MATHEMATICAL FRACTIONAL MODELING OF TRANSPOT PHENOMENA OF VISCOUS FLUID-FLOW BETWEEN TWO PLATES

by

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This work is about the mass and heat transfer flow for adhesive fluid between two upright plates pulled apart by a distance, d. Fractional model of the considered problem is developed after making governing equations dimensionless. Laplace transform technique is utilized to acquire analytical solutions and some graphics are presented to see the physical behavior of embedded parameters.

Key words: *MHD, viscous fluid, fractional modeling, heat and mass transfer*

Introduction

In recent time, a major role has been played by fractional dynamical equations in modeling of non-typical behavior and memory effect that are familiar features of natural happening. For precise modeling of those systems that require precise modeling of damping, the fractional derivatives models are used. In the last few years, a momentous role has been played by fractional calculus in different fields such as mechanics, electricity, chemistry, biology, and economics, heat transfer, nanotechnology Miller and Ross [1]. The most recent developments of fractional calculus in the application point of view can be seen in the following references [2-5]. Baleanu, *et al.* [6] introduce a new fractional parameter known as Constant proportional Caputo. Also provide its definition and properties. A fractional model obtained through generalized Fourier and Ficks laws for heat transfer problem for vertical channel and solutions obtained by means of Laplace transform method.

Mathematical methodology

Consider the unsteady free convection of a dense viscous fluid trickling between two upright parallel plates with persistent concentration and temperature slope. The x-axis is taken along one of the plate which is fastened in perpendicularly upwards direction and y-axis is normal to the T_d and concentration C_d. When the value of time, t, is greater than zero and y is equal to zero, T_w and C_w are the value of temperature and concentration respectively, producing

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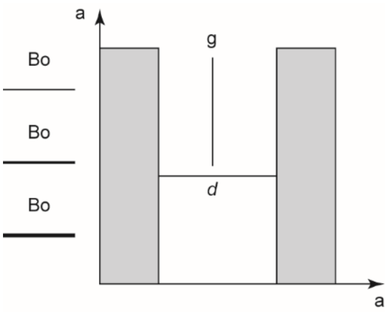

Figure 1. Geometry of the problem

the flow of free convection streams as shown in the fig. 1. In accordance with regular Boussinesq's approximation the dominating equations for unsteady flow.

$$\frac{\partial U(t,y)}{\partial t} = \upsilon \frac{\partial^2 U(t,y)}{\partial y^2} + \text{g}\beta_1 (T - T_d) + + \text{g}\beta_2 (C - C_d) - \sigma \frac{\beta_0}{\rho} U(t,y) \tag{1}$$

$$\rho C_p \frac{\partial T(t,y)}{\partial t} + \frac{\partial q}{\partial y} = 0 \tag{2}$$

$$q(t,y) - k \, {}^{\text{CPC}}D_t^{\beta} \frac{\partial T(t,y)}{\partial y} = 0, \quad 1 \geq \beta \geq 0 \tag{3}$$

$$\frac{C(t,y)}{\partial t} + D \frac{\partial j}{\partial y} = 0 \tag{4}$$

$$J(t,y) + \text{D}_n \, {}^{\text{CPC}}\text{D}_t^{\gamma} \frac{\partial C(t,y)}{\partial y} = 0, \quad 1 < \gamma \leq 0 \tag{5}$$

Suitable initial and boundary conditions are:

$$0 \geq t : U(t,y) = 0, \quad T(t,y) = T_d, \quad C(t,y) = C_d, \quad d \geq y \geq 0 \tag{6}$$

$$0 < t : U(t,y) = 0, \quad T(t,y) = T_w, \quad C(t,y) = C_w, \quad \text{at} \quad y = 0 \tag{7}$$

$$0 < t : U(t,y) = 0, \quad T(t,y) = T_d, \quad C(t,y) = C_d, \quad \text{at} \quad y = d \tag{8}$$

Problem solving with constant proportional Caputo time fraction derivative

The given set of variables is being introduced to construct the problem dimensionless:

$$y^* = \frac{y}{d}, \quad t^* = \frac{\upsilon t}{d^2}, \quad u^* = \frac{\upsilon U}{d^2 \text{g}\beta_1 (T_w - T_d)}, \quad \theta = \frac{T - T_d}{T_w - T_d}, \quad C^* = \frac{C - C_d}{C_w - C_d}$$
$$q^* = \frac{q}{q_0}, \quad j^* = \frac{j}{j_0}, \quad M = \frac{\sigma \beta_0^2 d^2}{\nu \rho}, \quad \text{Gr} = \frac{\text{g}\beta_1 (T_w - T_d) d^3}{\nu^2} \tag{9}$$
$$Gm = \frac{\text{g}\beta_2 (C_w - C_d) d^3}{\nu^2}, \quad N = \frac{Gm}{\text{Gr}}, \quad \text{Pr} = \frac{\mu C_p}{K}, \quad Sc = \frac{\nu}{D}$$

Momentum eq. (1) in dimensionless form by utilizing non-dimensional variables given in eq. (9):

$$\frac{\partial U(t,y)}{\partial t} = \frac{\partial^2 U(t,y)}{\partial y^2} + \theta(t,y) + NC(t,y) - MU(t,y) \tag{10}$$

The general fractional model for energy and diffusion equations [7] and Henery *et al.* [8]:

$$\frac{\partial T(t,y)}{\partial t}-\frac{1}{\mathrm{Pr}}\,{}^{\mathrm{CPC}}\mathrm{D}_t^{\beta}\,\frac{\partial^2 C(t,y)}{\partial y^2}=0 \tag{11}$$

where Pr is the dimensionless Pradndtl number:

$$\frac{\partial C(t,y)}{\partial t}-\frac{1}{\mathrm{Sc}}\,{}^{\mathrm{CPC}}\mathrm{D}_t^{\gamma}\,\frac{\partial^2 C(t,y)}{\partial y^2}=0 \tag{12}$$

where Sc the dimensionless Schmidt number and conditions are

$$0\ge t: U(t,y)=\theta(t,y)=C(t,y)=0,\quad 1\ge y\ge 0 \tag{13}$$

$$0<t: U(t,y)=0,\quad \theta(t,y)=C(t,y)=1,\quad \text{at}\quad y=0 \tag{14}$$

$$0<t: U(t,y)=\theta(t,y)=C(t,y)=0,\quad \text{at}\quad y=1 \tag{15}$$

Temperature field

Applying Laplace transform on eq. (11) with conditions $(13)_2$ and $(15)_2$, solution for temperature obtained as:

$$\begin{aligned}\theta(t,y)=1&+\sum_{k=0}^{\infty}\sum_{i_1=1}^{\infty}\sum_{j_1=1}^{\infty}\frac{(-a\sqrt{\mathrm{Pr}})^{i_1}(-1)^{j_1}[k_1(\beta)]^{j_1}t^{\frac{\beta i_1}{2}-\frac{i_1}{2}+j_1}\Gamma\left(\frac{i_1}{2}+j_1\right)}{i_1!\,j_1![k_0(\beta)]^{\frac{i_1}{2}+j_1}\Gamma\left(\frac{i_1}{2}\right)\Gamma\left(1+\frac{\beta i_1}{2}-\frac{i_1}{2}+j_1\right)}-\\&-\sum_{k=0}^{\infty}\sum_{i_2=1}^{\infty}\sum_{j_2=1}^{\infty}\frac{(-b\sqrt{\mathrm{Pr}})^{i_2}(-1)^{j_2}[k_1(\beta)]^{j_2}t^{\frac{\beta i_2}{2}-\frac{i_2}{2}+j_2}\Gamma\left(\frac{i_2}{2}+j_2\right)}{i_2!\,j_2![k_0(\beta)]^{\frac{i_2}{2}+j_2}\Gamma\left(1+\frac{\beta i_2}{2}-\frac{i_2}{2}+j_2\right)\Gamma\left(\frac{i_2}{2}\right)}\end{aligned} \tag{16}$$

Concentration field

Appling Laplace on eq. (12) with conditions $(13)_3$ and $(15)_3$, solution for concentration:

$$\begin{aligned}C(t,y)=1&+\sum_{l=0}^{\infty}\sum_{i_3=1}^{\infty}\sum_{j_3=1}^{\infty}\frac{(-c\sqrt{\mathrm{Sc}})^{i_3}(-1)^{j_3}[k_1(\gamma)]^{j_1}t^{\frac{\gamma i_3}{2}-\frac{i_3}{2}+j_3}\Gamma\left(\frac{i_3}{2}+j_3\right)}{i_3!\,j_3![k_0(\gamma)]^{\frac{i_3}{2}+j_3}\Gamma(1+\frac{\gamma i_3}{2}-\frac{i_3}{2}+j_3)\Gamma\left(\frac{i_3}{2}\right)}-\\&-\sum_{l=0}^{\infty}\sum_{i_4=1}^{\infty}\sum_{j_3=1}^{\infty}\frac{(-d\sqrt{\mathrm{Pr}})^{i_2}(-1)^{j_4}[k_1(\gamma)]^{j_4}t^{\frac{\gamma i_2}{2}-\frac{i_4}{2}+j_4}\Gamma\left(\frac{i_4}{2}+j_4\right)}{i_4!\,j_4![k_0(\gamma)]^{\frac{i_4}{2}+j_4}\Gamma\left(1+\frac{\gamma i_2}{2}-\frac{i_4}{2}+j_4\right)\Gamma\left(\frac{i_4}{2}\right)}\end{aligned} \tag{17}$$

Velocity field

Appling Laplace transform on eq. (10) with conditions $(13)_1$ and $(14)_1$, we will get:

$$\bar{U}(s,y)=\frac{1}{s}\sum_{k=0}^{\infty}\frac{e^{-2k\sqrt{\frac{\Pr s}{(\frac{k_1(\beta)}{s}+k_0(\beta))s^{\beta}}}}-e^{-(2k+2)\sqrt{\frac{\Pr s}{\left[\frac{k_1(\beta)}{s}+k_0(\beta)\right]s^{\beta}}}}}{\frac{\Pr s}{\left[\frac{k_1(\beta)}{s}+k_0(\beta)\right]s^{\beta}}-(s+M)}\left[\frac{\sin h(1-y)\sqrt{s+M}}{\sin h\sqrt{s+M}}\right]+$$

$$+\frac{N}{s}\sum_{l=0}^{\infty}\frac{e^{-2l\sqrt{\frac{\mathrm{Sc}\, s}{\left[\frac{k_1(\gamma)}{s}+k_0(\gamma)\right]s^{\gamma}}}}-e^{-(2l+2)\sqrt{\frac{\mathrm{Sc}\, s}{\left[\frac{k_1(\gamma)}{s}+k_0(\gamma)\right]s^{\gamma}}}}}{\frac{\mathrm{Sc}s}{\left[\frac{k_1(\gamma)}{s}+k_0(\gamma)\right]s^{\gamma}}-(s+M)}\left[\frac{\sin h(1-y)\sqrt{s+M}}{\sin h\sqrt{s+M}}\right]- \tag{18}$$

$$-\frac{1}{s}\sum_{k=0}^{\infty}\frac{e^{-a\sqrt{\frac{\Pr s}{\left[\frac{k_1(\beta)}{s}+k_0(\beta)\right]s^{\beta}}}}-e^{-b\sqrt{\frac{\Pr s}{\left[\frac{k_1(\beta)}{s}+k_0(\beta)\right]s^{\beta}}}}}{\frac{\Pr s}{\left[\frac{k_1(\beta)}{s}+k_0(\beta)\right]s^{\beta}}-(s+M)}+\frac{N}{s}\sum_{l=0}^{\infty}\frac{e^{-c\sqrt{\frac{\mathrm{Sc}\, s}{\left[\frac{k_1(\gamma)}{s}+k_0(\gamma)\right]s^{\gamma}}}}-e^{-d\sqrt{\frac{\mathrm{Sc}\, s}{\left[\frac{k_1(\gamma)}{s}+k_0(\gamma)\right]s^{\gamma}}}}}{\frac{Scs}{\left[\frac{k_1(\gamma)}{s}+k_0(\gamma)\right]s^{\gamma}}-(s+M)}$$

The inverse Laplace transform can be obtained numerically.

Graphical outcomes

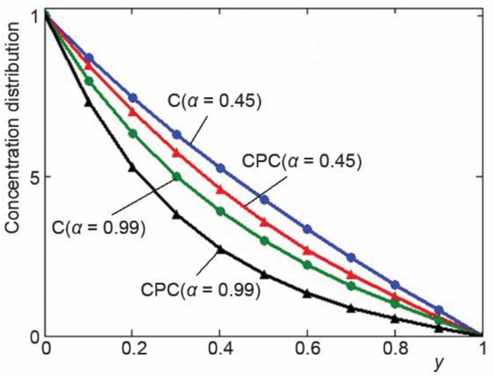

Figure 2. Comparison between results obtained by (C) and (CPC) fractional operators for concentration field when t = 4, Sc = 5, $k_1(\gamma)$ = 0.04, and $k_0(\gamma)$ = 0.3

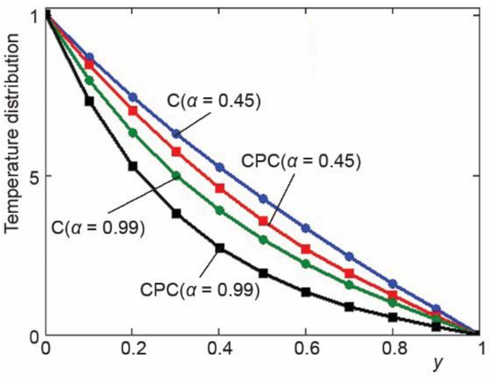

Figure 3. Comparison between results obtained by (C) and (CPC) fractional operators for temperature field when t = 4, Pr =5, $k_1(\beta)$ = 0.4, and $k_0(\beta)$ = 0.3

Conclusions

To observe the consequence of fractional parameters on concentration, temperature, and velocity fields with comparison of different fractional operators (CPC) and (C), some graphs has been plotted and as result key points are obtained as follows.

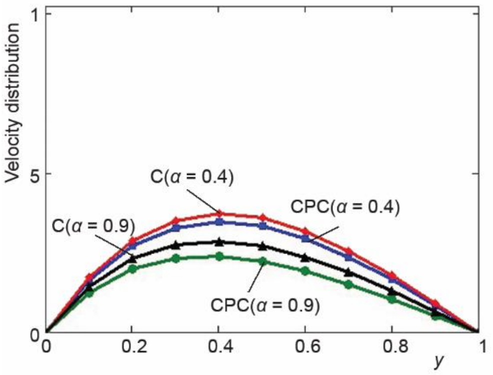

Figure 4. Comparison between results obtained by (C) and (CPC) fractional operators for velocity field when t = 9, Pr = 6.2, Sc = 6.2, $k_1(\gamma)$ = 0.02, $k_0(\gamma)$ = 0.6, M = 2, and N = 6

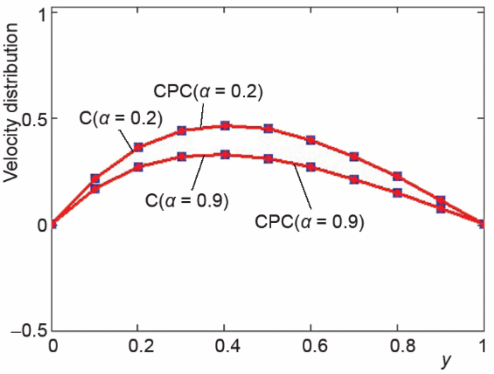

Figure 5. Comparison between results obtained by (C) and (CPC) fractional operators for velocity field when t = 9, Pr = 6.2, Sc = 6.2, $k_1(\gamma)$ = 0, $k_0(\gamma)$ = 1, M = 2, and N = 6

- It is concluded from the figs. 2-4 constant proportional fractional operator can play the role in decaying the fluid properties like temperature, concentration and velocity in efficient way for different fractional parameter values.
- It is concluded that (CPC) and (C) coincide each other while comparing with [5] when $k_1(\beta) = 0$, $k_1(\beta) = 1$. It is due to the fact that by taking this limit on constant proportional Caputo, it reduces to Caputo only. This fact can be observed for velocity field and presented in fig. 5.

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