# ANALYTICAL SOLUTION OF TANK DRAINAGE FLOW FOR ELECTRICALLY CONDUCTING NEWTONIAN FLUID

by

***Kamran Nazir MEMON[a], Muhammad Kamran ALAM[b*], Jamel BAILI[c,d], Zeeshan NAWAZ[e], Abdul Hanan SHIEKH[f], and Hijaz AHMAD[g]***

[a] Department of Mathematics and Statistics, QUEST, Nawabshah, Pakistan
[b] Department of Pure and Applied Mathematics, The University of Haripur, Haripur, Pakistan
[c] Department of Computer Engineering, College of Computer Science, King Khalid University, Abha, Saudi Arabia
[d] Higher Institute of Applied Science and Technology of Sousse (ISSATS), Cite Taffala (Ibn Khaldoun) Sousse, University of Souse, Souse,Tunisia
[e] SABIC Technology and Innovation, Khraj Highway, Riyadh, Saudi Arabia
[f] Department of Mathematics, CCSIS, Institute of Business Management, Karachi, Pakistan
[g] Section of Mathematics, International Telematic University Uninettuno, Corso Vittorio Emanuele II, Roma, Italy

Original scientific paper
https://doi.org/10.2298/TSCI21S2433M

*The electrically conducting, incompressible and isothermal Newtonian fluid-flow in unsteady tank drainage is studied. The perturbation method is employed to obtain solution and results have been compared with those of obtained by adomian decomposition method result. The results of adomian decomposition method are same as those of perturbation method. The Newtonian fluid solution is worked out with substitution $\varepsilon = 0$. Explicit expressions on behalf of velocity field, flow rate, relationship however will the times vary with length, average velocity and time needed for complete drainage are acquired. Impacts of different developing parameters on velocity profile, $v_z$, flow rate and depth of the tank, $H(t)$, are exhibited graphically.*

Key words*: electrically conducting fluids, Newtonian MHD fluid, analytical solution*

## Introduction

Tank drainage flow under gravitational forces is old and simple yet intricated problem. The flows of draining fluid under force of gravity have great importance, as it frequently appears in various industries. Examples include processing of fluids, immiscible gas applications, dams, waste water management and draining condensate into storage. The tank might be depleted by a simple opening or might be drained throughout a complete channeling framework of horizontanl and/or vertical pipes. The tank has usually a cylindrical form with vertical wall, however base may flat cone shaped hemispherical or other shape is likely. Considering various facts including, precise time and height measurement, friction losses and other end effects, the governing equations model the flow accurately [1, 2]. Some outstanding

* Corresponding author, e-mail: mkalam@uoh.edu.pk

reviews concerning series solutions and exact solutions have been given by Khaskheli *et al.* [3]. The Newtonian, power law and couple stress fluids have been used for tank drainage flow by [4-7] to study and analyze the problem for exact solution.

This research studies Newtonian MHD fluid-flow in a rectangular tank drainage setup. The governing equations are solved analytically, by perturbation using parameter $\varepsilon = 0$. The obtained solution leads to analysis of the various profiles and parameters, including fluid velocity, fluid-flow rate, instantaneous relation between fluid in tank and time. To best of knowledge, literature does not report analytical solution of the problem.

## Governing equations of fluid-flow

Setting aside the thermal effects, the incompressible viscous fluid-flow is modeled by the equations $\nabla V = 0$, and $\rho \mathrm{D}V/\mathrm{D}t = -\nabla p + \rho b + \nabla T + (J \times B)$. The symbol $\rho$ represents density, considered constant thoroughly, $V$ be the velocity, $p$ stand for the dynamic pressure, $b$ represents the body force, $T$ is the extra stress tensor, and D/D$t$ defines material derivatives. As a result Lorentz force per unit volume be $J \times B = [0, 0, -\sigma B_0^2 v_z]$, where $\sigma$ is the electrical conductivity, $B = [0, 0, B_0]$ be the uniform magnetic field, here $B_0$ be the magnetic field, as applied, and $J$ be the current density $J$, which is $J = \sigma\left[E + V \times B\right]$, and $\nabla \times B = \mu_0 J$.

Here $E$ is the electric field which is not considered in this study and $\mu_0$ be the magnetic permeability. The extra stress tensor defining a Newtonain fluid is specified by $T = \mu A_1$, here $A_1$ be the first Rivlin-Ericksen tensor identified as $A_1 = \nabla V + (\nabla V)^T$.

## Tank drainage flow

A cylindrical tank, as depicted in fig. 1 and is described in [3], is considered with radius, $R_T$, and diameter, $D$. The tank contains an isothermal, incompressible electrically conducting Newtonian fluid till height, $H_0$. The radius and length of the pipe, causing drainage under gravity, is $R$ and $L$. Height of the fluid in tank is function of time and is denoted by $H(t)$. Flow is driven by pressure of the fluid and gravity. The solution is aimed at determining the shear stress on the walls of pipe and flow rate.

The governing equation describing flow is:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) - \frac{\sigma B_0^2}{\eta} v_z(r) = -\frac{\rho \mathrm{g}}{\eta}\left[\frac{H(t)}{L} + 1\right] \tag{1}$$

The related boundary conditions are:

$$\text{at} \quad r = 0, \quad \frac{\partial v_z}{\mathrm{d}r} = 0, \quad \text{free space boundary condition} \tag{2}$$

$$\text{at} \quad r = R, \quad v_z = 0 \quad \text{no-slip boundary condition} \tag{3}$$

For details of derivation of equations, reader is referred to [1, 3].

## Adomian decomposition method

First, we use adomian decomposition method (ADM) to solve above second order differential equation. Let us define:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) = L$$

such that $L$ is invertible operator and is given as:

$$L^{-1}(*) = \int r^{-1}\left[\int r(*)\mathrm{d}r\right]\mathrm{d}r$$

In operator form, can be written:

$$Lv_z = \frac{\sigma B_0^2}{\eta} v_z(r) - \frac{\rho \mathrm{g}}{\eta}\left[\frac{H(t)}{L} + 1\right] \tag{4}$$

Applying $L^{-1}$on both sides, we get:

$$v_z = -\frac{\rho \mathrm{g}}{4\eta}\left[\frac{H(t)}{L} + 1\right] r^2 + A\ln r + B + \frac{\sigma B_0^2}{\eta} L^{-1} v_z \tag{5}$$

By Adomian substitution:

$$v_z(r) = \sum_{n=0}^{\infty} v_n \tag{6}$$

Using in eq. (6) in (5), we get the following solution:

$$v_0 = -\frac{\rho \mathrm{g}}{4\eta}\left[\frac{H(t)}{L} + 1\right] r^2 + A\ln r + B \tag{7}$$

Similarly, next term we can find by utilizing recursive relation:

$$v_i = \frac{\sigma B_0^2}{\eta} L^{-1} v_{i-1}, \quad i \ge 1 \tag{8}$$

Using solution (6) in the boundary conditions (2) and (3), we get:

$$\text{at} \quad r = R, \quad v_0 = v_1 = v_2 = v_3 = ... = 0 \tag{9}$$

$$\text{at} \quad r = 0 \quad \frac{\partial v_0}{\mathrm{d}r} = \frac{\partial v_1}{\mathrm{d}r} = \frac{\partial v_2}{\mathrm{d}r} = \frac{\partial v_3}{\mathrm{d}r} = ... = 0 \tag{10}$$

Substitute the solution of velocity obtained by solving zeroth order from eq. (7), first order for selecting $i$ = 1 in (8) and for second order take $i$ = 2 with related conditions in eq. (6), the considerable calculations is:

$$v_z = \frac{\rho \mathrm{g}}{4\eta}\left[\frac{H(t)}{L} + 1\right]\left[(R^2 - r^2) + \frac{\sigma B_0^2}{16\eta}(4r^2R^2 - r^4 - 3R^4) + \right.$$
$$\left. + \frac{\sigma^2 B_0^4}{576\eta^2}(9r^4R^2 - r^6 - 27r^2R^4 + 19R^6)\right] \tag{11}$$

## Solution by perturbation method

Consider $\varepsilon = (\sigma B_0^2)/\eta$, where $\varepsilon$ is supposed as a small parameter. The velocity profile $v_z(r, \varepsilon)$, in terms of power, can be specified:

$$v_z(r, \varepsilon) \approx v_0(r) + \varepsilon v_1(r) + \varepsilon^2 v_2(r) + .... \tag{12}$$

By utilizing eq. (12) into eqs. (1) to (3) and collecting the coefficients of similar powers of $\varepsilon$, we end up with $0^{th}$, $1^{st}$, and $2^{nd}$ order problem, given below, along with boundary conditions:

$$\varepsilon^0 : \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}v_0}{\mathrm{d}r}\right) = -\frac{\rho \mathrm{g}}{\eta}\left[\frac{H(t)}{L}+1\right] \tag{13}$$

with related boundary conditions, $(\mathrm{d}v_0/\mathrm{d}r)=0$ at $r=0$, and $v_0=0$ at $r=R$:

$$\varepsilon^1 : \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}v_1}{\mathrm{d}r}\right) - v_0 = 0 \tag{14}$$

with conditions, $(\mathrm{d}v_1/\mathrm{d}r)=0$ at $r=0$, and $v_1=0$ at $r=R$, and:

$$\varepsilon^2 : \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}v_2}{\mathrm{d}r}\right) - v_1 = 0 \tag{15}$$

with associated conditions, $(\mathrm{d}v_2/\mathrm{d}r)=0$ at $r=0$, and $v_2=0$ at $r=R$.

Substitute the solution of velocity obtained by solving $0^{th}$ order, $1^{st}$ order, $2^{nd}$ order with related conditions in eq. (12), the considerable calculations is:

$$v_z = \frac{\rho \mathrm{g}}{4\eta}\left[\frac{H(t)}{L}+1\right]\left[(R^2-r^2)+\frac{\varepsilon}{16}(4r^2R^2-r^4-3R^4)+ + \frac{\varepsilon^2}{576}(9r^4R^2-r^6-27r^2R^4+19R^6)\right] \tag{16}$$

It is called attention to that on the off chance that the perturbation parameter is set to be $\varepsilon = 0$ in eq. (16), we recuperate the answer for the similar problem with Newtonian fluid, which is presented in [3].

## Flow rate, average velocity, and time-depth relation

The flow rate expression, per unit width is denoted by $Q$, average velocity, $\bar{V}$ , and mass balance over the entire tank formula is given in [1, 5, 7] by using these formulas in after using (26), we obtain:

$$Q = \frac{\rho \mathrm{g}\pi}{8\eta}\left[\frac{H(t)}{L}+1\right]\left(R^4-\frac{\varepsilon}{6}R^6+\frac{33\varepsilon^2}{1152}R^8\right) \tag{17}$$

$$\bar{V} = \frac{\rho \mathrm{g}}{8\eta}\left[\frac{H(t)}{L}+1\right]\left(R^2-\frac{\varepsilon}{6}R^4+\frac{33\varepsilon^2}{1152}R^6\right) \tag{18}$$

$$H(t) = e^{\frac{-\rho \mathrm{g}t}{8\eta R_t^2 L}\left(R^4-\frac{\varepsilon}{6}R^6+\frac{33\varepsilon^2}{1152}R^8\right)}(H_0+L)-L \tag{19}$$

$$t = \frac{-8\eta R_T^2 L}{\rho g\left(R^4 - \frac{\varepsilon}{6}R^6 + \frac{33\varepsilon^2}{1152}R^8\right)}\ln\left(\frac{H(t)+L}{H_0+L}\right) \tag{20}$$

Remark: Time of efflux (Time required for complete drainage) is obtained by taking $H(t) = 0$ in eq. (19). Taking $\varepsilon = 0$ in eq. (20), the solution is worked out [1].

## Discussion on results

In this study, unsteady flow an isothermal, incompressible and electrically conducting Newtonian fluid is considered under gravity and hydrostatic pressure, in a tank drainage setup. The various influencing parameters have been studied, which affect the characteristics of fluid-flow, *i.e.* velocity profile, $v_z$, flow rate, $Q$, and depth, $H(t)$. Figures 1-6 represent the velocity profile for various parameter settings. The relationship between effects of the electrical conductivity, $\sigma$, and velocity, applied magnetic field, $B_0$, and velocity, density, $\rho$, and velocity, velocity and dynamic viscosity, $\eta$, and velocity are depicted in figs. 1-4, respectively. The flow rate against various depths, $H(t)$, is presented in fig. 5, which shows higher flow rate near walls of the pipe. Figure 6 explains the fact that larger the radius of the tank, the depth will increase. From here, it can be concluded that larger the radius of the tank it will take more time to empty the tank completely.

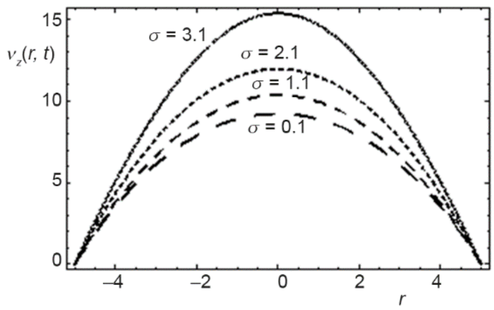


**Figure 1. Velocity profile for different $\sigma$, when $\eta$ = 11.5 poise, $\rho$ = 0.78 g/cm$^3$, $R$ = 5 cm, $L$ = 10 cm, $B_0$ = 1, and $H(t)$ = 20 cm**

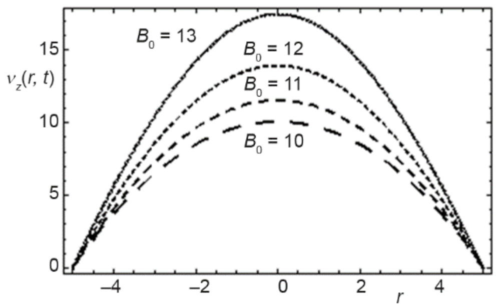


**Figure 2. Effect of $B_0$ on velocity profile, when $\eta$ = 11.5 poise, $\rho$ = 0.78 g/cm$^3$, $R$ = 5 cm, $L$ = 10 cm, $\sigma$ = 0.02, and $H(t)$ = 20 cm**

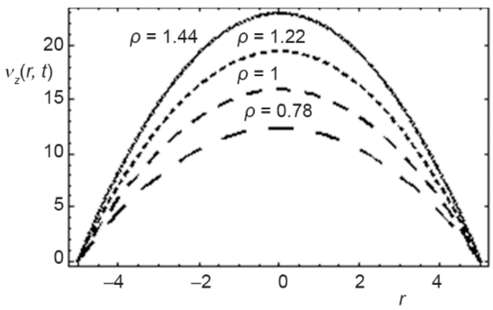


**Figure 3. Velocity profile for different values of $\rho$, when $\eta$ = 11.5 poise, $R$ = 5 cm, $L$ = 10 cm, $H(t)$ = 20 cm, $\sigma$ = 0.01, and $B_0$ = 0.25**

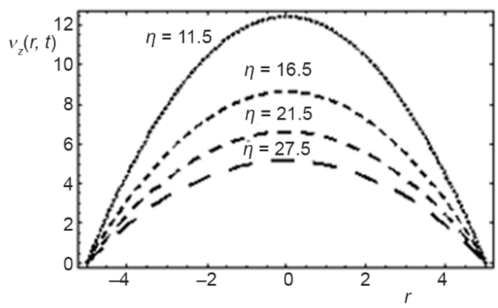


**Figure 4. Velocity profile $\eta$, when $\rho$ = 0.78 g/cm$^3$, $R$ = 5 cm, $L$ = 10 cm, $H(t)$ = 20 cm, $\sigma$ = 0.1, and $B_0$ = 0.25**

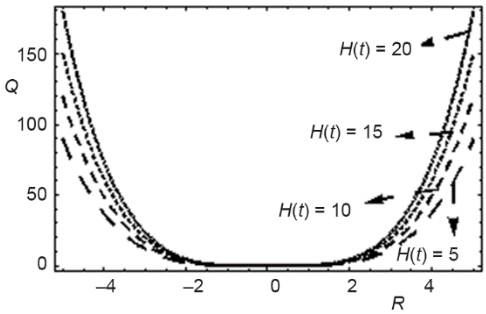


**Figure 5. Flow rate against $H(t)$, when $\eta = 31.5$ poise, $\rho = 0.78$ g/cm$^3$, $L = 10$ cm, $\sigma = 0.1$, and $B_0 = 0.25$**

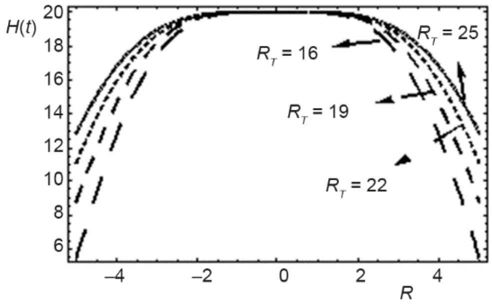


**Figure 6. Effect of $R_T$ on depth w.r.t "$R$", when $\eta = 0.6$ poise, $t = 1$, $H_0 = 20$ cm, $L = 10$ cm, $\rho = 1.38$ g/cm$^3$, $\sigma = 0.1$, and $B_0 = 0.25$**

The results summarize that the electrical conductivity, $\sigma$, applied magnetic field, $B_0$, depth, $H(t)$, pipe radius, $R$, density, $\rho$, are directly proportional to velocity, where length of the pipe, $L$, and dynamic viscosity, $\eta$, are inversely proportional to velocity. An increase in earlier parameters leads to increase in velocity, whereas decrease in later parameters decreases veclocity.

## Concluding remarks

The exact solution, aiming at analyzing velocity under gravity and hydrostatic force, was obtained from governing equations. These equations are used to model unsteady, incompressible, isothermal fluid draining through pipe attached in tank drainage under force of gravity and pressure. Here it is noted that for the perturbation parameter $\varepsilon = 0$, solution of the problem reduces to the Newtonian solution without MHD [4]. In equation (19), the depth of the fluid at various time instants is expressed. This is inevitable to mention that lower the dynamic viscosity, higher the velocity of the fluid, and fluid will take less time in complete drainage as compare to thicker.

## Acknowledgements

The Authors extend their thanks to the Deanship of Scientific Research at King Khalid University for funding this work through the big research groups under grant number RGP. 1/155/42.

## Funding

This research work was supported by the Deanship of Scientific Research at King Khalid University under Grant number RGP. 1/155/42.

## References

[1] Memon, K. N., The Hydrodynamics of Gravity Vessel Drainage of Non-Newtonian liquids, Ph. D. thesis, Mehran University of Engineering and Technology, Jamshoro, Pakistan, 2018
[2] Shah, S. M., *et al.*, Exact Solution for PTT Fluid on a Vertical Moving Belt for Lift with Slip Condition, *Indian Journal of Science and Technology, 12* (2019), 30, pp. 1-9
[3] Khaskheli, M. A., *et al.*, Tank Drainage for an Electrically Conducting Newtonian Fluid with the use of the Bessel Function, *Eng. Technol. Appl. Sci. Res., 10* (2020), 2, pp. 5377-5381

[4] Hrymak, A. N., Applied Fluid Mechanics by T. C. Papanastasiou, Prentice-Hall, Englewood Cliffs, NJ, 1994, 520 pp., *AIChE Journal*, *41* (1995), 10, pp. 2342-2342
[5] Memon, K. N., *et al.*, Exact Solution of Tank Drainage for Newtonian Fluid with Slip Condition, *Sindh Uni. Res. Jour.*, *49* (2017), 2, p. 5
[6] Bird, R. B., *et al.*, Transport phenomena, *John Wiley and Sons*, *7* (1961), 2, pp. 5J-6J
[7] Memon, K. N., *et al.*, Exact Solution of Unsteady Tank Drainage for Ellis Fluid, *Journal of Applied Fluid Mechanics*, *11* (2018), 6, pp. 1629-1636

Paper submitted: July 5, 2021
Paper revised: July 23, 2021
Paper accepted: August 7, 2021