BIHARMONIC HEAT EQUATION WITH GRADIENT NON-LINEARITY ON L^p SPACE

by

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In this paper, we deal with the biharmonic heat equation with gradient non--linearity. Under the suitable condition of the initial datum, we show that the global unique existence of the mild solution. The main technique in the paper is to use Banach's fixed point theorem in combination with the L^p – L^q evaluation of biharmonic operator.

Key words: *biharmonic heat equation, L^p – L^q estimate, global existence, banach fixed point theorem*

Introduction

In this paper, we are interested to study the following biharmonic heat equation:

$$\begin{aligned} z_t + \Delta^2 z &= \psi(t)G(z,\nabla z), \quad (x,t)\in \Omega\times(0,T) \\ z = \frac{\partial z}{\partial \nu} &= 0, \quad (x,t)\in \partial\Omega\times(0,T) \\ z(x,0) &= f(x), \quad x\in\Omega \end{aligned} \tag{1}$$

where ν is the unit outward normal on $\partial\Omega$. The higher order parabolic equation, namely the fourth order parabolic equations are used in many practical application models. They occur in the Cahn-Hillard equation, image segmentation, epithelial thin film growth, surface diffusion current equation [1-6]. We list some other applications of quadratic PDE. Issues related to airfoils, bridge panels, floor systems, and window glazing are being modeled as panels bearing different types of end supports modeled as a quaternary PDE [7]. If the biharmonic operator Δ^2 is replaced by a second order operator $-\Delta$, the problem (1) is called problem is called classical heat equation [8-15]. The results of the classical thermal equations are plentiful and varied. To our best knowledge, there is limited previous study on the biharmonic heat equation with a such non-linearity.

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We explain in detail why our study of the problem is interesting.

- The appearance of quaternary operators and gradient terms is one of the main difficulties in research.
- The solution space we consider is the L^p space, and the technique of studying global solutions in this space is not simple. Our main idea is to use the lemma about $L^p - L^q$ estimate of Ishige-Kawakami-Kobayashi [16]. Another remark is that if we just use the standard as in Weissler [17], then we get only local solution. However, we do not follow in his method. In the paper, we follow the technique of the paper of Atienza. This technique is also studied in the work of Tuan and Carabullo [14].

Preliminary solutions and definitions of mild roots contains two main results of the article. The first result is about the existence and unique local solution. The second result is related to the convergence of the solution when the parameter k approaches 0.

Preliminaries

For each number $s \geq 0$, we define the following space:

$$H^s(\Omega) = \left\{ f = \sum_{k=1}^{\infty} \left[\int_{\Omega} f(x) \mathrm{e}_k(x) \mathrm{d}x \right] \mathrm{e}_k \in L^2(\Omega) : \sum_{k=1}^{\infty} \left[\int_{\Omega} f(x) \mathrm{e}_k(x) \mathrm{d}x \right]^2 \lambda_k^{2s} < \infty \right\} \tag{2}$$

and the norm of $f \in H^s(\Omega)$:

$$\| f \|_{H^s(\Omega)}^2 = \sum_{k=1}^{\infty} \left[\int_{\Omega} f(x) \mathrm{e}_k(x) \mathrm{d}x \right]^2 \lambda_k^{2s} \tag{3}$$

By a simple calculation, we get the following ordinary differential equation with Riemann-Liouville:

$$\frac{\mathrm{d}\left[\int_{\Omega} z(x,t) \mathrm{e}_k(x) \mathrm{d}x \right]}{\mathrm{d}t} = \lambda_k^2 \left[\int_{\Omega} z(x,t) \mathrm{e}_k(x) \mathrm{d}x \right] + \psi(t) \int_{\Omega} G[z(t), \nabla z(t)] \mathrm{e}_k(x) \mathrm{d}x$$

Multiplying bothsides to $\mathrm{e}^{-\lambda_k^2 t}$ and taking the integral from 0 to t, we get the definition of u in the below.

Definition 1 The function w is called a mild solution of Problem (1) if it satisfies:

$$z(t) = \mathrm{e}^{-t\Delta^2} f + \int_0^t \mathrm{e}^{-(t-\theta)\Delta^2} G[z(\theta), \nabla z(\theta)] \mathrm{d}\theta \tag{4}$$

where $e^{-t\Delta^2}$ is biharmonic heat semigroup and defined by the following Fourier series:

$$\mathrm{e}^{-t\Delta^2} f = \sum_{k=1}^{\infty} \mathrm{e}^{-t\lambda_k^2} \left[\int_{\Omega} f(x) \mathrm{e}_k(x) \mathrm{d}x \right] \mathrm{e}_k$$

for any $f \in L^2(\Omega)$.

First we state the following *Lemma* which will be useful in our main results (this lemma can be found in [18], *Lemma 8*, page 9).

Lemma 1 Let $a > -1,\ b > -1$ such that $a + b \geq -1,\ \theta > 0,$ and $t \in (0,T)$. For $\mu > 0,$ the following limit holds:

$$\lim_{\mu\to\infty}\left[\sup_{t\in(0,T)} t^{\theta}\int_0^1 r^a(1-r)^b \mathrm{e}^{-\mu t(1-r)}\mathrm{d}r\right] = 0$$

Lemma 2 There exists a positive constant C depending on p, q such that for any $1 \leq p \leq q$ then:

$$\|\nabla^j \mathrm{e}^{-t\Delta^2}\varphi\|_{L^q(\Omega)} \leq C(p,q)t^{-\frac{N}{4}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{j}{4}}\|\varphi\|_{L^p(D)},\quad t>0,\quad \varphi\in L^p(\Omega) \tag{5}$$

Proof. The proof can be found in [15].
Theorem 1 Let G be such:

$$G[z(.,t),\nabla z(.,t)] = \psi(t)F(z) + \psi(t)F(\nabla z)$$

Let F satisfy the following condition:

$$\|F(z_1)-F(z_2)\|_{L^q(\Omega)} \leq K_f\,\|z_1 - z_2\|_{L^p(\Omega)},\quad 1\leq q\leq p \tag{6}$$

where $1 \leq q \leq p$ and $1/q - 1/p < 3/N.$ Let us assume that ψ satisfies:

$$\sup_{0\leq t\leq T}|\psi(t)| \leq Ct^{\delta}$$

where $\delta < 3/4 - N/2(1/q - 1/p)$. Let us assume that $f \in L^q(\Omega)$ then Problem has a unique global existence in $X_{\mathrm{d},m}[(0,T];L^p(\Omega)]$ if m enough large. Here d satisfies:

$$\frac{N}{4}\left(\frac{1}{q}-\frac{1}{p}\right) \leq \mathrm{d} < \min\left[1-\delta, \frac{3}{4}-\frac{N}{4}\left(\frac{1}{q}-\frac{1}{p}\right)-\delta\right] \tag{7}$$

In addition, the mild solution $z \in L^r[0,T;L^p(\Omega)]$ and:

$$\|z\|_{L^r[0,T;L^p(\Omega)]} \leq C(p,q,r,m,T)T^{d-\frac{N}{4}\left(\frac{1}{q}-\frac{1}{p}\right)}\|f\|_{L^q(\Omega)} \tag{8}$$

for $1 < r < 1/\mathrm{d}$.

Proof. Let us define the space $X_{\mathrm{d},m}[(0,T];L^p(\Omega)]$ denotes the weighted space of all functions v, here $v \in X_{\mathrm{d},m}[(0,T];L^p(\Omega)]$ such:

$$\|f\|_{X_{\mathrm{d},m}[(0,T];L^p(\Omega)]} := \sup_{t\in(0,T]} t^{\mathrm{d}}\mathrm{e}^{-mt}\|f(t,\cdot)\|_{L^p(\Omega)} < \infty$$

where $p > 0.$ Let us set the following function:

$$Jz(t) = \mathrm{e}^{-t\Delta^2}f + \int_0^t \mathrm{e}^{-(t-\theta)\Delta^2}\psi(\theta)G[z(\theta),\nabla z(\theta)]\mathrm{d}\theta \tag{9}$$

If $z(t) = 0$ then:

$$\| Jz(t) \|_{L^p(\Omega)} = \| \mathrm{e}^{-t\Delta^2} f \|_{L^p(\Omega)} \le C(p,q) t^{-\frac{N}{4}\left(\frac{1}{q}-\frac{1}{p}\right)} \| f \|_{L^q(\Omega)} \tag{10}$$

This implies that the following estimate hold:

$$t^{\mathrm{d}} \mathrm{e}^{-mt} \| Jz(t) \|_{L^p(\Omega)} \le C(p,q) t^{d-\frac{N}{4}\left(\frac{1}{q}-\frac{1}{p}\right)} \| f \|_{L^q(\Omega)} \tag{11}$$

Since $d - N/4(1/q - 1/p) \ge 0$, we deduce that $Jz(t) \in X_{d,m}[(0,T]; L^p(\Omega)]$ if $z(t) = 0$. Let us take any functions $z_1, z_2 \in L^p(\Omega)$. Then we get the following estimate:

$$\| Jz_1(.,t) - Jz_2(.,t) \|_{L^p(\Omega)} =$$

$$= \left\| \int_0^t \mathrm{e}^{-(t-\theta)\Delta^2} \{[G(z_1(\theta), \nabla z_1(\theta)] - G[z_2(\theta), \nabla z_2(\theta)]\} \mathrm{d}\theta \right\|_{L^p(\Omega)} \le$$

$$\le \left\| \int_0^t \mathrm{e}^{-(t-\theta)\Delta^2} \psi_1(\theta) \{F[z_1(\theta)] - F[z_2(\theta)]\} \mathrm{d}\theta \right\|_{L^p(\Omega)} +$$

$$+ \left\| \int_0^t \mathrm{e}^{-(t-\theta)\Delta^2} \psi_2(\theta) \nabla \{F[z_1(\theta)] - F[z_2(\theta)]\} \mathrm{d}\theta \right\|_{L^p(\Omega)} = M_1 + M_2 \tag{12}$$

Let us to treat the first term M_1. By using *Lemma 2*, since $1 \le q \le p$, we get:

$$M_1 \le K_f C_1 C(p,q) \int_0^t \theta^{-\delta} (t-\theta)^{-\frac{N}{4}\left(\frac{1}{q}-\frac{1}{p}\right)} \left\| F[z_1(\theta)] - F[z_2(\theta)] \right\|_{L^q(\Omega)} \mathrm{d}\theta \tag{13}$$

This follows from (6):

$$t^{\mathrm{d}} \mathrm{e}^{-mt} M_1 \le K_f t^{\mathrm{d}} \mathrm{e}^{-mt} C_1 C(p,q) \int_0^t \theta^{-\delta} (t-\theta)^{-\frac{N}{4}\left(\frac{1}{q}-\frac{1}{p}\right)} \left\| z_1(\theta) - z_2(\theta) \right\|_{L^p(\Omega)} \mathrm{d}\theta \le$$

$$\le t^{\mathrm{d}} K_f C_1 C(p,q) \int_0^t \theta^{-\delta-\mathrm{d}} (t-\theta)^{-\frac{N}{4}\left(\frac{1}{q}-\frac{1}{p}\right)} \mathrm{e}^{-m(t-\theta)} \left[\theta^{\mathrm{d}} \mathrm{e}^{-m\theta} \left\| z_1(\theta) - z_2(\theta) \right\|_{L^p(\Omega)} \right] \mathrm{d}\theta \le$$

$$\le K_f C_1 C(p,q) t^{\mathrm{d}} \left[\int_0^t \theta^{-\delta-\mathrm{d}} (t-\theta)^{-\frac{N}{4}\left(\frac{1}{q}-\frac{1}{p}\right)} \mathrm{e}^{-m(t-\theta)} \mathrm{d}\theta \right] \| z_1 - z_2 \|_{X_{\mathrm{d},m}[(0,T);L^p(\Omega)]} \tag{14}$$

We continute to provide the estimation of M_2. By using *Lemma 2*, since $1 \le q \le p$ and together with (6), we derive:

$$t^{\mathrm{d}}\mathrm{e}^{-mt}M_2 \le C_1C(p,q)t^{\mathrm{d}}\mathrm{e}^{-mt}\int_0^t \theta^{-\delta}(t-\theta)^{-\frac{N}{4}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{4}}\left\|F[z_1(\theta)]-F[z_2(\theta)]\right\|_{L^q(\Omega)}\mathrm{d}\theta \le$$

$$\le K_f C_1C(p,q)t^{\mathrm{d}}\mathrm{e}^{-mt}\int_0^t \theta^{-\delta}(t-\theta)^{-\frac{N}{4}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{4}}\left\|z_1(\theta)-z_2(\theta)\right\|_{L^p(\Omega)}\mathrm{d}\theta \le$$

$$\le K_f C_1C(p,q)t^{\mathrm{d}}\left[\int_0^t \theta^{-\delta-\mathrm{d}}(t-\theta)^{-\frac{N}{4}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{4}}\mathrm{e}^{-m(t-\theta)}\mathrm{d}\theta\right]\| z_1-z_2 \|_{X_{\mathrm{d},m}[(0,T];L^p(\Omega)]} \tag{15}$$

It is obvious to see that if $0<\theta\le t\le T$ then we confirm:

$$(t-\theta)^{-\frac{N}{4}\left(\frac{1}{q}-\frac{1}{p}\right)} = (t-\theta)^{-\frac{N}{4}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{4}}(t-\theta)^{1/4} \le T^{1/4}(t-\theta)^{-\frac{N}{4}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{4}}$$

This observations together with (12), (14), and (15) allows us to derive:

$$\sup_{t\in[0,T]} t^{\mathrm{d}}\mathrm{e}^{-mt}\| Jz_1(.,t)-Jz_2(.,t)\|_{L^p(\Omega)} \le \sup_{t\in(0,T]} t^{\mathrm{d}}\mathrm{e}^{-mt}\left(M_1+M_2\right) \le$$

$$\le K_f(T^{1/4}+1)C_1C(p,q)\cdot$$

$$\cdot\left[\sup_{t\in[0,T]} t^{\mathrm{d}}\int_0^t \theta^{-\delta-\mathrm{d}}(t-\theta)^{-\frac{N}{4}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{4}}\mathrm{e}^{-m(t-\theta)}\mathrm{d}\theta\right]\| z_1-z_2 \|_{X_{\mathrm{d},m}[(0,T];L^p(\Omega)]} \tag{16}$$

For our next purpose, we continue to show:

$$\sup_{t\in[0,T]} t^{\mathrm{d}}\int_0^t \theta^{-\delta-\mathrm{d}}(t-\theta)^{-\frac{N}{4}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{4}}\mathrm{e}^{-m(t-\theta)}\mathrm{d}\theta \to 0, \quad \text{if } m\to+\infty \tag{17}$$

Indeed, by change variable $\theta = tz$, we find:

$$t^{\mathrm{d}}\int_0^t \theta^{-\delta-\mathrm{d}}(t-\theta)^{-\frac{N}{4}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{4}}\mathrm{e}^{-m(t-\theta)}\mathrm{d}\theta =$$

$$= t^{1-\delta-\frac{N}{4}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{4}}\int_0^1 z^{-\delta-\mathrm{d}}(1-z)^{-\frac{N}{4}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{4}}\mathrm{e}^{-mt(1-z)}\mathrm{d}z \tag{18}$$

We need the requirements of *Lemma 2*. Indeed, we easily to verify:

$$1-\delta-\frac{N}{4}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{4}>0,\ -\delta-d>-1$$

$$-\frac{N}{4}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{4}>-1,\ -\delta-d-\frac{N}{4}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{4}>-1 \tag{19}$$

These conditions ensure that (16) holds. Therefore, if m enough large then we have:

$$K_f(T^{1/4}+1)C_1C(p,q)\left[\sup_{t\in[0,T]} t^{\mathrm{d}}\int_0^t \theta^{-\delta-\mathrm{d}}(t-\theta)^{-\frac{N}{4}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{4}}\mathrm{e}^{-m(t-\theta)}\mathrm{d}\theta\right]\le\frac{1}{2} \tag{20}$$

So, the mapping J is a contraction mapping on the space $X_{\mathrm{d},m}[(0,T];L^p(\Omega)]$ if m enough large. Using Banach fixed point theorem, we deduce that J has a fixed point $u\in X_{\mathrm{d},m}[(0,T];L^p(\Omega)]$. It is obvious to see that z is a mild solution of Problem (1). We have in view of (16) and (20):

$$\|z\|_{X_{\mathrm{d},m}[(0,T];L^p(\Omega)]}=\|Jz\|_{X_{\mathrm{d},m}[(0,T];L^p(\Omega)]}\le$$

$$\le C(p,q)T^{\mathrm{d}-\frac{N}{4}\left(\frac{1}{q}-\frac{1}{p}\right)}\|f\|_{L^q(D)}+\frac{1}{2}\|z\|_{X_{\mathrm{d},m}[(0,T];L^p(\Omega)]} \tag{21}$$

which allows us to conclude:

$$t^{\mathrm{d}}\mathrm{e}^{-mt}\|z(t,\cdot)\|_{L^p(\Omega)}\le\|z\|_{X_{\mathrm{d},m}[(0,T];L^p(\Omega)]}\le 2C(p,q)T^{\mathrm{d}-\frac{N}{4}\left(\frac{1}{q}-\frac{1}{p}\right)}\|f\|_{L^q(\Omega)} \tag{22}$$

Therefore, multiply both sides of the above expression by $t^{-\mathrm{d}}\mathrm{e}^{mt}$, we get:

$$\|z(t,\cdot)\|_{L^p(\Omega)}\le 2C(p,q)\mathrm{e}^{mT}T^{\mathrm{d}-\frac{N}{4}\left(\frac{1}{q}-\frac{1}{p}\right)}t^{-\mathrm{d}}\|f\|_{L^q(\Omega)} \tag{23}$$

Since $1<r<1/\mathrm{d}$, we know that the proper integral $\int_0^T t^{-\mathrm{d}r}\mathrm{d}t<+\infty$. Hence, we follows from (23) that $z\in L^r[0,T;L^p(\Omega)]$ and:

$$\|z\|_{L^r[0,T;L^p(\Omega)]}\le C(p,q,r,m,T)T^{\mathrm{d}-\frac{N}{4}\left(\frac{1}{q}-\frac{1}{p}\right)}\|f\|_{L^q(\Omega)} \tag{24}$$

The proof of our *Theorem* is completed.

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