THE THERMAL PROPERTIES OF WATER-BASED HYBRID NANOFLUID (CU-AL₂O₃) BEYOND AN INCLINED PLANE

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The thermal properties of radiating water-based hybrid nanofluid with nanoparticles Cu-Al₂O₃ over an inclined shrinking plane are investigated. The governing equations in this model are transformed into similarity equations. Then, the boundary value problem solver (bvp4c) in Matlab software is used numerically to solve these similarity equations. It has been discovered that utilizing Matlab software; the dual numerical solution occurs for certain values of the nanoparticle volume fraction and the suction parameter. Therefore, the skin friction coefficient and Nusselt number increase due to the effect of radiation and suction parameter. As a result of the findings, we were able to identify that the increasing nanoparticle volume fraction and the suction parameter cause the reliable numerical findings for velocity profile to enhance. When the first solution of suction parameter is increased, the skin friction coefficient and the local Nusselt number increase. Meanwhile, in the presence of the radiation parameter, the temperature of both solutions rises.

Key words: Hybrid Nanofluid, Thermal Radiation, Shrinking inclined plane, Alumina, Copper, Suction

1. Introduction

Thermal properties have significant contributions in different kinds of engineering and industrial applications. Wheelchair ramp and water heater using solar are examples of inclination sheets in an engineering application. The previous research paper considered the case of inclination case on the flat plane [1,2]. However, research on the inclined plane is sparse. Hence, this article investigates the flow on an inclined plane at various angles using boundary layer approximations. In addition, the heat and mass transfer bounded by inclined shrinking/stretching have been reported [3-4].

Nanofluids were first introduced by Choi and Eastman [5], whereas the thermal conductivity variations on nanofluids were published [6-8]. The most recent research on nanofluids was conducted by [9-14], who included the following factors: i) magnetohydrodynamics case (MHD) [9-11], ii)
different shape of container, such as trapezoidal cavity [9] and rectangular cavity [12], iii) double-diffusive convection flow [10], iv) with the existence of chemical reaction [13], and v) the occurrence of two heat sources [14]. Nowadays, researchers are more interested in hybrid nanofluids, which are considered revolutionary new in fluid dynamics. This type of nanofluids comprises two different nanoparticles dispersed in the base fluid [15]. Many scholars have explored further explorations of hybrid nanofluids in various flow configurations [16-21]. In their fluid flow model, these publications [16-21] considered a magnetic field, buoyancy force, or stagnation point impact. Meanwhile, the effect of various nanoparticles in hybrid nanofluid has been described clearly in these publications [22-25], by considering these nanoparticles: Cu (copper), Al₂O₃ (alumina), TiO₂ (titania), molybdenum disulfide (MoS₂), Ag (silver), and CuO (copper(II) oxide). Furthermore, thermal radiation can substantially impact temperature profiles and heat transfer rate at high pressure and temperature. Wakif et al. [26] explored the effects of thermal radiation on the thermomagneto-hydrodynamic stability of moving alumina nanoparticles in hybrid nanofluids containing copper oxide. Waqas et al., [27] has also considered thermal radiation with magnetized flow over a vertical stretching cylinder in a hybrid nanofluid. They discovered that higher values of the porosity variable increase the temperature profile and suppress the fluid flow.

According to the existing literature, there has been no previous investigation in hybrid nanofluid flow on an inclined shrinking plane in the presence of thermal radiation and suction. A mathematical description of current research is built by referencing prior research from Waini et al. [15] to bridge the gap with the existing literature. Thus, utilizing an inclined plane, this paper will extend the analysis of the flat plane model by Waini et al. [15]. This paper investigates the effect of nanoparticle volume fraction parameter and temperature-dependent viscosity on the Nusselt number, skin friction coefficient, velocity profile, and temperature profile.

2. Mathematical Formulation

The $x$-$y$ Cartesian coordinate of the fluid flow model is depicted in Figure 1. The shrinking plane is projected by $\alpha$ from the reference plate located at $x$-plane, and the $y$-axis parallel to the plane when $y = 0$. This model is under the influence of gravitational acceleration $g$. In addition, the plane velocity is denoted by $\mu_{w}$, and the shrinking rate is indicated by $\lambda$. The velocity component of the hybrid nanofluids in directions along $x$ and $y$ are indicated with $u$ and $v$. In this early formulation, a nanoparticle is neglected [11/15] since the mixture of the nanoparticles and the base fluid is stable. In this study, it was discovered that the shrinking case has occurred when $\lambda < 0$, whereas the plane remained static when $\lambda = 0$.

The following formulations are used for the governing boundary layer equations of motion and the energy equation [28-31]:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\mu_{nf}}{\rho_{nf}} \frac{\partial^2 u}{\partial y^2} + g \beta (T - T_{\infty}) \cos \alpha,$$
The boundary conditions are as follows
\[ v = v_w(x), \quad u = u_w(x)\lambda, \quad T = T_w \quad \text{at} \quad y = 0, \]
\[ u \to 0, \quad T \to T_\infty \quad \text{as} \quad y \to \infty. \] (4)

where \( \mu_{hnf} \) denoted the dynamic viscosity of the hybrid nanofluids, \( \rho_{hnf} \) is the density of the hybrid nanofluid, \( \beta_T \) denoted the coefficient of the thermal expansion, \( T \) is the temperature, \( T_\infty \) is assumed as surface temperature, \( k_{hnf} \) is the thermal conductivity of the hybrid nanofluid, and \( (\rho C_p)_{hnf} \) is the heat capacity of the hybrid nanofluids. For this study, the radiation flux, \( q_r \) depicted from [32-33]

\[ q_r = \frac{4\sigma^* \partial T^4}{3K^*}, \] (5)

where \( \sigma^* \) is called the Stefan-Boltzmann constant and \( K^* \) is the Rosseland absorption coefficient of the approximation. Besides, \( T^4 \) is the term where it can be joint as a linear function of temperature and can be expanded with Taylor series method. Next, the higher order is removed [32-33], and the result is

\[ T^4 = 4T^3T - 3T^4. \] (6)

Hence, the energy equation (3) is expressed as follows
Here, the fluid velocity is low due to the laminar flow, and viscous heat dissipation is expected to be minimal. Das [34] gives the effective basic nanofluids properties as follows:

\[
\rho_{nf} = (1 - \phi)\rho_f + \phi \rho_s, \quad \mu_{nf} = \frac{\mu_f}{(1 - \phi)^{2.5}},
\]

\[
(\rho C_p)_{nf} = (1 - \phi)(\rho C_p)_f + \phi(\rho C_p)_s,
\]

\[
\frac{\kappa_{nf}}{\kappa_f} = \frac{(2\kappa_f + \kappa_s) - 2\phi(\kappa_f - \kappa_s)}{(2\kappa_f + \kappa_s) + \phi(\kappa_f - \kappa_s)},
\]

where \(\rho\) is the density, \(\mu\) is the viscosity, \(\kappa\) is the effective thermal conductivities, \(\phi\) is the solid volume fraction, the specific heat capacity is defined as \((\rho C_p)_s\). From Eq. (8), subscripts \(nf\), \(f\), and \(s\) refer to the nanofluid, base fluid and the nanoparticle, respectively.

Subsequently, the particular form of thermo-physical characteristics [15] is used. The thermal parameters of the hybrid nanofluid that will be used in this study are listed in Table 1, where subscript \(hnf\) refer to the symbols owned by hybrid nanofluid. Table 2 also includes the thermophysical characteristics of nanoparticles and the base fluid. The physical properties related to water (base fluid), \(Al_2O_3\) and Cu for hybrid nanoparticles are given in Table 2 [35-36].

The similarity variables are used to reduce the above-mentioned governing equations depicted from [15] and

\[
u = \frac{v_f}{L^2} x^\frac{1}{3} f'(\eta), \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \quad \eta = y x^{-\frac{1}{3}},
\]

where \(\eta\) represents the similarity variable and \(L\) is the length of characteristic.

We assumed

\[
u_w(x) = \left(\frac{v_f}{L^2}\right) x^\frac{1}{3}, \quad \nu_w = -\frac{1}{3} \frac{v_f}{L^2} x^{-\frac{1}{3}} [2f(\eta) - \eta f'(\eta)] S,
\]

where \(S\) is the steady mass flux parameter, with \(S > 0\) indicating suction.

In our study, substituting Eqs (7), (8), (9) and (10) into Eqs. (2) and (7) give the following equations

\[
\frac{\mu_{hnf}}{\mu_f} 3f'''' + 2ff'' - f'^2 + \varepsilon \theta \cos k = 0.
\]
The boundary conditions are transformed as given

\[ f(0) = S, \quad f'(0) = \lambda, \quad \theta(0) = 1, \]

\[ f'(\infty) \to 0, \quad \theta'(\infty) \to 0. \]  

(14)

where \( R = 4\sigma T_\infty^3/(\kappa_f) \) for the radiation parameter, \( Pr = \mu_f \left( C_p \right)_f / \kappa_f \) for Prandtl number and \( \varepsilon \) is the mixed convection parameter. The primes in Eqs. (11) and (12) denote the function differentiation with respect to \( \eta \). The skin friction coefficients, \( C_f \) and the Nusselt number, \( Nu_x \), are the physical parameters of primary interest in practical applications, which are defined as

\[ C_f = \frac{\tau_w}{\rho f u_w^2}, \quad Nu_x = \frac{x q_w}{k_f(T_w - T_\infty)} \]  

(15)

and given by

\[ \tau_w = \mu_{hf} \left( \frac{\partial u}{\partial y} \right)_{y=0}, \quad q_w = -k_{hf} \left( \frac{\partial T}{\partial y} \right)_{y=0} + (q_r)_{y=0} \]  

(16)

where \( \tau_w \) denote as the shear stress along the plate and \( q_w \) denotes the heat flux from the plate. The skin friction and Nusselt number can be ordered as follows by substituting Eqs. (8), (9) and (7) into Eq. (14) and (15) to get

\[ C_f Re_x^2 = \frac{\mu_{hf}}{\mu_f} f''(0), \quad Nu_x Re_x^2 = -\left( \frac{k_{hf}}{k_f} + \frac{4}{3} Rd \right) \theta'(0). \]  

(17)

where \( Re_x = u_w(x)x/\nu_f \) is called the local Reynold numbers.

Table 1 Thermo-physical properties of hybrid nanofluid [15-37]

<table>
<thead>
<tr>
<th>Properties</th>
<th>Hybrid Nanofluid</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dynamic viscosity</td>
<td>[ \mu_{hf} = \frac{\mu_f}{(1 - \phi_1)^{2.5}(1 - \phi_2)^{2.5}} ]</td>
</tr>
</tbody>
</table>
| Thermal conductivity| \[ \frac{k_{hf}}{k_f} = \frac{(2\kappa_{nf} + \kappa_{s2}) - 2\phi_f (\kappa_{nf} - \kappa_{s2})}{(2\kappa_{nf} + \kappa_{s2}) + \phi_f (\kappa_{nf} - \kappa_{s2})} \]  

where

\[ \frac{k_{hf}}{k_f} = \frac{(2\kappa_f + \kappa_{s1}) - 2\phi_1 (\kappa_f - \kappa_{s1})}{(2\kappa_f + \kappa_{s1}) + \phi_1 (\kappa_f - \kappa_{s1})} \]
\[ \rho_{hnf} = (1 - \phi_2)[(1 - \phi_1)\rho_f + \phi_1\rho_{s1}] + \phi_2\rho_{s2}, \]

\[ (\rho C_p)_{hnf} = (1 - \phi_2)[(1 - \phi_1)(\rho C_p)_f + \phi_1(\rho C_p)_{s1}] + \phi_2(\rho C_p)_{s2}. \]

Table 2. Thermo-physical properties [35-36].

<table>
<thead>
<tr>
<th></th>
<th>( \rho ) (kg/m(^3))</th>
<th>( C_p ) (J/kg K)</th>
<th>( \kappa ) (W/m K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water</td>
<td>997.1</td>
<td>4179</td>
<td>0.613</td>
</tr>
<tr>
<td>( \text{Al}_2\text{O}_3 )</td>
<td>3970</td>
<td>765</td>
<td>40</td>
</tr>
<tr>
<td>Cu</td>
<td>8933</td>
<td>385</td>
<td>400</td>
</tr>
</tbody>
</table>

3. Result and Discussion

Eq (12) and Eq (13) has been analysed with inclined permeable shrinking sheet using bvp4c Matlab software with the boundary conditions in Eq (14). This study adds \( \text{Al}_2\text{O}_3 \) (first nanoparticles) with Cu (second nanoparticles) to form a water-based hybrid nanofluid. The first and second nanoparticles are denoted by \( \phi_1 \) and \( \phi_2 \), respectively. The present study shows several fixed parameters which have been identified: Prandtl number \( Pr = 6.2 \), first nanoparticle volume fraction \( \phi_1 = 0.01 \), second nanoparticle volume fraction \( \phi_2 = 0.06 \), thermal radiation \( R = 1.0 \), suction \( S = 2.2 \), shrinking sheet \( \lambda = -1.0 \), inclined angle alpha \( \alpha = 45^\circ \), and mixed convection parameter \( \varepsilon = 0.05 \). The results presented here are subjected to the various values of copper nanoparticles (Cu), instead of alumina (\( \text{Al}_2\text{O}_3 \)) [15]. The various values of parameters such \( \phi_2 \), \( S \), and \( R \) have been observed for the velocity and temperature profiles in Figure 2 until Figure 6, respectively. Moreover, the skin friction and Nusselt number also will be displayed in Figure 7 and Figure 8, with the impacts of \( \phi_2 \) and \( S \). Dual solutions are introduced in the graph denoted as first solution (solid line) and second solution (dashed line).

Figure 2 and Figure 3 presents the dual solutions of velocity profiles \( f'(\eta) \) and temperature profiles \( \theta(\eta) \), with distinct values \( \phi_2 \). The obtained results in Figure 2 and Figure 3 show that increases \( \phi_2 \) leads to a rise in fluid velocity and temperature profiles for the first solution, while the velocity profile decreases for the second solution. This result is due to the presence of the inclined permeable shrinking sheet, which makes the first solution slightly thicker than the second solution in different \( \phi_2 \).

Next, Figure 4 shows the \( f'(\eta) \) and \( \theta(\eta) \) for the difference in \( S = 2.15, 2.20, 2.25 \). From the graph observation, Figure 4 displays that increasing \( S \) can cause an increase in velocity profiles. For the initial solutions, the temperature profile in Figure 5 is rising. On the other hand, Figure 4 differs in the second solution where the decrease in velocity profile and temperature profile identified due to an increase in suction parameter, \( S \). This finding is because suction creates greater resistance in fluid flow as stated by [37].
Furthermore, the graph of temperature profile with different values of thermal radiation, $R$ are shown in Figure 6. As a result, with an increase in $R$, the value of temperature profile, $\theta(\eta)$ will also increase for dual solutions. It can be identified that the thermal boundary layer thickness is enhanced due to the increase in $R$. Therefore, the higher the values of $R$ indicates that more radiative heat energy is pumped into the flow field causes the increment in temperature.

Next, Figures 7 and 8 describe the skin friction coefficient, $f''(0)$ and Nusselt number with various values of nanoparticle volume fraction for $\phi_2$. Figure 7 demonstrate the various values of $S$ on $f''(0)$ when dealing with different values of $\phi_2$. From Figure 7, the intersection point is introduced as a critical suction point $S_c$ and can be seen smoothly intersecting for the first solution and second solution. From the observation, the increase in suction parameter, $S$ and nanoparticle volume fraction for $\phi_2$ leads to the increase in skin friction coefficient for first solution. This result is because mass suction can aid the passage of hybrid fluid particles toward the wall while also causing the flow to move more slowly as proved by [17]. Besides, the second solution is vice versa from the first solution from observation. We can see that as $\phi_2$ increases, the solution domain grows and the crucial values $S_c$ shift to the left. Furthermore, for the upper branch solutions, the values of $f''(0)$ grow as $S_c$ increases. As a result, the value of critical suction for dual solutions from Figure 7 are $2.1142\ (\phi_2 = 0.06)$, $2.0871\ (\phi_2 = 0.08)$ and $2.0686\ (\phi_2 = 0.1)$.

Fig.2. Various values of $\phi_2$ for $f'(\eta)$  
Fig.3. Various values of $\phi_2$ for $\theta(\eta)$  
Fig.4. Various values of $S$ for $f'(\eta)$  
Fig.5. Various values of $S$ for $\theta(\eta)$
From Figure 8, the Nusselt number has been presented when $\phi_2 = 0.06$ and $S_c = 2.1142$. The illustration of Nusselt number for other $\phi_2$ ($\phi_2 = 0.08$ and 0.1) are not attached here due to the pages limitation. However, their variation have remained the same (the first solution decreases when it is approaching the critical point $S_c$, whereas the second solution enhances). The critical suction points for Figure 8 have remained the same as Figure 7: $S_c = 2.1142$, $S_c = 2.0871$ and $S_c = 2.0686$ when $\phi_2 = 0.06, 0.08, 0.1$ respectively. From this information, the increase in $\phi_2$ and $S$ leads to the increase in the Nusselt number for the first solution, while the Nusselt number in the second solution is decreased.

Fig.6. Various values of $R$ for $\theta(\eta)$

Fig.7. Various values of $\phi_2$ for $f''(0)$ with $S$
4. Conclusion

The governing boundary layer equations have been reduced to ordinary differential equations with the appropriate similarity transformation before being solved using bvp4c and programmed in MATLAB software. In this study, the first and second solutions indicate the different solutions in the graph for various parameters. As a result, we can conclude that the velocity profile increases for the first solutions when the nanoparticle volume fraction $\phi_2$ and suction parameter $S$ also increase. It was observed that as the suction parameter was increased, the temperature of both solutions decreased. While the increase in temperature profile leads to the rise in thermal radiation, $R$. The effects of the thermal radiation parameter, $R$ thicken the thermal boundary layer for both solutions, as well. Moreover, the skin friction coefficient and Nusselt number increase, which make the suction parameter and nanoparticle volume fraction $\phi_2$ increases.

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Nomenclature

$\alpha$ - thermal diffusivity [m$^2$/s$^1$]
$\beta_T$ - coefficient of the thermal expansion [$\mu$m m$^{-1}$K$^{-1}$]
$C_f$ - skin friction coefficient
$C_p$ - specific heat at constant pressure [Jkg$^{-1}$K$^{-1}$]
$(\rho C_p)_{hnf}$ - heat capacitance of the hybrid nanofluid [JK$^{-1}$m$^{-3}$]
$f(\eta)$ - dimensionless stream function
$g$ - gravitational acceleration [m/s$^2$]
$k_{hnf}$ - thermal conductivity of the hybrid nanofluid [Wm$^{-1}$K$^{-1}$]
$k'$ - Roseland: absorption coefficient [m$^{-1}$]
$L$ - characteristics length of the surface [m]
$Nu_x$ - local Nusselt number
$Re_x$ - local Reynolds number
$Pr$ - Prandtl number
$q_r$ - radiative heat flux component in y directions [Wm$^{-2}$]
$q_w$ - surface heat flux
$R$ - radiation parameter
$S$ - constant mass flux
$T$ - fluid temperature [K]
$T_w$ - surface temperature [K]
$T_\infty$ - ambient temperature [K]
$u, v$ - velocity of components in the $x$ and $y$ directions [ms$^{-1}$]
$u_w$ - velocity of shrinking surface [ms$^{-1}$]
$u_\infty$ - velocity of the mainstream [ms$^{-1}$]
$v_w$ - velocity of the wall mass transfer [ms$^{-1}$]
$x, y$ - Cartesian coordinates [m]

**Greek Symbol**

\(\eta\) - similarity variable
\(\theta\) - dimensionless temperature
\(\kappa\) - effective thermal conductivity [Wm$^{-1}$K$^{-1}$]
\(\varepsilon\) - mixed convection parameter.
\(\lambda\) - constant shrinking parameter
\(\mu_{nf}\) - dynamic viscosity (hybrid nanofluid) [kgm$^{-1}$s$^{-1}$]
\(\rho_{nf}\) - density (hybrid nanofluid) [kgm$^{-3}$]
\(\nu_f\) - kinematic viscosity of the base fluid [m$^2$s$^{-1}$]
\(\sigma\) - constant of Stefan-Boltzmann [Wm$^{-2}$K$^{-4}$]
\(\phi_1\) - nanoparticle volume fractions (Al$_2$O$_3$ for Alumina)
\(\phi_2\) - nanoparticle volume fractions (Cu for Copper)
\(\tau_w\) - skin friction [kgm$^{-1}$s$^{-2}$]

**Subscripts**

\(f\) - fluid
\(nf\) - nanofluid
\(hnf\) - hybrid nanofluid
\(s\) - nanoparticles

**Superscript**

' - differentiation with respect to $\eta$

**References**


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