SOLUTION OF BURGERS' EQUATION APPEARS IN FLUID MECHANICS BY MULTISTAGE OPTIMAL HOMOTOPY ASYMPTOTIC METHOD

by

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In this article, we approximate analytical solution of Burgers' equations using the Multistage homotopy asymptotic method which are utilized in modern physics and fluid mechanics. The suggested algorithm is an accurate and simple to-utilize semi-analytic tool for non-linear problems. In the current research we investigation the efficiency and accuracy of the method for the solution of non-linear PDE for large time span. Numerical comparison with the variational iteration method shows the efficacy and accuracy of the proposed method.

Key words: *multistage optimal homotopy asymptotic method, Burgers' equation*

Introduction

Non-linear PDE are utilized in numerous fields, for example, plasma physics, hydrodynamic, and non-linear optic, *etc*. It is observed that in most of the cases it is hard to tackle non-linear problems, especially analytically. Numerous strategies have been developed by the researchers for the non-linear problems wherein the Perturbation procedures [1], which depended on the existence of large or small parameters, to be specific the perturbation quantities. Tragically, numerous non-linear problems in physical sciences do not contain such sort of perturbation quantities by any means. To over come such types of difficulties some non-perturbative procedures which are free of small parameters are proposed in [2]. Be that as it may, both perturbative and non-perturbative strategies couldn't give a straightforward method to adjust or control the rate and region of convergence of approximate series [3]. To over come this trouble,

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another analytical technique was proposed by Herisanu and Marinca [4], known as the optimal homotopy asymptotic method (OHAM) which has been effectively employed to numerous non-linear problems in heat transfer and fluid mechanics [5, 6].

The OHAM is an approximate analytical method which can be used with ease like to homotopy analysis method, however, with more degree of flexibility. Many authors have shown that the suggested procedure is accurate and reliable, and calculated the solutions of complex problems see [4] and the references their in. To circumvent the short time span limitation of OHAM, a new modification is made which is based on the standard OHAM and called it the multistage optimal homotopy asymptotic method (MOHAM).

On the other hand, the Burgers' equation which is known as a basic convection diffusion equation and generally appears in fluid mechanics, additionally, it is utilized for describing shock waves' structure [7]. The purpose of the current research is to utilized efficiently the MOHAM to find out the approximate analytical solutions for the Burgers' equation.

Description of MOHAM

This section is devoted to the basic principles of the OHAM as given in [4]. Consider the initial-value problem:

$$L_i\left[U_i\left(y,\tau\right)\right]+N_i\left[U_i\left(y,\tau\right)\right]=0,\ i=1,2,...,N \tag{1}$$

with initial condition

$$U_i(y,\alpha)=\alpha_i \tag{2}$$

where $U_i(y, \tau)$ is the unknown function whereas L_i and N_i denote the linear operator, non-linear operator, respectively, additionally, y and τ are the independent variables. A homotopy map $h_i[v_i(y, \tau, q), q]$: $R \times [0, 1] \to R$ which satisfies:

$$(1-q)\left\{L_i\left[v_i(y,\tau,q)\right]-U_{i,0}(\tau)\right\}=H_i(q,\tau)\left\{L_i\left[v_i(y,\tau,q)\right]+N_i\left[v_i(y,\tau,q)\right]\right\} \tag{3}$$

can be constructed. Here $q \in [0, 1]$ and $y, \tau \in R$ where $H_i(q) \neq 0$ is an auxiliry funtion, $H_i(0) = 0$ for $q = 0$, and $v_i(y, \tau, q)$ is an unknown function. It is understood that $v_i(y, \tau, 0) = U_{i,0}(\tau)$ holds for $q = 0$ and $v_i(y, \tau, 0) = U_{i,0}(\tau)$ holds for $q = 1$. In the same manner q changes from 0 to 1, the solution $v_i(y, \tau, q)$ changes from $U_{i,0}(y, t)$ to $U_i(y, \tau)$ where $U_{i,0}(y, \tau)$ is the initial guess which is known and calculated from eq. (2) for $q = 0$:

$$L_i\left[U_{i,0}(y,\tau)\right]=0 \tag{4}$$

Now, the auxiliary function $H_i(q)$ has been chosen in the following manner:

$$H_i(q)=C_{1,j}q+C_{2,j}q^2+C_{3,j}q^3+\ldots \text{ or } H_i(q,\tau)=C_{1,j}q+C_{2,j}\tau q^2+C_{3,j}\tau^2q^3+\ldots \tag{5}$$

where $C_{1,j}$, $C_{2,j}$, $C_{3,j}$,... denote convergence control parameters (CCP) and can be find out later. In order to find the required approximate solution, the Taylor's series are utilized in the accompanying form to expand $v_i(y, \tau, q, C_k)$ about q:

$$v_i(y,\tau,q,C_k)=U_0(y,\tau)+\sum_{k=1}^{\infty}U_{i,k}(y,\tau,C_1,C_2,\ldots,C_k)q^k \tag{6}$$

Define the vectors:

$$\vec{C}_i = \{C_1, C_2, \ldots, C_i\},\ \vec{U}_{i,s} = \left\{U_{i,0}(y,\tau), U_{i,1}(y,\tau,C_1), \ldots,\ U_{i,s}(y,\tau,\vec{C}_s)\right\}$$

where $s = 1, 2, 3, \ldots$, setting eq. (6) into eq. (3) and to the linear equations which are given below, we proceed by comparing coefficient q. Also, the zeroth-order problem is given by eq. (4) whereas the first- and second-order problems are given:

$$\begin{gathered} L_i(U_{i,1}(y,\tau)) = C_1 N_0(\vec{U}_{i,0}),\ \ U_{i,1}(a) = 0 \ \text{ and} \\ L_i(U_{i,2}(y,\tau)) - L_i(U_{i,1}(\tau)) = C_2 N_{i,0}(\vec{U}_{i,0}) + C_{1,j}[L_i(U_{i,1}(y,\tau)) + N_{i,1}(\vec{U}_{i,1})], U_{i,2}(a) = 0 \end{gathered} \tag{7}$$

The general equations for $U_{i,k}(\tau)$ are:

$$\begin{gathered} L_i(U_{i,k}(y,\tau)) - L_i(U_{i,k-1}(y,\tau)) = C_{k,j} N_{i,0}(U_{i,0}(\tau)) + \sum_{m=1}^{k-1} C_{i,m}[L_i(U_{i,k-m}(y,\tau)) + \\ + N_{i,1}(\vec{U}_{i,k-1})],\ U_{i,k}(a) = 0 \end{gathered} \tag{8}$$

where $k = 2, 3, \ldots$ and $N_{i,m}[U_0(y,\tau), U_{i,1}(y,\tau), \ldots, U_{i,m}(y,\tau)]$ is the coefficient of q^m in the expansion of $N_i[v_i(y,\tau,q)]$ about q which is known as embedding parameter:

$$N_i[v_i(y,\tau,q)] = N_{i,0}[U_{i,0}(y,\tau)] + \sum_{m=1}^{\infty} N_{i,m}(\vec{U}_{i,m}) q^m \tag{9}$$

As it is notice that the convergence of the series given in eq. (9) heavily depends on the CCP $C_1, C_2, C_3, \ldots$, if it is convergent at $q = 1$, then:

$$v_i(y,\tau,C_k) = U_{i,0}(\tau) + \sum_{k=1}^{\infty} U_k(y,\tau,C_1,C_2,\ldots,C_k) \tag{10}$$

The result of the m^{th} order approximation is given:

$$\tilde{U}(y,\tau,C_1,C_2,C_3,\ldots,C_k) = U_{y,0}(y,\tau) + \sum_{k=1}^{\infty} U_k(y,\tau,C_1,C_2,\ldots,C_k). \tag{11}$$

Substituting eq. (11) into eq. (1) gives the accompanying residual:

$$\begin{gathered} R_i(y,\tau,C_{1,j},C_{2,j},C_{3,j},\ldots,C_{m,j}) = L\left[\tilde{U}_i(y,\tau,C_{1,j},C_{2,j},C_{3,j},\ldots,C_{m,j})\right] + \\ + N\left[\tilde{U}_i(y,\tau,C_{1,j},C_{2,j},C_{3,j},\ldots,C_{m,j})\right] \end{gathered} \tag{12}$$

where $U(y,\tau)$ will represent the exact solution when $R_i = 0$. It is noticed that such a case will not happen for non-linear problems, yet we can limit the function:

$$J_i(C_{1,j},C_{2,j},C_{3,j},\ldots,C_{m,j}) = \int_{\tau_j}^{\tau_{j+h}} R_i^2(y,\tau,C_{1,j},C_{2,j},C_{3,j},\ldots,C_{m,j})\mathrm{d}\tau \tag{13}$$

where the length and the number of subintervals $[\tau_j,\ \tau_{j+1}]$ is denoted by h and $N = [T/h]$, respectively. Next, changing the initial approximation in each subinterval from the previous one, we can solve eq. (13) at $j = 0, 1, \ldots, N$. For instant, we define $\alpha = U(\tau_j)$ in the subinterval $[\tau_j,\ \tau_{j+1}]$. The unknown CCP $C_{i,j}(i = 1, 2, 3, \ldots, m, j = 0, 1, \ldots, N)$ can be determined from the solution of the below given system of equations:

$$\frac{\partial J}{\partial C_{1,j}} = \frac{\partial J}{\partial C_{2,j}} = \ldots = \frac{\partial J}{\partial C_{m,j}} = 0 \tag{14}$$

and hence, the approximate analytic solution is

$$\tilde{U}(y,\tau)=\begin{cases} U_1(y,\tau), & \tau_0 \le \tau < \tau_1 \\ U_2(y,\tau), & \tau_1 \le \tau < \tau_2 \\ & \vdots \\ U_N(y,\tau), & \tau_{N-1} \le \tau < T \end{cases} \tag{15}$$

Proceeding with thusly, we effectively calculate the initial value problems' solution analytically for large value of T. It merits referencing that the MOHAM convert to the standard OHAM when $j = 0$. It is also essential to mention that MOHAM gives an easy way to adjust and control the convergence region by means of the auxiliary function $H_i(q)$ involving many CCP $C_{i,j}$'s. Then again, the proposed method overcomes the main difficulty, due to the large computational domain, in calculating the solution of problems.

3 Implementation of proposed scheme

The suggested MOHAM is implemented in the section the Burgers' equation show the effectiveness and validity of the algorithm, furthermore, the initial-boundary conditions can be computed easily in accordance to the exact solution throughout the paper.

Test Problem 1. Consider the Burgers' equation in the form:

$$u_\tau + \frac{1}{2}u_y^2 - u_{yy} = 0 \tag{16}$$

with analytical/exact solution

$$u(y,\tau)=\frac{y}{1+\tau}$$

Now, to solve this problem using the proposed MOHAM, we select the linear and non-linear operators:

$$L[u(y,\tau,q)] = u_\tau(y,\tau,q), \quad N[u(y,\tau,q)] = \frac{1}{2}u_y^2(y,\tau,q) - u_{yy}(y,\tau,q) \tag{17}$$

Now, in this case the auxiliary function $H_i(q)$ is taking as $H_i(q) = (C_{1,jq} + C_{2,j}q^2)$ where $C_{1,j}$, $C_{2,j}$ are unknown constant to be computed. Using the producer as described in section *Description MOHAM* by taking step-size $h = 0.1$ and starting with $\tau_0 = 0$ to $\tau_{10} = T = 1$. Various order initial value problems and their respective solutions for the first subinterval are given:

Zeroth order problem:

$$\frac{\partial u_0}{\partial \tau} = y, \; u_0(y,0) = y \tag{18}$$

with solution

$$u_0(y,\tau) = y + \tau y \tag{19}$$

First order problem:

$$\frac{\partial u_1}{\partial \tau} = \left(1 + C_{1,j}\right)\frac{\partial u_0}{\partial \tau} - C_{1,j}\frac{\partial^2 u_0}{\partial y^2} + \frac{1}{2}C_{1,j}\frac{\partial u_0^2}{\partial y} - y, u_1(y,0) = 0 \tag{20}$$

Their solution:

$$u_1(y,\tau,C_{1,j}) = \frac{1}{3}C_{1,j}(6\tau + 3\tau^2 + \tau^3)y \tag{21}$$

Second order problem:

$$\frac{\partial u_2}{\partial \tau} = C_{2,j}\left(\frac{\partial u_0}{\partial \tau} - \frac{\partial^2 u_0}{\partial y^2}\right) + \left(1 + C_{1,j}\right)\frac{\partial u_1}{\partial \tau} - C_{1,j}\frac{\partial^2 u_1}{\partial y^2} + C_{1,j} + $$
$$+ C_{1,j}\frac{\partial u_0}{\partial y}\frac{\partial u_1}{\partial y} + \frac{1}{2}C_{1,j}\frac{\partial u_0^2}{\partial y}, \quad u_2(y,0) = 0 \tag{22}$$

with solution

$$u_2(y,\tau,C_{1,j},C_{2,j}) = \frac{1}{15}\left(15C_{1,j}^2\tau^2 + 15C_{1,j}^2\tau^3 + 5C_{1,j}^2\tau^4 + C_{1,j}^2\tau^5 + 30C_{1,j}\tau y + \right.$$
$$\left. + 30C_{1,j}^2\tau y + 30C_{2,j}\tau y + 15C_{1,j}\tau^2 y + 15C_{1,j}^2\tau^2 y + 15C_{2,j}\tau^2 y + 5C_{1,j}\tau^3 y + 5C_{1,j}^2\tau^3 y + 5C_{2,j}\tau^3 y\right) \tag{23}$$

Hence, second order MOHAM solution for the first subintervals:

$$\tilde{u}(y,\tau,C_{1,j},C_{2,j}) = u_0(y,\tau) + u_1(y,\tau,C_{1,j}) + u_2(y,\tau,C_{1,j},C_{2,j}) \tag{24}$$

which is

$$\tilde{u}(y,\tau,C_{1,j},C_{2,j}) = y + \tau y + \frac{1}{3}C_{1,j}\left(6\tau + 3\tau^2 + \tau^3\right)y + \frac{1}{15}\left(15C_{1,j}^2\tau^2 + \right.$$
$$+ 15C_{1,j}^2\tau^3 + 5C_{1,j}^2\tau^4 + C_{1,j}^2\tau^5 + 30C_{1,j}\tau y + 30C_{1,j}^2\tau y +$$
$$\left. + 30C_{2,j}\tau y + 15C_{1,j}\tau^2 y + 15C_{1,j}^2\tau^2 y + 15C_{2,j}\tau^2 y + 5C_{1,j}\tau^3 y + 5C_{1,j}^2\tau^3 y + 5C_{2,j}\tau^3 y\right) \tag{25}$$

Following the procedure given in [8], the CCP $C_{i,j}$'s are computed which are tabulated in tab. 1. Using the values in eq. (24), we obtained MOHAM solution of eq. (16) in the first subinterval. Similar procedure is adopted to obtain second order MOHAM solution of eq. (16) in the remaining subintervals.

Table 1. The convergent control constants $C_{i,j}$'s for *Test Problem 1*

j	$C_{1,j}$	$C_{2,j}$
1	$-1.9788331762752687 \cdot 10^{-14}$	–0.90676563109569270
2	0.62540682310629560	–1.12571489364935200
3	0.21193168797057305	0.00726393611476467
4	0.19450625550430434	0.19899076239866692
5	0.29145271561972920	0.29145271561972920
6	0.12497642360686809	0.63418142392997800
7	0.07195217538052939	0.86700586125546590
8	–2.4562952902766180	1.92596034308506670
9	–0.0723139834460049	1.34782573055597290
10	–0.1681035122380220	1.56555026900759530

Table 2. Results the second order MOHAM for *Test Problem 1*

τ	y	Exact	MOHAM (present method)	Ab.Err (MOHAM)	Ab. Err (VIM)
	0	0	0	0	0
0.1	0.1	0.090909	0.090927	$1.9 \cdot 10^{-5}$	$9.1 \cdot 10^{-13}$
0.2	0.2	0.166667	0.166707	$14.1 \cdot 10^{-5}$	$3.4 \cdot 10^{-9}$
0.3	0.3	0.230769	0.230690	$7.9 \cdot 10^{-5}$	$4.1 \cdot 10^{-7}$
0.4	0.4	0.285714	0.285656	$15.8 \cdot 10^{-5}$	$1.2 \cdot 10^{-5}$
0.5	0.5	0.333333	0.333373	$3.9 \cdot 10^{-5}$	$1.6 \cdot 10^{-4}$
0.6	0.6	0.375000	0.375213	$2.1 \cdot 10^{-4}$	$1.4 \cdot 10^{-3}$
0.7	0.7	0.411765	0.412179	$4.1 \cdot 10^{-4}$	$8.1 \cdot 10^{-3}$
0.8	0.8	0.444444	0.445262	$8.2 \cdot 10^{-4}$	$3.8 \cdot 10^{-2}$
0.9	0.9	0.473684	0.474225	$5.4 \cdot 10^{-4}$	$1.5 \cdot 10^{-1}$
1.0	1.0	0.500000	0.500230	$2.3 \cdot 10^{-4}$	$5.0 \cdot 10^{-1}$

A comparison between the absolute error of second-order MOHAM and nine order of VIM are presented in tab. 2. It is quite clear from tab. 2 that solution obtained from second order MOHAM is become better and better than nine order approximate solution of VIM when time span increased.

Conclusion

In this research, the MOHAM is utilized to obtain the analytical approximate solutions of the Burgers' equation. The results of the proposed method are compared with exact and variational iteration method (VIM), and it was found that the MOHAM is more effective than VIM. One advantage of the MOHAM is ease and straightforward calculations and the second one is, the reduction in the size of computational domain. In light of these results, we concluded that, the MOHAM is an accurate and efficient for calculating approximate analytical solution of the non-linear partial differential equations which utilized in modern physics and fluid mechanics.

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