FIVE–POINT THIRTY–TWO OPTIMAL ORDER ITERATIVE METHOD FOR SOLVING NON-LINEAR EQUATIONS

by

Malik Zaka ULLAH[a] and Fayyaz AHMAD[b*]

[a] Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia
[b] Department of Mathematics, School of Science, National Textile University, Faisalabad, Pakistan

Original scientific paper
https://doi.org/10.2298/TSCI21S2401U

A five-point thirty-two convergence order derivative-free iterative method to find simple roots of non-linear equations is constructed. Six function evaluations are performed to achieve optimal convergence order $2^{6-1} = 32$ *conjectured by Kung and Traub [1]. Secant approximation to the derivative is computed around the initial guess. High order convergence is attained by constructing polynomials of quotients for functional values.*

Key words: *non-linear equations, derivative-free, iterative method, simple roots, optimal order of convergence, Kung-Trub conjecture*

Introduction

Non-linear problems are unavoidable in science and technology because the majority of phenomena in nature have non-linear behavior. Non-linear equations belong to non-linear problems, and their analytical solution is hard to find in general. Due to this limitation, iterative methods are good candidates to find the approximation solution of non-linear equations. The effectiveness of an iterative method for computing the solution of non-linear equations can be measured by the Ostrowski efficiency index [2]. Optimal efficiency index can be obtained in Kung and Traub conjecture (K-T) [1]. According to the K-T conjecture, an iterative method to find a simple root of non-linear equations is optimal if it reaches an order of convergence $p = 2^{n-1}$, where n is the number of functional evaluations per cycle.

The classical Newton's method (NM) has quadratic convergence to find the simple root, and it attains the optimal convergence order in the sense of K-T conjecture. In NM, we use two functional evaluations and hence obtain optimal convergence order $2^{2-1} = 2$. Quadratic convergence means when we are in the vicinity of the root, the error in the approximation of root at step k is the square of the error in the approximation of root at step $k + 1$. Generally, we observe very fact convergence in this case. But when we deal with a non-linear equation, it is hard to compute an explicit derivative expression for the derivative then NM is out of the question. In such a situation, we are looking for derivative-free optimal convergence order methods. Steffensen's method (SM) [3] is a derivative-free optimal convergence method that also offers quadratic convergence. Fourth-order derivative-free optimal order four iterative is established in [4, 5]. Other Steffensen type optimal order iterative

* Corresponding author, e-mail: fayyaz.ahmad@ntu.edu.pk

methods for finding simple roots can be found [6-8]. Authors use interpolation polynomial to enhance the convergence order. Cordero *et al.* [9] constructed a technique to obtaining derivative-free methods with optimal convergence order in the sense of K-T conjecture. In the aforementioned article, the authors provided Steffensen type methods of optimal order four, eight, and sixteen using Padé approximation. Interested readers can consult articles [9-11] and references in them.

Thirty-two order optimal method

We constructed a five-point optimal convergence order thirty-two iterative method for computing simple roots of smooth non-linear equations. Six function evaluations are used to achieve the optimal convergence order in the sense of K-T conjecture. Let f be a smooth non-linear function and α be a simple root of $f(x) = 0$, *i.e.*, $f(\alpha) = 0$, and $f'(\alpha) \neq 0$. Our proposed optimal order iterative method is FM_{32}:

$$\mathrm{FM}_{32} = \begin{cases} \text{Initialguess} = x \\ w = x + a f_x \\ \mathrm{d}f = \dfrac{f_w - f_x}{w - x} \\ y_1 = x - \dfrac{f_x}{\mathrm{d}f} & \text{Convergence order} = 2, \quad \text{Number of function evaluations} = 2 \\ y_2 = y_1 - p_2 \dfrac{f_{y_1}}{\mathrm{d}f} & \text{Convergence order} = 4, \quad \text{Number of function evaluations} = 3 \\ y_3 = y_2 - p_3 \dfrac{f_{y_2}}{\mathrm{d}f} & \text{Convergence order} = 8, \quad \text{Number of function evaluations} = 4 \\ y_4 = y_3 - p_4 \dfrac{f_{y_3}}{\mathrm{d}f} & \text{Convergence order} = 16, \quad \text{Number of function evaluations} = 5 \\ y_5 = y_4 - p_5 \dfrac{f_{y_4}}{\mathrm{d}f} & \text{Convergence order} = 32, \quad \text{Number of function evaluations} = 6 \\ x = y_5 \end{cases}$$

where $f_x = f(x)$, $f_w = f(w)$, $f_{y_1} = f(y_1)$, $f_{y_2} = f(y_2)$, $f_{y_3} = f(y_3)$, $f_{y_4} = f(y_4)$, $t_0 = f_{y_1}/f_w$, $t_1 = f_{y_1}/f_x$, $t_2 = f_{y_2}/f_w$, $t_3 = f_{y_2}/f_x$, $t_4 = f_{y_2}/f_{y_1}$, $t_5 = f_{y_3}/f_w$, $t_6 = f_{y_3}/f_x$, $t_7 = f_{y_3}/f_{y_1}$, $t_8 = f_{y_3}/f_{y_2}$, $t_9 = f_{y_4}/f_w$, $t_{10} = f_{y_4}/f_x$, $t_{11} = f_{y_4}/f_{y_1}$, $t_{12} = f_{y_4}/f_{y_2}$, $t_{13} = f_{y_4}/f_{y_3}$, $B = 1/(1 + a\mathrm{d}f)$ and polynomial of t_i's are:

$$p_2 = 1 + t_0 + t_1$$

$$p_3 = 1 + t_0 + t_1 + 2t_2 + 2t_3 + t_4 + t_0 t_1 - t_0^3 - t_0^2 t_1 - t_0 t_1^2 - t_1^3$$

$$p_4 = 1 + (t_1^5 t_4 - t_1^4 t_4 - 2t_1^3 t_4^2 - t_1^3 t_4 - t_1^3 t_8 + t_1^2 t_4^2 - 2t_1 t_4^3 - t_1^3 + t_1^2 t_4 + t_1^2 t_8 + t_1 t_4^2 + t_1^2 + t_1 t_8 + t_1 +$$

$$2t_3 + 4t_6)B + (t_1^5 t_4 - t_1^4 t_4 - 2t_1^3 t_4^2 - t_1^3 t_4 - t_1^3 t_8 - t_1^3)B^2 + (t_1^5 t_4 - t_1^4 t_4 - t_1^3 t_4^2 - t_1^3 t_4 - t_1^3 t_8 - t_1^3)B^3 +$$

$$+t_1+2t_3+t_4+4t_6+2t_7+t_8+t_8t_1-t_1^3+t_4^2t_1-t_4^3-t_4t_1^3-t_8t_1^3-2t_4^3t_1-$$

$$-t_4^2t_1^3+t_4t_1^5+B^4t_4t_1^5+B^5t_4t_1^5$$

$$p_5=1-t_1^7t_4^2t_8+t_1^5t_4^2t_8+2t_1^3t_4^4t_8+2t_1t_4^5t_8-t_1t_4^4t_8-3t_1t_4^3t_8^2-2t_1t_{13}t_4^3-2t_1t_4^3t_8-t_8t_4^2t_1^7B^6+t_8t_1+$$

$$+t_4^2t_1-t_4t_1^3-t_8t_1^3+t_4t_1^5-2t_4^3t_1-t_4^2t_1^3-t_8t_4t_1^3-t_8t_4^2t_1^3+t_{13}t_4t_1^5-2t_{13}t_8t_1^3-t_1^3+t_8+t_4-$$

$$t_8t_4^2t_1^7B^7+8t_1t_{13}t_4t_8+t_8^3t_1^3+2t_4t_1+2t_8t_4+t_{13}t_4+2t_{13}t_8-t_1t_8^3+t_8^2t_4-t_{13}t_4t_1^3+t_8t_4t_1^5+t_{13}t_1-$$

$$t_{13}t_1^3+(-t_1^7t_4^2t_8+t_1^6t_4^2t_8+t_1^5t_4^3t_8+t_1^5t_4^2t_8-2t_1^4t_4^3t_8+3t_1^3t_4^4t_8+t_1^5t_{13}t_4+t_1^5t_4t_8-t_1^4t_4^2t_8-2t_1^3t_4^3t_8-$$

$$2t_1^3t_4^2t_8^2+t_1^5t_4-t_1^4t_{13}t_4-t_1^4t_4t_8-2t_1^3t_{13}t_4^2-2t_1^3t_4^2t_8+t_1^3t_8^3-t_1^4t_4-t_1^3t_{13}t_4-2t_1^3t_{13}t_8-2t_1^3t_4^2-$$

$$t_1^3t_4t_8-t_1^3t_{13}-t_1^3t_4-t_1^3t_8-t_1^3)B^2+(-t_1^7t_4^2t_8+t_1^6t_4^2t_8+t_1^5t_4^3t_8+t_1^5t_4^2t_8-t_1^4t_4^3t_8+2t_1^3t_4^4t_8+$$

$$t_1^5t_{13}t_4+t_1^5t_4t_8-t_1^4t_4^2t_8-t_1^3t_4^3t_8+t_1^5t_4-t_1^4t_{13}t_4-t_1^4t_4t_8-t_1^3t_{13}t_4^2-t_1^3t_4^2t_8+t_1^3t_8^3-t_1^4t_4-t_1^3t_{13}t_4-$$

$$2t_1^3t_{13}t_8-t_1^3t_4^2-t_1^3t_4t_8-t_1^3t_{13}-t_1^3t_4-t_1^3t_8-t_1^3)B^3+(-t_1^7t_4^2t_8+t_1^6t_4^2t_8+t_1^5t_4^3t_8+t_1^5t_4^2t_8+t_1^5t_{13}t_4+$$

$$+(-t_1^7t_4^2t_8+t_1^6t_4^2t_8+t_1^5t_4^2t_8+t_1^5t_{13}t_4+t_1^5t_4t_8+t_1^5t_4)B^5+2t_{13}t_4t_1+2t_{13}t_8t_1-t_8t_4^3t_1^3+4t_8t_4t_1-$$

$$t_{13}t_4^2t_1^3+t_1^5t_4t_8-t_1^4t_4^2t_8-t_8t_4^5-t_8t_4^3+(-t_1^7t_4^2t_8+t_1^6t_4^2t_8+t_1^5t_4^3t_8+t_1^5t_4^2t_8-t_1^4t_4^3t_8+3t_1^3t_4^4t_8+$$

$$t_1^5t_{13}t_4+t_1^5t_4t_8-t_1^4t_4^2t_8-2t_1^3t_4^3t_8-2t_1^3t_4^2t_8^2-t_1^2t_4^4t_8+2t_1t_4^5t_8+t_1^5t_4-t_1^4t_{13}t_4-t_1^4t_4t_8-2t_1^3t_{13}t_4^2-$$

$$2t_1^3t_4^2t_8+t_1^3t_8^3+t_1^2t_4^3t_8-t_1t_4^4t_8-3t_1t_4^3t_8^2-t_1^4t_4-t_1^3t_{13}t_4-2t_1^3t_{13}t_8-2t_1^3t_4^2-t_1^3t_4t_8+t_1^2t_{13}t_4^2+$$

$$t_1^2t_4^2t_8-t_1^2t_8^3-2t_1t_{13}t_4^3-2t_1t_4^3t_8+t_1t_4^2t_8^2-4t_1t_4t_8^3-t_1^3t_{13}-t_1^3t_4-t_1^3t_8+t_1^2t_{13}t_4+2t_1^2t_{13}t_8+$$

$$t_1^2t_4^2+t_1^2t_4t_8+t_1t_{13}t_4^2+8t_1t_{13}t_4t_8-2t_1t_4^3+t_1t_4^2t_8+2t_1t_4t_8^2-t_1t_8^3-t_1^3+t_1^2t_{13}+t_1^2t_4+t_1^2t_8+$$

$$2t_1t_{13}t_4+2t_1t_{13}t_8+t_1t_4^2+4t_1t_4t_8+t_1^2+t_1t_{13}+2t_1t_4+t_1t_8+t_1)B+t_1t_4^2t_8^2-4t_1t_4t_8^3+t_1t_{13}t_4^2+$$

$$t_1t_4^2t_8+2t_1t_4t_8^2+4t_{13}t_8t_4+t_1-t_{13}t_4^3-2t_8^3t_4-t_8^3+t_{13}-t_8^2t_4^3-t_4^3$$

Theorem 1 Let $\alpha \in I$ be a simple root of a sufficiently differentiable function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ in an open interval *I*. If x_0 is in the vicinity of *α*, then the method FM_{32} has optimal convergence order 32.

Proof. Let e_x be an error in *x i.e.* $e_x = x - \alpha$, $c_1 = f'(\alpha)$, and $c_j = f^{(j)}(\alpha)/[j!f'(\alpha)]$ for $j = 1, 2, 3, 4, 5, 6$. The proof of an order of convergence is established by using MAPLE. We obtain the following error equation:

$$e_{y_5} = c_1{}^2a^2 + 5c_1a + 5c_1{}^6a^6 + 25c_1{}^5a^5 + 237c_1{}^4a^4 + 985c_1{}^3a^3 + 1895c_1{}^2a^2 + 1683c_1a + 561c_2{}^8 -$$

$$-6c_3c_1{}^6a^6 + \frac{47c_1{}^5a^5}{2} + \frac{1177c_1{}^4a^4}{6} + \frac{2239c_1{}^3a^3}{3} + \frac{4127c_1{}^2a^2}{3} + 1203c_1a + 401c_1a + 1c_2{}^6 +$$

$$+4c_4c_1{}^2a^2 + 5c_1a + 5c_1{}^2a^2 + 17/2c_1a + 17/2c_1a + 1^2c_2{}^5 + 11c_1{}^4a^4c_3{}^2 + \frac{169c_1{}^3a^3}{11}c_3{}^2 -$$

$$-\frac{c_5}{169} + \frac{778c_1{}^2a^2}{11}c_3{}^2 - \frac{3c_5}{389} + \frac{1218c_1a}{11}c_3{}^2 - \frac{5c_5}{609} + \frac{609c_3{}^2}{11} - \frac{5c_5}{11}c_1a + 1^2c_2{}^4 -$$

$$-9c_3c_4c_1a + 1^3c_1{}^2a^2 + \frac{59c_1a}{9} + \frac{59}{9}c_2{}^3 - 6c_1a + 1^3c_1{}^2a^2c_3{}^3 + 8c_3{}^3 - 1/48c_3c_5 -$$

$$-1/48c_4{}^2ac_1 + 8c_3{}^3 - 1/6c_3c_5 - 1/6c_4{}^2c_2{}^2 + 3c_3{}^2c_4c_1a + 1^4c_2 + c_3{}^4c_1a + 1^4c_1a + 1c_1{}^4a^4 +$$

$$+14c_1{}^3a^3 + 59c_1{}^2a^2 + 90c_1a + 45c_2{}^4 + -3c_1{}^3c_3a^3 - 22c_1{}^2c_3a^2 - 38c_1c_3a - 19c_3c_2{}^2 +$$

$$+c_4c_1a + 1^2c_2 + c_3{}^2c_1a + 1^2c_1{}^2a^2 + 5c_1a + 5c_2{}^2 - c_1c_3a - c_3c_2c_1{}^{12}a^{12} + 46c_1{}^{11}a^{11} +$$

$$+988c_1{}^{10}a^{10} + 12724c_1{}^9a^9 + 105177c_1{}^8a^8 + 569656c_1{}^7a^7 + 2045391c_1{}^6a^6 + 4906906c_1{}^5a^5 +$$

$$+7867690c_1{}^4a^4 + 8309000c_1{}^3a^3 + 5543491c_1{}^2a^2 + 2117286c_1a + 352881c_1{}^2a^2 + 5c_1a +$$

$$+5^2c_2{}^{16} - 10c_3c_1{}^{12}a^{12} + \frac{241c_1{}^{11}a^{11}}{5} + \frac{5286c_1{}^{10}a^{10}}{5} + \frac{134157c_1{}^9a^9}{10} + \frac{213281c_1{}^8a^8}{2} +$$

$$+\frac{2760832c_1{}^7a^7}{5} + \frac{9526764c_1{}^6a^6}{5} + \frac{22166539c_1{}^5a^5}{5} + \frac{34785142c_1{}^4a^4}{5} + \frac{14491245c_1{}^3a^3}{2} +$$

$$+\frac{47956127c_1{}^2a^2}{10} + \frac{9126786c_1a}{5} + \frac{1521131}{5}c_1a + 1c_1{}^2a^2 + 5c_1a + 5c_2{}^{14} + 10c_4c_1a + 1^2c_1{}^2a^2 +$$

$$+5c_1a + 5^2c_1{}^8a^8 + \frac{167c_1{}^7a^7}{5} + \frac{4609c_1{}^6a^6}{10} + \frac{6577c_1{}^5a^5}{2} + \frac{25525c_1{}^4a^4}{2} + \frac{54877c_1{}^3a^3}{2} +$$

$$+\frac{65281c_1{}^2a^2}{2} + 20138c_1a + \frac{10069}{2}c_2{}^{13} + 40c_1a + 1^2c_1{}^{12}a^{12}c_3{}^2 + \frac{254c_1{}^{11}a^{11}}{5}c_3{}^2 - \frac{5c_5}{2032} +$$

$$+\frac{8937c_1{}^{10}a^{10}}{8}c_3{}^2 - \frac{58c_5}{14895} + \frac{137427c_1{}^9a^9}{10}c_3{}^2 - \frac{1295c_5}{274854} + \frac{4183597c_1{}^8a^8}{40}c_3{}^2 - \frac{21607c_5}{4183597} +$$

$$+\frac{103379137c_1{}^7a^7}{20}c_3{}^2 - \frac{112279c_5}{20758274} + a^6\frac{34557833c_3{}^2}{20} - \frac{47843c_5}{5}c_1{}^6 + \frac{19562206c_1{}^5a^5}{5}c_3{}^2 -$$

$$-\frac{219307c_5}{39124412} + \frac{120451369c_1{}^4a^4}{20}c_3{}^2 - \frac{1359007c_5}{240902738} + \frac{12387785c_1{}^3a^3}{2}c_3{}^2 - \frac{56099c_5}{9910228} + a^2 -$$

$$-\frac{92317c_5}{4}+\frac{40696989c_3^2}{10}c_1^2+\frac{7720692c_1a}{5}c_3^2-\frac{58425c_5}{10294256}+\frac{1286782c_3^2}{5}-\frac{11685c_5}{8}c_2^{12}-$$

$$-74c_1^8a^8c_3c_4+\frac{1211c_1^7a^7}{37}c_3c_4-\frac{c_6}{2422}+\frac{31779c_1^6a^6}{74}c_3c_4-\frac{20c_6}{31779}+\frac{212635c_1^5a^5}{74}c_3c_4-$$

$$-\frac{153c_6}{212635}+\frac{391695c_1^4a^4}{37}c_3c_4-\frac{589c_6}{783390}+\frac{1630843c_1^3a^3}{74}c_3c_4-\frac{1245c_6}{1630843}+\frac{1907541c_1^2a^2}{74}c_3c_4-$$

$$-\frac{1465c_6}{1907541}+\frac{584540c_1a}{37}c_3c_4-\frac{45c_6}{58454}+\frac{146135c_3c_4}{37}-\frac{225c_6}{74}c_1a+1^3c_1^2a^2+5c_1a+5c_2^{11}-$$

$$-85c_1a+1^3c_1^{10}a^{10}c_3^3+\frac{804c_1^9a^9}{17}c_3^3-\frac{8c_3c_5}{1005}-\frac{7c_4^2}{1340}+\frac{74918c_1^8a^8}{85}c_3^3-\frac{451c_3c_5}{37459}-$$

$$-\frac{303c_4^2}{37459}\quad 2em+\frac{730386c_1^7a^7}{85}c_3^3-\frac{5180c_3c_5}{365193}-\frac{2395c_4^2}{243462}+\frac{4159632c_1^6a^6}{85}c_3^3-\frac{5307c_3c_5}{346636}-$$

$$-\frac{946c_4^2}{86659}+\frac{14573201c_1^5a^5}{85}c_3^3-\frac{231696c_3c_5}{14573201}-\frac{168672c_4^2}{14573201}+\frac{32104249c_1^4a^4}{85}c_3^3-\frac{520444c_3c_5}{32104249}-$$

$$-\frac{384258c_4^2}{32104249}+\frac{44447606c_1^3a^3}{85}c_3^3-\frac{364008c_3c_5}{22223803}-\frac{38733c_4^2}{3174829}+\frac{37478714c_1^2a^2}{85}c_3^3-$$

$$-\frac{308577c_3c_5}{18739357}-\frac{230964c_4^2}{18739357}+\frac{3515575c_1a}{17}c_3^3-\frac{11604c_3c_5}{703115}-\frac{8703c_4^2}{703115}+\frac{703115c_3^3}{17}-$$

$$-\frac{11604c_3c_5}{17}-\frac{8703c_4^2}{17}c_2^{10}+205c_1^8a^8c_3^2c_4+\frac{6431c_1^7a^7}{205}c_3^2c_4-\frac{5c_3c_6}{6431}-\frac{12c_4c_5}{6431}+$$

$$+\frac{79214c_1^6a^6}{205}c_3^2c_4-\frac{46c_3c_6}{39607}-\frac{118c_4c_5}{39607}+\frac{498662c_1^5a^5}{205}c_3^2c_4-\frac{669c_3c_6}{498662}-\frac{894c_4c_5}{249331}+$$

$$+\frac{1753771c_1^4a^4}{205}c_3^2c_4-\frac{2497c_3c_6}{1753771}-\frac{6844c_4c_5}{1753771}+\frac{3546088c_1^3a^3}{205}c_3^2c_4-\frac{5185c_3c_6}{3546088}-\frac{515c_4c_5}{126646}+$$

$$+\frac{4083296c_1^2a^2}{205}c_3^2c_4-\frac{6045c_3c_6}{4083296}-\frac{605c_4c_5}{145832}+\frac{497360c_1a}{41}c_3^2c_4-\frac{37c_3c_6}{24868}-\frac{26c_4c_5}{6217}+$$

$$+\frac{124340c_3^2c_4}{41}-\frac{185c_3c_6}{41}-\frac{520c_4c_5}{41}c_1a+1^4c_2^9+106c_1^8a^8c_3^4+\frac{4343c_3a^7c_1^7}{106}c_3^3-\frac{72c_3c_5}{4343}-$$

$$-\frac{92c_4^2}{4343}+\frac{63439c_1^6a^6}{106}c_3^4-\frac{1536c_3^2c_5}{63439}-\frac{2062c_3c_4^2}{63439}+\frac{c_6c_4}{63439}+\frac{c_5^2}{63439}+\frac{223926c_1^5a^5}{53}c_3^4-$$

$$+\frac{6271c_3^2c_5}{223926}-\frac{2921c_3c_4^2}{74642}+\frac{c_6c_4}{37321}+\frac{c_5^2}{37321}+\frac{1698171c_1^4a^4}{106}c_3^4-\frac{16926c_3^2c_5}{566057}-\frac{72911c_3c_4^2}{1698171}+$$

$$+\frac{56c_6c_4}{1698171}+\frac{56c_5^{\,2}}{1698171}+\frac{1794680c_1^{\,3}a^3}{53}c_3^{\,4}-\frac{11073c_3^{\,2}c_5}{358936}-\frac{16149c_3c_4^{\,2}}{358936}+\frac{13c_6c_4}{358936}+\frac{13c_5^{\,2}}{358936}+$$

$$+\frac{2115115c_1^{\,2}a^2}{53}c_3^{\,4}-\frac{66223c_3^{\,2}c_5}{2115115}-\frac{97342c_3c_4^{\,2}}{2115115}+\frac{16c_6c_4}{423023}+\frac{16c_5^{\,2}}{423023}+\frac{1300190c_1a}{53}c_3^{\,4}-$$

$$-\frac{20472c_3^{\,2}c_5}{650095}-\frac{30183c_3c_4^{\,2}}{650095}+\frac{5c_6c_4}{130019}+\frac{5c_5^{\,2}}{130019}+\frac{650095c_3^{\,4}}{106}+\frac{25c_6c_4}{106}-\frac{10236c_3^{\,2}c_5}{53}-$$

$$-\frac{30183c_3c_4^{\,2}}{106}+\frac{25c_5^{\,2}}{106}c_1a+1^4c_2^{\,8}-267c_1a+1^5c_3^{\,3}c_4a^6c_1^{\,6}+\frac{6458c_1^{\,5}a^5}{267}c_3^{\,3}c_4-\frac{4c_6c_3^{\,2}}{3229}-$$

$$-\frac{39c_3c_4c_5}{6458}-\frac{9c_4^{\,3}}{6458}+\frac{55816c_1^{\,4}a^4}{267}c_3^{\,3}c_4-\frac{25c_6c_3^{\,2}}{13954}-\frac{517c_3c_4c_5}{55816}-\frac{67c_4^{\,3}}{27908}+\frac{73052c_1^{\,3}a^3}{89}c_3^{\,3}c_4-$$

$$-\frac{37c_6c_3^{\,2}}{18263}-\frac{2371c_3c_4c_5}{219156}-\frac{325c_4^{\,3}}{109578}+\frac{410678c_1^{\,2}a^2}{267}c_3^{\,3}c_4-\frac{436c_6c_3^{\,2}}{205339}-\frac{4723c_3c_4c_5}{410678}-\frac{1325c_4^{\,3}}{410678}+$$

$$+\frac{120440c_1a}{89}c_3^{\,3}c_4-\frac{13c_6c_3^{\,2}}{6022}-\frac{283c_3c_4c_5}{24088}-\frac{10c_4^{\,3}}{3011}-\frac{1415c_3c_4c_5}{267}-\frac{260c_6c_3^{\,2}}{267}+\frac{120440c_3^{\,3}c_4}{267}-$$

$$-\frac{400c_4^{\,3}}{267}c_2^{\,7}-78c_1^{\,6}a^6c_3^{\,5}+\frac{421c_3^{\,2}a^5c_1^{\,5}}{13}c_3^{\,3}-\frac{23c_3c_5}{842}-\frac{22c_4^{\,2}}{421}+\frac{12932c_1^{\,4}a^4}{39}c_3^{\,5}-\frac{1013c_3^{\,3}c_5}{25864}-$$

$$-\frac{2035c_3^{\,2}c_4^{\,2}}{25864}+\frac{c_6c_4}{12932}+\frac{c_5^{\,2}}{12932}c_3+\frac{3c_4^{\,2}c_5}{25864}+\frac{56093c_1^{\,3}a^3}{39}c_3^{\,5}-\frac{5009c_3^{\,3}c_5}{112186}-\frac{10417c_3^{\,2}c_4^{\,2}}{112186}+$$

$$+\frac{7c_6c_4}{56093}+\frac{7c_5^{\,2}}{56093}c_3+\frac{21c_4^{\,2}c_5}{112186}+\frac{109934c_1^{\,2}a^2}{39}c_3^{\,5}-\frac{937c_3^{\,3}c_5}{19988}-\frac{26c_3^{\,2}c_4^{\,2}}{263}+\frac{8c_6c_4}{54967}+\frac{8c_5^{\,2}}{54967}c_3+$$

$$+\frac{12c_4^{\,2}c_5}{54967}+\frac{32755c_1a}{13}c_3^{\,5}-\frac{3121c_3^{\,3}c_5}{65510}-\frac{6611c_3^{\,2}c_4^{\,2}}{65510}+\frac{c_6c_4}{6551}+\frac{c_5^{\,2}}{6551}c_3+\frac{3c_4^{\,2}c_5}{13102}\quad 2em+$$

$$+\frac{32755c_3^{\,5}}{39}-\frac{3121c_3^{\,3}c_5}{78}-\frac{6611c_3^{\,2}c_4^{\,2}}{78}+\frac{5c_5^{\,2}}{39}+\frac{5c_6c_4}{39}c_3+\frac{5c_4^{\,2}c_5}{26}c_1a+1^5c_2^{\,6}+$$

$$+166c_3c_3^{\,3}c_4a^4c_1^{\,4}+\frac{1385c_1^{\,3}a^3}{83}c_3^{\,3}c_4-\frac{c_6c_3^{\,2}}{554}-\frac{18c_3c_4c_5}{1385}-\frac{19c_4^{\,3}}{2770}+\frac{13567c_1^{\,2}a^2}{166}c_3^{\,3}c_4-$$

$$-\frac{34c_6c_3^{\,2}}{13567}-\frac{260c_3c_4c_5}{13567}-\frac{149c_4^{\,3}}{13567}+\frac{10797c_1a}{83}c_3^{\,3}c_4-\frac{29c_6c_3^{\,2}}{10797}-\frac{224c_3c_4c_5}{10797}-\frac{130c_4^{\,3}}{10797}+$$

$$+\frac{10797c_3^{\,3}c_4}{166}-\frac{29c_6c_3^{\,2}}{166}-\frac{112c_3c_4c_5}{83}-\frac{65c_4^{\,3}}{83}c_1a+1^6c_2^{\,5}+35c_1^{\,4}a^4c_3^{\,6}+\frac{761c_3^{\,3}a^3c_1^{\,3}}{35}c_3^{\,3}-$$

$$-\frac{28c_3c_5}{761}-\frac{66c_4^{\,2}}{761}+\frac{4286c_1^{\,2}a^2}{35}c_3^{\,6}-\frac{112c_5c_3^{\,4}}{2143}-\frac{277c_4^{\,2}c_3^{\,3}}{2143}+\frac{c_6c_4}{4286}+\frac{c_5^{\,2}}{4286}c_3^{\,2}+\frac{3c_3c_4^{\,2}c_5}{4286}+$$

$$+\frac{c_4^{\,4}}{4286}+\frac{1410c_1a}{7}c_3^{\,6}-\frac{196c_5c_3^{\,4}}{3525}-\frac{488c_4^{\,2}c_3^{\,3}}{3525}+\frac{c_6c_4}{3525}+\frac{c_5^{\,2}}{3525}c_3^{\,2}+\frac{c_3c_4^{\,2}c_5}{1175}+\frac{c_4^{\,4}}{3525}+$$

$$+\frac{705c_3^{\,6}}{7}-\frac{28c_5c_3^{\,4}}{5}-\frac{488c_4^{\,2}c_3^{\,3}}{35}+1/35c_5^{\,2}+1/35c_6c_4c_3^{\,2}+\frac{3c_3c_4^{\,2}c_5}{35}+1/35c_4^{\,4}c_1a+1^6c_2^{\,4}-$$

$$-50c_1^{\,2}a^2c_3^{\,3}c_4+\frac{43c_1a}{5}c_3^{\,3}c_4-\frac{c_6c_3^{\,2}}{430}-\frac{9c_3c_4c_5}{430}-\frac{3c_4^{\,3}}{215}+\frac{43c_3^{\,3}c_4}{5}-\frac{1}{50}c_6c_3^{\,2}-\frac{9c_3c_4c_5}{50}-$$

$$-\frac{3c_4^{\,3}}{25}c_3^{\,2}c_1a+1^7c_2^{\,3}-9c_1^{\,2}a^2c_3^{\,3}+\frac{94c_1a}{9}c_3^{\,3}-\frac{2c_3c_5}{47}-\frac{11c_4^{\,2}}{94}+\frac{94c_3^{\,3}}{9}-4/9c_3c_5-$$

$$-\frac{11c_4^{\,2}}{9}c_3^{\,4}c_1a+1^7c_2^{\,2}+6c_3^{\,6}c_4c_1a+1^8c_2+c_3^{\,8}c_1a+1^8e^{32}+O(e^{33})$$

Numerical results

To validate our proposed iterative method FM_{32}, we adopt the following definition of approximated computational order of convergence (ACOC):

$$\text{ACOC}=\frac{\log[|\,f(x_{k-2})/f(x_{k-1})\,|]}{\log[|\,f(x_{k-1})/f(x_k)\,|]}$$

where x_i is the sequence of approximations to root α, which is computed by using FM_{32}. We choose six non-linear functions to compute the computational order of convergence. Table 1 list the functions and their simple roots. In tab. 2, we have shown that the approximated computational order of convergence is thirty-two, which supports our theoretically computed order of convergence. The effect of the parameter *a* of the accuracy of root approximation is depicted in figs. 1(a) and 1(b). The higher value of the graph corresponds to higher accuracy. We conclude that the smaller parameter values *a* give us better accuracy in approximating the first-order derivative. The smaller value *a* improves the accuracy of approximation of the root.

Table 1. Non-linear equations and simple roots

Non-linear functions	Simple roots
$f_1(x)=x^4\cos(x^2)-x^5\ln(1+x^2-\pi)+\pi^2$	1.77245
$f_2(x)=x\exp(x^2)-\sin(x)^2+3\cos(x)$	– 0.8077428
$f_3(x)=x^3-10$	2.154434
$f_4(x)=\sin(x)^2-x^2+1$	1.40449
$f_5(x)=(x+2)\exp(x)-1$	– 0.44285
$f_6(x)=(x-1)^3-2$	2.25992

Table 2. Approximated computational order of convergence, *a* = 00.01

	f_1	f_2	f_3	f_4	f_5	f_6
Initial guess	1.5	–1.2	2.15	1.40	–0.44	2.25
$f(x_1)$	2.36e-1	4.85	6.16e-2	1.11e-2	4.70e-3	4.69e-2
$f(x_2)$	3.82e-25	1.41e-5	5.45e-72	2.65e-62	6.30e-68	1.68e-51
$f(x_3)$	5.16e-849	2.96e-171	8.83e-2314	2.28e-2102	8.65e-2375	4.50e-1733
ACOC	31.9456	29.9191	32.0011	32.0023	31.9991	32.0058
	32.0000	31.9999	32.0000	32.0001	32.0000	32.0001

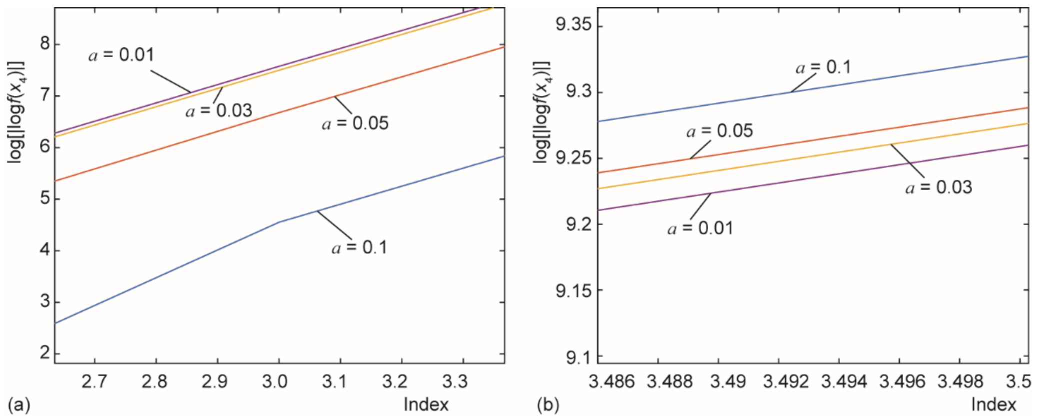

Figure 1. Effect *a* on the accuracy of root approximation; (a) effect of the parameter *a* on the accuracy of root approximation $f_1(x) = 0$ and (b) effect of the parameter *a* on the accuracy of root approximation $f_4(x) = 0$

Conclusions

Multi-point numerical iterative methods are numerically stable to compute the simple roots. We proposed a five-point numerical iterative method to compute the simple roots of non-linear equations. Our proposed method is numerically stable because it uses the information at six different points. It has thirty-two order of convergence that is optimal in the sense of K-T conjecture. Our numerical iterative method does not require the evaluation of the analytical derivative of the functions. We use the finite difference technique to approximate the derivative. The discretization of the first-order derivative introduces a parameter *a*. The small value of the parameter *a* can better approximate the first-order derivative. The parameter *a* can change the dynamics of the numerical iterative method by changing its path of convergence. To validate our numerical iterative method, we choose six non-linear equations to find the simple roots. In all numerical simulations, the value of the parameter *a* was 0.01. All the initial guesses are reported in tab. 2. We performed three iterations in all numerical simulations.

Acknowledgment

This project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, under grant no. G: 351-130-1439. The authors, therefore, acknowledge with thanks DSR for technical and financial support.

References

[1] Kung, H. T., Traub, J. F., Optimal Order of One-Point and Multi-Point Iteration, *Journal of the Association for Computing Machinery*, *21* (1974), 4, pp. 643-651
[2] Ostrowski, A. M., *Solutions of Equations and Systems of Equations*, Academic Press, New York, USA, 1966
[3] Ortega, J. M., Rheinboldt, W. C., *Iterative Solutions of Non-linear Equations in Several Variables*, Academic Press, New York, USA, 1970
[4] Ren, H., *et al*., A Class of Two-Step Steffensen Type Methods with Fourth-Order Convergence, *Applied Mathematics and Computation*, *209* (2009), 2, pp. 206-210
[5] Feng, X., He, Y., High Order Iterative Methods without Derivatives for Solving Non-linear Equations, *Applied Mathematics and Computation*, *186* (2007), 2, pp. 1617-1623
[6] Cordero, A., Torregrosa, J. R., A Class of Steffensen type Methods with Optimal Order of Convergence, *Applied Mathematics and Computation*, *217* (2011), 19, pp. 7653-7659
[7] Liu, Z., *et al*., A Variant of Steffensen's Method of Fourth-Order Convergence and Its Applications, *Applied Mathematics and Computation*, *216* (2010), 7, pp. 1978-1983
[8] Zheng, Q., *et al*., A Steffensen-Like Method and Its Higher-Order Variants, *Applied Mathematics and Computation*, *214* (2009), 1, pp. 10-16
[9] Cordero, A., *et al*., A New Technique to Obtain Derivative-Free Optimal Iterative Methods for Solving Non-linear Equations, *Journal of Computational and Applied Mathematics*, *252* (2013), Nov., pp. 95-102
[10] Soleymani, F., *et al*., A Class of Three-Step Derivative-Free Root-Solvers with Optimal Convergence Order, *Journal of Applied Mathematics*, *2012* (2012), ID 568740
[11] Cordero, A., Torregrosa, J. R., Variants of Newton's Method Using Fifth-Order Quadrature Formulas, *Applied Mathematics and Computation*, *190* (2007), 1, pp. 686-698

Paper submitted: July 15, 2021
Paper revised: July 28, 2021
Paper accepted: August 2, 2021