\beta}) = -c_1[{}_{L}\tilde{I}_{+}^{\alpha}\mathrm{H}(t^{\beta}) + {}_{L}\tilde{I}_{-}^{\alpha}\mathrm{H}(t^{\beta})] = -c_1[{}_{M}\tilde{I}_{+}^{\alpha}\Phi(t^{\beta}) + {}_{M}\tilde{I}_{-}^{\alpha}\mathrm{H}(t^{\beta})] = $$
$$ = -\frac{\beta c_1}{\Gamma(\alpha)}\int_{-\infty}^{\infty}\frac{\tau^{\beta-1}}{\left|t^{\beta}-\tau^{\beta}\right|^{\alpha}}\mathrm{H}(\tau^{\beta})\mathrm{d}\tau = \frac{\beta}{2\Gamma(\alpha)\cos\left(\frac{\pi\alpha}{2}\right)}\int_{-\infty}^{\infty}\frac{\tau^{\beta-1}}{\left|t^{\beta}-\tau^{\beta}\right|^{\alpha}}\mathrm{H}(\tau^{\beta})\mathrm{d}\tau \tag{32} $$

where

$$ {}_{M}\tilde{I}_{+}^{\alpha}\mathrm{H}(t^{\beta}) = {}_{L}\tilde{I}_{+}^{\alpha}\mathrm{H}(t^{\beta}) = \frac{\beta}{\Gamma(\alpha)}\int_{-\infty}^{t}\frac{\tau^{\beta-1}}{(t^{\beta}-\tau^{\beta})^{\alpha-1}}\mathrm{H}(\tau^{\beta})\mathrm{d}\tau \tag{33} $$

and

$$ {}_{M}\tilde{I}_{-}^{\alpha}\mathrm{H}(t^{\beta}) = {}_{L}\tilde{I}_{-}^{\alpha}\mathrm{H}(t^{\beta}) = \frac{\beta}{\Gamma(\alpha)}\int_{t}^{\infty}\frac{\tau^{\beta-1}}{(\tau^{\beta}-t^{\beta})^{\alpha-1}}\mathrm{H}(\tau^{\beta})\mathrm{d}\tau \tag{34} $$

Anomalous diffusion models

Example 1. We now consider the anomalous diffusion models within the RTFD with respect to the MIF:

$$ (-\Delta)_{\theta}^{\frac{\alpha}{2}}\mathrm{H}_{\theta}(x,t) = \frac{\partial^{2}\mathrm{H}_{\theta}(x,t)}{\partial x^{2}} \tag{35} $$

subject to the initial and boundary conditions

$$ \mathrm{H}_{\theta}(x,0) = q(x) \tag{36} $$

and

$$ \mathrm{H}_{\theta}(0,t) = p(t) \tag{37} $$

where the Riesz-type fractional partial derivative (RTFPD) with respect to the MIF is:

$$ (-\Delta)_{\theta}^{\frac{\alpha}{2}}\mathrm{H}_{\theta}(x,t) = \frac{c_{1}}{\Gamma(1-\alpha)}\left[\frac{1}{\theta^{(1)}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right]\int_{-\infty}^{\infty}\frac{\mathrm{H}_{\theta}(x,t)}{\left|\theta(t)-\theta(\tau)\right|^{\alpha}}\theta^{(1)}(\tau)\mathrm{d}\tau \tag{38} $$

Example 2. We now suggest the anomalous diffusion models within the RTFD with respect to the PLF:

$$ (-\Delta)^{\frac{\alpha}{2}}\mathrm{H}(x,t) = \frac{\partial^{2}\mathrm{H}(x,t)}{\partial x^{2}} \tag{39} $$

subject to the initial and boundary conditions:

$$ \mathrm{H}(x,0) = q(x) \tag{40} $$

and

$$ \mathrm{H}(0,t) = p(t) \tag{41} $$

where the RTFPD with respect to the PLF reads:

$$ (-\Delta)^{\frac{\alpha}{2}}\mathrm{H}(x,t) = \frac{1}{\Gamma(1-\alpha)}\left(t^{1-\beta}\frac{\mathrm{d}}{\mathrm{d}t}\right)\int_{-\infty}^{\infty}\frac{\tau^{\beta-1}}{\left|t^{\beta}-\tau^{\beta}\right|^{\alpha}}\mathrm{H}(x,\tau^{\beta})\mathrm{d}\tau \tag{42} $$

Conclusion

In the present work we suggested that the RTFD with respect to PLF with the aid of the MTFD with respect to PLF. The anomalous diffusion models with respect to MIF were obtained in detail. The new RTFD formula is proposed as a mathematical tool to model the power-law behaviors of the real world problems.

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Nomenclature

t – time, [s]

$H(x, t)$ – distribution function, [–]

x – co-ordinates, [m]

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