# VARIATIONAL PRINCIPLE FOR A GENERALIZED RABINOWITSCH LUBRICATION

by

***Hongjin MA*** *

School of Science, Xi'an University of Architecture and Technology, Xi'an, China

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*This paper adopts Rotem and Shinnar's modification of the Rabinowitsch fluid model for the 1-D non-Newtonian lubrication problem, a variational principle is established by the semi-inverse method, and a generalized Reynolds-type equation is obtained. This article opens a new avenue for the establishment of Reynolds-type equation of complex lubrication problems.*

Key words: *Lagrange functional, variational theory, Euler-Lagrange equation, Reynolds equation, He's semi-inverse method*

## Introduction

Non-Newtonian fluids appear everywhere in engineering applications, for examples, polymer solutions [1, 2], nanofluids [3, 4] , Walters' B fluid [5], Reiner-Rivlin fluid [6], phase change material [7], and peristaltic flow [8]. Zuo [9] suggested a fractal rheological model for non-Newtonian fluids, and Liang and Wang [10] gave a fractal viscoelastic element for various non-Newtonian fluids.

The Rabinowitsch fluid model is widely used in the non-Newtonian lubrication theory [11-18]. This paper considers a 1-D slide bearing as illustrated in fig. 1.

The Rabinowitsch 1929 model assumes the following constitutive relationship [19]:

$$\sigma_{xz} + a\sigma_{xz}^3 = \eta \frac{\partial u}{\partial z} \tag{1}$$

where $\sigma_{xz}$ is the shear stress, $u$ – the velocity, $\eta$ – the viscosity coefficient, and $a$ – the Rabinowitsch parameter for non-Newtonian property.

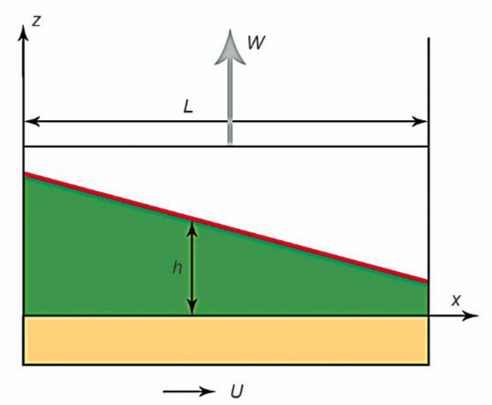


**Figure 1. Geometric structure of a sliding surface**

A more generalized modification of eq. (1) was suggested by Rotem and Shinnar in the form [20]:

$$\sigma_{xz} + \sum_{n=1}^{N} a_n \sigma_{xz}^{2n+1} = \eta \frac{\partial u}{\partial z} \tag{2}$$

* Author's, e-mail: hjma@xauat.edu.cn

where $a_n (n=1\sim N)$ are constants. In this paper we adopt the following one with a cubic-quinary non-linearity:

$$\sigma_{xz} + a_1\sigma_{xz}^3 + a_2\sigma_{xz}^5 = \eta\frac{\partial u}{\partial z} \tag{3}$$

where $a_1$ and $a_2$ are Rabinowitsch parameters.

### Variational formulation

The governing equations for 1-D non-Newtonian lubrication can be expressed [18]:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{4}$$

$$\frac{\partial p}{\partial x} = \frac{\partial \sigma_{xz}}{\partial z} \tag{5}$$

$$\frac{\partial p}{\partial z} = 0 \tag{6}$$

where $u$, $w$ are velocities in $x$- and $z$-directions, respectively, and $p$ – the pressure.

Equation (1) is for the mass conservation, while eqs. (5) and (6) are the moment conservation in $x$- and $z$-directions, respectively.

The boundary conditions are:

$$z = 0 : u = U, \quad w = 0 \tag{7}$$

$$z = h : u = 0, \quad w = W \tag{8}$$

In this section, we search for a variational formulation for the aforementioned lubrication problem. The variational principle is widely used in engineering to establish a needed differential model for practical problems. Various variational formulations were appeared in literature for various problems, for examples, water waves [21-24], non-linear vibration system [25, 26], Burgers equation [27], Benney-Lin equation [28], solitary waves [29-31]. In this paper, the semi-inverse method [32-34] is applied to establish a needed variational formulation.

According to the semi-inverse method [32-34], we construct the following trial-functional:

$$J(u, w, p, \sigma_{xz}) = \int_0^L\int_0^h \left[ u\frac{\partial p}{\partial x} + w\frac{\partial p}{\partial z} + F(u, w, \sigma_{xz}) \right] \mathrm{d}x\mathrm{d}z \tag{9}$$

where $F$ is unknown yet, it will be determined step by step. Alternative approaches to construction of the trial-functional are available in [35, 36].

The stationary condition with respect to $p$ is eq. (4), and we obtain the following two equations with respect to $u$ and $w$, respectively:

$$\frac{\partial p}{\partial x} + \frac{\delta F}{\delta u} = 0 \tag{10}$$

$$\frac{\partial p}{\partial z}+\frac{\delta F}{\delta w}=0 \tag{11}$$

where $\delta F/\delta u$ is the variational derivative defined:

$$\frac{\delta F}{\delta u}=\frac{\partial F}{\partial u}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)-\frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right) \tag{12}$$

In view of eqs. (5) and (6), we can convert eqs. (10) and (11) into the following forms:

$$\frac{\delta F}{\delta u}=-\frac{\partial p}{\partial x}=-\frac{\partial \sigma_{xz}}{\partial z} \tag{13}$$

$$\frac{\delta F}{\delta w}=-\frac{\partial p}{\partial z}=0 \tag{14}$$

We, therefore, can determine $F$:

$$F=-u\frac{\partial \sigma_{xz}}{\partial z}+f(\sigma_{xz}) \tag{15}$$

where $f$ is unknown yet. Now eq. (9) becomes:

$$J(u,w,p,\sigma_{xz})=\int_0^L\int_0^h\left[u\frac{\partial p}{\partial x}+w\frac{\partial p}{\partial z}-u\frac{\partial \sigma_{xz}}{\partial z}+f(\sigma_{xz})\right]\mathrm{d}x\mathrm{d}z \tag{16}$$

Now the stationary condition with respect to $\sigma_{xz}$ is:

$$\frac{\partial u}{\partial z}+\frac{\delta f}{\delta \sigma_{xz}}=0 \tag{17}$$

By eq. (3), we have:

$$\frac{\delta f}{\delta \sigma_{xz}}=-\frac{\partial u}{\partial z}=-\frac{1}{\eta}(\sigma_{xz}+a_1\sigma_{xz}^3+a_2\sigma_{xz}^5) \tag{18}$$

From eq. (18), we have:

$$f=-\frac{1}{\eta}\left(\frac{1}{2}\sigma_{xz}^2+\frac{1}{4}a_1\sigma_{xz}^4+\frac{1}{6}a_2\sigma_{xz}^6\right) \tag{19}$$

Equation (16) is updated:

$$J(u,w,p,\sigma_{xz})=\int_0^L\int_0^h\left[u\frac{\partial p}{\partial x}+w\frac{\partial p}{\partial z}-u\frac{\partial \sigma_{xz}}{\partial z}-\frac{1}{\eta}\left(\frac{1}{2}\sigma_{xz}^2+\frac{1}{4}a_1\sigma_{xz}^4+\frac{1}{6}a_2\sigma_{xz}^6\right)\right]\mathrm{d}x\mathrm{d}z \tag{20}$$

By a similar derivation as given in [18], the boundary integral can be incorporated. We assume that:

$$J(u,w,p,\sigma_{xz})=\int_0^L\int_0^h\left[u\frac{\partial p}{\partial x}+w\frac{\partial p}{\partial z}-u\frac{\partial \sigma_{xz}}{\partial z}-\frac{1}{\eta}\left(\frac{1}{2}\sigma_{xz}^2+\frac{1}{4}a_1\sigma_{xz}^4+\frac{1}{6}a_2\sigma_{xz}^6\right)\right]\mathrm{d}x\mathrm{d}z+$$

$$+\int_0^L g(p,\sigma_{xz})\mathrm{d}x \tag{21}$$

where $g$ is unknow. The natural conditions are:

$$u\big|_{x=0}^{x=h}+\frac{\delta g}{\delta p}=0 \tag{22}$$

$$-u\big|_{x=0}^{x=h}+\frac{\delta g}{\delta \sigma_{xz}}=0 \tag{23}$$

In view of the boundary conditions of eqs. (7) and (8), we have:

$$\frac{\delta g}{\delta p}=-u\big|_{x=0}^{x=h}=-[W(h)-W(0)] \tag{24}$$

$$\frac{\delta g}{\delta \sigma_{xz}}=u\big|_{x=0}^{x=h}=U \tag{25}$$

From eqs. (24) and (25), we obtain:

$$g=-[W(h)-W(0)]p+U\sigma_{xz} \tag{26}$$

We finally obtain the following generalized variational principle:

$$J(u,w,p,\sigma_{xz})=\int_0^L\int_0^h\left[u\frac{\partial p}{\partial x}+w\frac{\partial p}{\partial z}-u\frac{\partial \sigma_{xz}}{\partial z}-\frac{1}{\eta}\left(\frac{1}{2}\sigma_{xz}^2+\frac{1}{4}a_1\sigma_{xz}^4+\frac{1}{6}a_2\sigma_{xz}^6\right)\right]\mathrm{d}x\mathrm{d}z-$$

$$-\int_0^L[W(h)-W(0)]p\mathrm{d}x+\int_0^L U\sigma_{xz}\mathrm{d}x \tag{27}$$

*Proof.* The Lagrange function of eq. (27) is:

$$\lambda=u\frac{\partial p}{\partial x}+w\frac{\partial p}{\partial z}-u\frac{\partial \sigma_{xz}}{\partial z}-\frac{1}{\eta}\left(\frac{1}{2}\sigma_{xz}^2+\frac{1}{4}a_1\sigma_{xz}^4+\frac{1}{6}a_2\sigma_{xz}^6\right) \tag{28}$$

and the stationary conditions of eq. (27) are:

$$\frac{\partial \lambda}{\partial \xi}-\frac{\partial}{\partial x}\left(\frac{\partial \lambda}{\partial \xi_x}\right)-\frac{\partial}{\partial z}\left(\frac{\partial \lambda}{\partial \xi_z}\right)=0 \tag{29}$$

where $\xi$ implies to $u$, or $w$, or $p$ or $\sigma_{xz}$, and its subscription implies the partial derivative, *e.g.* $\xi_x=\partial\xi/\partial x$. The stationary conditions with respective to $u$, $w$, $p$, and $\sigma_{xz}$ are given, respectively:

$$-\frac{\partial \sigma_{xz}}{\partial z}+\frac{\partial p}{\partial x}=0 \tag{30}$$

$$\frac{\partial p}{\partial z}=0 \tag{31}$$

$$-\frac{\partial u}{\partial x}-\frac{\partial w}{\partial z}=0 \tag{32}$$

$$-\frac{1}{\eta}(\sigma_{xz}+a_1\sigma_{xz}^3+a_2\sigma_{xz}^5)+\frac{\partial u}{\partial z}=0 \tag{33}$$

It is obvious that eqs. (30)-(33) are the governing equations.

## Reynolds-type equation

Similar to the derivation as given in [18], we obtain the following constrained variational principle:

$$J(\sigma_{xz})=\int_0^L\int_0^h\left[-\frac{1}{\eta}\left(\frac{1}{2}\sigma_{xz}^2+\frac{1}{4}a_1\sigma_{xz}^4+\frac{1}{6}a_2\sigma_{xz}^6\right)\right]\mathrm{d}x\mathrm{d}z-$$
$$-\int_0^L[W(h)-W(0)]p\,\mathrm{d}x+\int_0^L U\sigma_{xz}\mathrm{d}x \tag{34}$$

which is subject to eqs. (3), (4), and (6).

Integrating eq. (5) with respect to $z$ and identifying the integration constant, we have:

$$\sigma_{xz}=\left(z-\frac{h}{2}\right)\frac{\partial p}{\partial x} \tag{35}$$

Substituting eq. (35) into eq. (34), and integrating from $z=0$ to $z=h$, we have:

$$J(p)=\int_0^L\left\{-\frac{1}{\eta}\left[\frac{1}{24}h^3\left(\frac{\partial p}{\partial x}\right)^2+\frac{1}{320}a_1h^5\left(\frac{\partial p}{\partial x}\right)^4+\frac{1}{2680}a_2h^7\left(\frac{\partial p}{\partial x}\right)^6\right]\right\}\mathrm{d}x-$$
$$-\int_0^L[W(h)-W(0)]p\,\mathrm{d}x+\int_0^L\frac{1}{2}Uh\frac{\partial p}{\partial x}\mathrm{d}x \tag{36}$$

Now the stationary condition of eq. (36) is:

$$\frac{\partial p}{\partial x}\left\{\frac{1}{\eta}\left[\frac{1}{12}h^3\frac{\partial p}{\partial x}+\frac{1}{80}a_1h^5\left(\frac{\partial p}{\partial x}\right)^3+\frac{1}{448}a_2h^7\left(\frac{\partial p}{\partial x}\right)^5\right]\right\}-\left[W(h)-W(0)\right]-\frac{\partial}{\partial x}\left(\frac{1}{2}Uh\right)=0 \tag{37}$$

Note that:

$$W(h)-W(0)=\frac{\partial h}{\partial t} \tag{38}$$

Finally we obtain the following Reynolds-type equation:

$$\frac{\partial p}{\partial x}\left\{\frac{1}{\eta}\left[\frac{1}{12}h^3\frac{\partial p}{\partial x}+\frac{1}{80}a_1h^5\left(\frac{\partial p}{\partial x}\right)^3+\frac{1}{448}a_2h^7\left(\frac{\partial p}{\partial x}\right)^5\right]\right\}=\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}\left(\frac{1}{2}Uh\right) \tag{39}$$

## Discussion and conclusion

In case $a_2 = 0$, eq. (37) reduces to:

$$\frac{\partial p}{\partial x}\left\{\frac{1}{\eta}\left[\frac{1}{12}h^3\frac{\partial p}{\partial x}+\frac{1}{80}a_1h^5\left(\frac{\partial p}{\partial x}\right)^3\right]\right\}=\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}\left(\frac{1}{2}Uh\right) \tag{40}$$

This is same as those in [14, 18]. When $a_1 = a_2 = 0$, eq. (37) reduces to the classic Reynolds equation [14, 18], which is:

$$\frac{\partial p}{\partial x}\left\{\frac{1}{\eta}\left[\frac{1}{12}h^3\frac{\partial p}{\partial x}+\frac{1}{80}a_1h^5\left(\frac{\partial p}{\partial x}\right)^3\right]\right\}=\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}\left(\frac{1}{2}Uh\right) \tag{41}$$

For the modification of eq. (2) given by Rotem *et al.* [20], the Reynolds-type equation can be derived by a similar manner discussed above, and it reads:

$$\frac{\partial p}{\partial x}\left\{\frac{1}{\eta}\left[\frac{1}{12}h^3\frac{\partial p}{\partial x}+\sum_{n=1}^{N}\frac{h^{2n+3}}{2^{2n+2}(2n+3)}a_n\sigma_{xz}^{2n+2}\right]\right\}=\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}\left(\frac{1}{2}Uh\right) \tag{42}$$

Lubrication arises everywhere including many food processes, and the present theoretical analysis is helpful to establish a mathematical model and design an optimal lubrication system.

To be concluded, this paper suggests a new approach to the establishment of a Reynolds-type equation for a complex lubrication problem.

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