NUMERICAL ANALYSIS OF FLOW AND HEAT TRANSFER IN A THIN FILM ALONG AN UNSTEADY STRETCHING CYLINDER

by

Azeem SHAHZAD[a], Bushra HABIB[a], Muhammad NADEEM[a], Muhammad KAMRAN[b], Hijaz AHMAD[c*], Muhammad ATIF[d], and Shafiq AHMAD[e]

[a] Department of Basic Sciences, University of Engineering and Technology, Taxila, Pakistan
[b] Department of Mathematics, COMSATS University Islamabad, Wah Campus, Pakistan
[c] Section of Mathematics, International Telematic University Uninettuno, Roma, Italy
[d] Department of Physics and Astronomy, College of Science, King Saud University, Riyadh, Saudi Arabia
[e] Industrial Engineering Department, College of Engineering, King Saud University, Riyadh, Saudi Arabia

Original scientific paper
https://doi.org/10.2298/TSCI21S2441S

In this framework, the boundary-layer mass and heat flow in a liquid film over an unsteady stretching cylinder are discussed under the influence of a magnetic field. By means of the similarity transformations the highly non-linear governing system of PDE is converted to ODE. We use the built-in function bvp4c in MATLAB to solve this system of ODE. The impact of distinctive parameters on velocity and temperature profile in the existence of an external magnetic field is depicted via graphs and deep analysis is also presented.

Key words: *boundary-layer, heat exchange, stretching cylinder, MATLAB, thin film*

Introduction

Over the previous decades, the flow and heat transfer both have been massively assessed by the analysts because of its colossal applications in modern designing. The affirmation of mass- and heat-flow inside the thin fluid film is of urgent significance in comprehending the covering procedure and in the structuring of different heat exchangers and the processing of chemical equipment. It additionally has various applications in modern designing procedures, for example, in the paper generation, materials produced by the expulsion procedure, the boundary-layer of a fluid film in the solidification scheme, foodstuff handling, cable covering, in conserving of cold metal plates, polymer coatings, streamlined expulsion of plastic sheet and hot moving, toughening and tinning of copper wires.

Thermal transport investigation owing to stretched surface through adjacent fluid is the prominent zone of enthusiasm for analyst these days because of its broad applications in various fields of science, engineering and specifically in the branch of chemical engineering (*e.g.* in the metallurgy, polymer molding contain condensing of liquid melt extended to the heat exchanging system). In these procedures, the rheology of the end product primarily rely on the

* Corresponding author, e-mail: hijaz555@gmail.com

heat exchange and stretching rate *e.g.* in synthetic fiber processing, the extrude from die is transformed into long strands or filament, then it is condensed through gradual cooling.

Sakiadis [1] investigated the phenomenon of laminar and turbulent flow past a compact surface with bounded length. Initially, Crane [2] considered the stationary 2-D flow of Newtonian fluid along with a stretched elastic sheet. Carragher *et al.* [3] had worked on the thermal transport of the viscous liquid over stretched surface into consideration that temperature difference amongst the surface and the adjoining liquid is comparable to a power of distance from a fixed point. Wang [4] was first who considered the dynamics of the thin liquid film on a stretched sheet and reduced the Navier-Stokes equations to a non-linear ODE by utilizing similarity transformation. Andersson [5] considered the non-Newtonian fluid-flow past a stretched sheet in the existence of a transversal magnetic field and found the exact solution. Liu and Andersson [6] proceeded his work and investigated the heat transport in a laminar fluid film on a horizontally stretched surface. Dandapat *et al.* [7] examined the thermocapillarity on the heat transfer and fluid-flow in a thin fluid film on a horizontally stretched sheet. In this problem, the forces on the surface of thermocapillary draw the fluid film extending alongside the stretching of sheet and a minimum local velocity of the film. Later Wang [8] extended his work and found the analytic result of the same problem via. applying homotopy analysis method. Dandapat and Maity [9] considered the effect of inertia and supposed the spontaneous stretching of the stationary surface. The results shown was that the quantity of initially dispersed fluid is ineffective on the final film thickness. It was also presented that the forced stretching causes non-uniformity in the liquid film across the stretched surface. Cortell [10] considered the thermophysical properties of a liquid with high viscosity past an arbitrarily stretched sheet.

In the literature survey, it is inferred that no attempt has so far been communicated to explore the thermophysical flow in a liquid film past a stretchable cylinder. The complexity of the problem raises due to stretching cylinder and liquid film. The current investigation is carried out in order to fill the gap for providing a reliable numerical scheme to solve liquid film problems over stretching surfaces.

Mathematical formulation

We consider a thin fluid film of viscous fluid of thickness, h, over a stretchable cylinder of radius, R, fig. 1. Motion in fluid is produced due to the stretching of the cylinder along axial z-direction of the cylindrical co-ordinates system. The $U(z,t) = bz/(1-\alpha_1 t)$ is the surface velocity, where b and α_1 are both greater than zero and has dimension [t^{-1}], $T = T_s = T_o - T_{ref}(bz^2/2\upsilon)(1-\alpha_1 t)^{-3/2}$ is the temperature of stretching cylinder, where T_o is the temperature at the slit and T_{ref} is taken as a constant reference temperature. The velocity $U(z, t)$ of cylinder imitates that the stretch rate $b/(1-\alpha_1 t)$ increases with time. The surface temperature $T_s(z, t)$ signifies that the temperature of the cylinder decreases from T_o at the slit is proportional to z^2 and the measure of temperature decreases along cylinder with time.

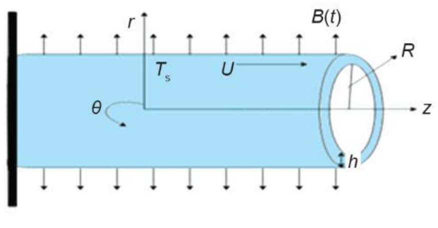

Figure 1. Schematic representation of a thin fluid film over a stretchable cylinder

The $B(t) = [B_o(t)]/(1-\alpha_1 t)^{1/2}$ is the uniform magnetic field delegated in the radial direction. The expressions $U(z, t)$, $T_s(z, t)$, $B(r, t)$ are chosen in such a way that they will help in constructing the new similarity transformations which will help in converting the governing PDE of momentum and heat equation into a system of ODE.

Governing equation and boundary conditions

In this portion, we will derive the equations of motion and heat by considering the unsteady axisymmetric flow of an incompressible viscous fluid across an elongating cylinder. The governing 2-D boundary-layer equation for velocity and temperature field is:

$$\frac{\partial}{\partial r}(ru)+\frac{\partial}{\partial z}(rw)=0 \tag{1}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right]-\frac{\sigma B_{\mathrm{o}}{}^2}{\rho(1-\alpha_1 t)}w \tag{2}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial r}+w\frac{\partial T}{\partial z}=\frac{\nu}{c_p}\left(\frac{\partial w}{\partial r}\right)^2+\alpha\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right] \tag{3}$$

In the mentioned equations velocity field is taken of the form $\vec{\mathrm{v}}=[u\ (r,z,t),0,w(r,z,t)]$, u and w are the radial and axial velocity components, respectively, $\nu=\mu/\rho$ – the kinematic viscosity, μ – the coefficient of fluid viscosity, ρ – the fluid density, $\alpha=\kappa/\rho c_p$ – the thermal diffusivity of the fluid, and T – the the temperature of the fluid.

The boundary conditions are:

$$w=U(z)=\frac{bz}{1-\alpha_1 t},\quad u=0\ ,\quad T=T_{\mathrm{s}}\quad \text{at}\quad r=R$$

$$\frac{\partial w}{\partial r}=0,\quad \frac{\partial T}{\partial r}=0,\quad u=\frac{\mathrm{d}h}{\mathrm{d}t}\quad \text{at}\quad r=R+h$$

Similarity transformation

By introducing the new similarity variables:

$$\psi=(U\nu z)^{\frac{1}{2}}Rf(\eta),\quad \eta=\frac{r^2-R^2}{2R}\left(\frac{U}{\nu z}\right)^{\frac{1}{2}},\quad \theta(\eta)=\frac{T-T_o}{-T_{\mathrm{ref}}\left(\dfrac{bz^2}{2\nu}\right)(1-\alpha_1 t)^{\frac{-3}{2}}} \tag{4}$$

where η is the independent variable and θ – the non-dimensional temperature. We define stream function $\psi(r,z)$:

$$u=-\frac{1}{r}\frac{\partial\psi}{\partial z},\quad w=\frac{1}{r}\frac{\partial\psi}{\partial r}$$

The velocity components (u and w) thus can be written:

$$u=-\frac{R}{r}\left(\frac{b\nu}{1-\alpha_1 t}\right)^{\frac{1}{2}}f(\eta)\quad \text{and}\quad w=\frac{bz}{(1-\alpha_1 t)}f'$$

By the definition of stream function, the continuity equation is satisfied correspondingly. Substituting the eq. (4) into eqs. (2) and (3), the governing equations and boundary conditions are reduced to:

$$(1+2C\eta) f'''+2Cf''-(M+S) f'-\frac{S\eta}{2} f''+ff''-(f')^2=0 \tag{5}$$

$$(1+2C\eta)\theta''+2C\theta'+\Pr[f\theta'-2f'\theta-\frac{S}{2}(3\theta+\eta\theta')+\mathrm{Ec}(1+2C\eta)f''^2]=0 \tag{6}$$

$$\left.\begin{array}{c} f(0)=0,\ f'(0)=\theta(0)=1 \\ f''(\beta)=\theta'(\beta)=0,\ f(\beta)=\dfrac{S\beta}{2} \end{array}\right\} \tag{7}$$

In the dimensionless eqs. (5) and (6), $\beta=[h+(h^2/2R)]\{b/[(1-\alpha_1 t)\nu]\}^{1/2}$ is the dimensionless film thickness, $C=\{[(1-\alpha_1 t)\nu]/bR^2\}^{1/2}$ – the curvature parameter, $M=(\sigma B_o{}^2)/(\rho b)$ – the magnetic parameter, $S=\alpha_1/b$ – the unsteadiness parameter, $\Pr=\nu/\alpha$ – the Prandtl number, and $\mathrm{Ec}=U^2/(C_p\Delta T)$ – the Eckert number. The local skin friction coefficient and the wall heat transfer coefficient (Nusselt number) are the physical quantities that are given by:

$$C_f=\frac{2\tau_w}{\rho U^2}=\frac{2\mu\left(\dfrac{\partial w}{\partial r}\right)_{r=R}}{\rho U^2}=2\mathrm{Re}^{\frac{-1}{2}} f''(0) \tag{8}$$

$$\mathrm{Nu}=\frac{zq_w}{\kappa(T_w-T_\infty)}=\frac{-z\kappa\left(\dfrac{\partial T}{\partial r}\right)_{r=R}}{-T_{\mathrm{ref}}\left(\dfrac{bz^2}{2\nu}\right)(1-\alpha_1 t)^{\frac{-3}{2}}}=-\kappa\mathrm{Re}^{\frac{1}{2}}\theta'(0) \tag{9}$$

where $\mathrm{Re}=Uz/\nu$ is the Reynold number.

Numerical results

The system of non-linear ODE (5) and (6) subject to eq. (7) are numerically solved by first transforming them into an initial value problem then an effective numerical method has been used for the numerical calculations namely bvp4c in MATLAB. A set of new variables is defined as

$$f=y_{(1)},\quad f'=y_{(2)},\quad f''=y_{(3)},\quad \theta=y_{(4)},\quad \theta''=y_{(5)}$$

So, the eqs. (4) and (5) take the form:

$$f'''=\frac{1}{1+2C\eta}\left\{My_{(2)}+Sy_{(2)}+\frac{S\eta}{2}y_{(3)}+[y_{(2)}]^2-2Cy_{(3)}-y_{(1)}y_{(3)}\right\} \tag{10}$$

$$\theta''=\frac{1}{1+2C\eta}\left(-2Cy_{(5)}-\Pr\left\{y_{(1)}y_{(5)}-2y_{(2)}y_{(4)}-\frac{S}{2}[3y_{(4)}+\eta y_{(5)}]+\mathrm{Ec}(1+2C\eta)[y_{(3)}]^2\right\}\right) \tag{11}$$

and related boundary conditions are:

$$y_{(1)}(0)=0, \quad y_{(2)}(0)=y_{(4)}(0)=1$$
$$y_{(3)}(\beta)=y_{(5)}(\beta)=0, \quad y_{(1)}(\beta)=\frac{s\beta}{2} \tag{12}$$

As already mentioned that the BVP is first transformed into an IVP and afterwards it is solved by bvp4c in MATLAB for distinct values of parameters involved in eqs. (5) and (6), namely Prandtl number, unsteadiness parameter, S, magnetic parameter, M, curvature parameter, C, and Eckert number. The suitable film thickness value is supposed to be the value for which $f(\beta)=(S\beta)/2$ holds and by using this value of β the IVP is solved finally.

Results and discussion

We studied the boundary-layer flow and thermal profile in a fluid film over an unsteadily stretched cylinder and observed the impact of various dimensionless parameters on energy and thermal boundary-layer structure in the existence of external magnetic field. Similarity transformations were utilized to modify the PDE into the dimensionless equations having distinct physical parameters. The numerical solutions were attained by using bvp4c. The drag coefficient, $f''(0)$, and local Nusselt number, $\theta'(0)$, are calculated and then given in the tabular form.

Figure 2 illustrates the curvature's parameter impact on velocity profile $f'(\eta)$ for S = 0.8 and M = 1. Curvature parameter and velocity varied directly. It can be detected from the graph that velocity boosts with the increment of C. It is because the radius of cylinder contracted when the values of C increases causing less interaction with surface area producing less resistance towards the particles of the fluid. Similarly, in fig. 5 the effect of curvature parameter C on temperature portrait for M = 1 and S = 0.8 is shown, that reveals increase in C enhances the velocity hence the kinetic energy that caused an increase in temperature.

The magnetic field on electrically conducting fluid exerts the resistance force known as Lorentz force, it reduces fluid velocity. The applied magnetic field caused the fluid to polarize and form dipoles, this resulted in flow restriction. Figure 3 represents the magnetic parameter, M, effect on velocity portrait $f'(\eta)$ for S = 0.8 and C = 0.5. By increasing magnetic parameter leads to reduction of velocity profile and film thickness of the fluid.

Impact of unsteadiness parameter on velocity and temperature portraits for C = 0.5 and M = 1 is examined in figs. 4 and 6. They depict that increase in S resulted increment in the velocity and temperature fields, as the dependence of S is on α and b. It may be concluded that the stretch velocity is an essential factor to determine the velocity portrait.

In fig. 7 impact of Eckert number on temperature portrait $\theta(\eta)$ for C = 0.5, S = 0.8, M = 1, and Pr = 0.7 is presented. As Eckert number gives the relation between the flow's kinetic energy and the boundary-layer enthalpy difference, so as Eckert number increases the kinetic energy of fluid particles also increases that cause the enhancement of temperature of the fluid.

In fig. 8 impact of Prandtl number on a temperature portrait for M = 1, S = 0.8, C = 0.5, and Ec = 0.5 is shown. It is observed that when Prandtl number is increased the temperature of the liquid declines. As a result the heat transfer rate limits the temperature of the flow, so at higher values of Prandtl number fluid has lower thermal conductivity. As Prandtl number and thermal diffusivity have an inverse relation that's why Prandtl number can be exploited in conducting flows to boost the thermal performance of the liquid. In tab. 1, we provide detailed analysis on the significance of different parameters on skin friction coefficients and Nusselt number.

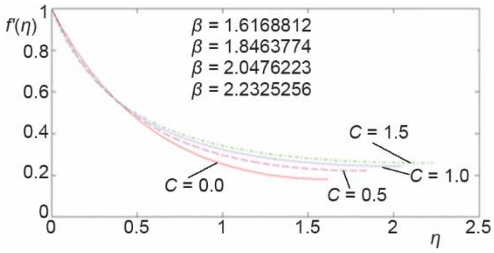

Figure 2. Impact of curvature parameter on velocity portrait $f'(\eta)$, when S = 0.8 and M = 1

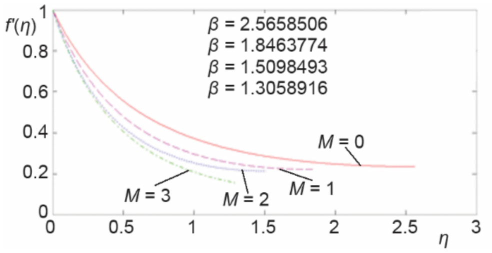

Figure 3. Impact of magnetic parameter on the velocity portrait $f'(\eta)$, when S = 0.8 and C = 0.5

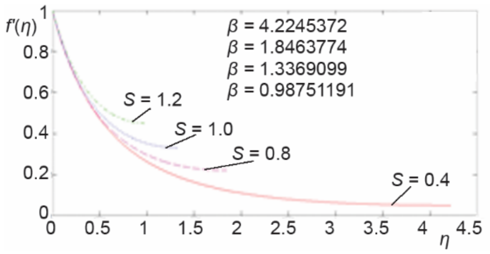

Figure 4. Impact of unsteadiness parameter on velocity portrait $f'(\eta)$, when C = 0.5 and M = 1

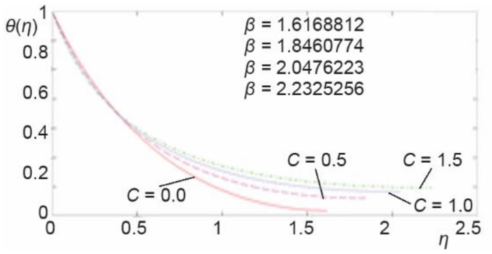

Figure 5. Impact of curvature parameter on temperature portrait $\theta(\eta)$, when S = 0.8 and M = 1

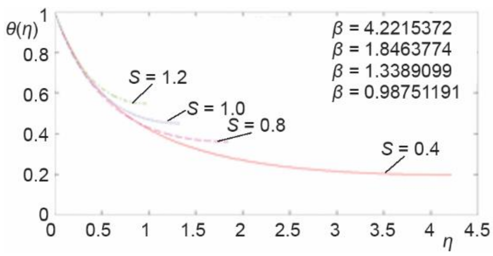

Figure 6. Impact of unsteadiness parameter on temperature portrait $\theta(\eta)$, when C = 0.5 and M = 1

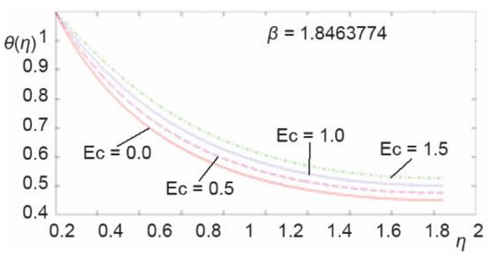

Figure 7. Impact of Eckert number on temperature portrait $\theta(\eta)$, when C = 0.5, S = 0.8, M = 1 and Pr = 0.7

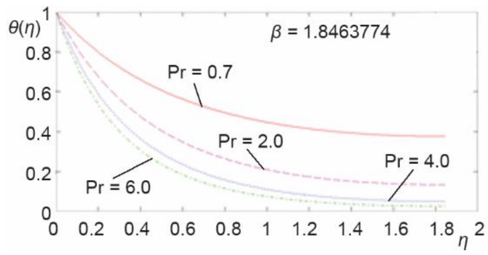

Figure 8. Influence of Prandtl number on temperature portrait $\theta(\eta)$, when S = 0.8, C = 0.5, M = 1 and Ec = 0.5

Table 1. Numerical values of skin friction coefficient and Nusselt number corresponding to various values of the physical parameter

S	M	C	β	Pr	Ec	$\theta'(0)$	$-f''(0)$
0.4	1	0.5	4.2215372	6	0.5	2.8207767	1.7277173
0.8			1.8463774			3.3104618	1.7771620
1.0			1.3389099			3.6047990	1.7617442
0.8	0	0.5	2.5658506			3.7423303	1.4242913
	1		1.8463774			3.3104618	1.7771620
	2		1.5098493			2.9387683	2.0630116
0.8	1	0	1.6168812			3.2908687	1.5893922
		0.5	1.8463774			3.3104618	1.7771620
		1	2.0476223			3.3478826	1.9464131
0.8	1	0.5	1.8463774	2		2.0626625	
				4		2.7902466	
				6		3.3104618	
				6	0	4.5342481	
				4	0.5	3.3104618	
				6	1	2.0866755	

Conclusions

We examined influence of magnetic field on the axisymmetric thin film flow and thermal transport of viscous fluid along a stretchable cylinder. The governing non-linear differential equations of momentum and heat are transformed using the similarity transformation into ODE and then numerically solved by means of bvp4c package. The influence of various dimensionless parameters is examined and graphically represented. Additionally, skin friction and Nusselt numbers are assessed and given in the tabular format. Some of the obtained results are the following.

- Increasing the curvature parameter, an increment is seen in temperature and velocity portraits.
- Rising estimations of unsteadiness parameter portray the increment in velocity and temperature profile and diminishes the fluid thickness.
- Eckert number has a tendency to reduce the Nusselt number.
- The coefficient of skin friction improves because of the increasing estimations of unsteadiness parameter, magnetic parameter, and curvature parameter while Nusselt number increments with the curvature parameter and Prandtl number.

Acknowledgment

Researchers Supporting Project number (RSP-2021/397), King Saud University, Riyadh, Saudi Arabia.

References

[1] Sakiadis, B. C, Boundary-Layer Behavior on Continuous Solid surFaces: I. Boundary-Layer Equations for Two Dimensional and Axisymmetric Flow, *AIChE Journal, 7* (1961), 1, pp. 26-28
[2] Crane, L. J, Flow Past a Stretching Plate, *Z. Angew. Math. Phys., 21 (*1970), 4, pp. 645-647
[3] Carragher, P., Crane, L. J, Heat Transfer on a Continuous Stretching Sheet, ZAMM – *Journal of Applied Mathematics and Mechanics, 62* (1982), 10, pp. 564-565
[4] Wang, C. Y., Liquid Film on an Unsteady Stretching Surface, *Quarterly of Applied Mathematics, 48* (1990), 4, pp. 601-610
[5] Andersson, H. I., MHD Flow of a Viscoelastic Fluid Past a Stretching Surface, *Acta Mechanica, 95* (1992), 1, pp. 227-230
[6] Liu, I. C., Andersson, H. I., Heat Transfer in a Liquid Film on an Unsteady Stretching Surface, *International Journal of Thermal Sciences, 47* (2008), 6, pp. 766-772
[7] Dandapat, B. S., *et al.*, Thermocapillarity in a Liquid Film on an Unsteady Stretching Surface, *International Journal of Heat and Mass Transfer, 46* (2003), 16, pp. 3009-3015
[8] Wang, C., Analytic Solutions for a Liquid Film on an Unsteady Stretching Surface, *Heat and Mass Transfer, 42* (2006), 8, pp. 759-756
[9] Dandapat, B. S., Maity, S., Flow of a Thin Liquid Film on an Unsteady Stretching Sheet, *AIP Physics of Fluid, 18* (2006), 10 ID 102101
[10] Cortell, R., Effects of Viscous Dissipation and Work Done by Deformation on the MHD Flow and Heat Transfer of a Viscoelastic Fluid Over a Stretching Sheet, *Physics Letters A, 357* (2006), 4-5, pp. 298-305

Paper submitted: July 15, 2021
Paper revised: July 28, 2021
Paper accepted: August 2, 2021